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# OBSERVABILITY OF A CLASS OF NONLINEAR SYSTEMS: A GEOMETRIC APPROACH

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Observability of a class of nonlinear systems: a geometric approach*)

by

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ABSTRACT

The purpose of the paper is twofold. Firstly, based on a set of invariants, a local canonical form for a locally weakly observable system without inputs is derived. Secondly a class of nonlinear control systems is introduced for which this set of invariants is unaffected by an arbitrary input function.

KEY WORDS & PHRASES: *Nonlinear control systems, observability, (invariant) distributions, canonical forms*

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*) This report will be submitted for publication elsewhere.

# 1. INTRODUCTION

We consider an affine nonlinear control system of the form (locally)

(1.1a) $\dot{x} = A(x) + \sum_{i=1}^{m} B_i(x)u_i := A(x) + B(x)u$

(1.1b) $y = C(x)$

where $x \in M$, $M$ is the analytic state manifold,
$u \in \mathbb{R}^m$, the input space,
$y \in N$, $N$ is the analytic output manifold,
$A, B_1, \ldots, B_m$ are analytic vector fields on $M$,
and $C : M \to N$ is an analytic surjective submersion.
During the last decade two of the basic notions of systems theory, namely controllability and observability, have been studied for such systems. From a mathematical point of view the nonlinear generalization of these notions have been dealt in a rather satisfactory way, see for example the basic paper of HERMANN & KRENER [7]. One of the important facts of that paper is that nonlinear controllability and observability really can be treated in a dual way. This is a well-known fact from linear systems theory. But in studying nonlinear observability (in the sense of [7]) one also encounters a phenomena that is completely different from linear systems. Consider the linear control system

(1.2a) $\dot{x} = Ax + Bu$

(1.2b) $y = Cx$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $A,B,C$, are matrices of appropriate dimension. Then the linear systems (1.2a,b) is observable - i.e. from knowledge of the input function and output function we can determine the (initial) state of the system - if the following geometric condition holds (see e.g. [19])

(1.3.) $\bigcap_{j=0}^{n-1} \text{Ker } CA^j = 0.$

While condition (1.3) is independent of the matrix B we have that if the system (1.2a,b) is observable then it is observable for any input function. (Hence observability for some input implies observability.) For the nonlinear system (1.1a,b) the situation is completely different. The analytic system (1.1a,b) is defined to be observable if there exists some input function such that, based on the knowledge of this input function and its corresponding output function one can exactly determine the initial state (locally), see SUSSMANN [17]. A priori it follows that, although the system (1.1a,b) is observable, we cannot decide whether or not a given input function distinguishes any two different initial states, which lie in a (small) neighbourhood of each other. Fortunately, for analytic systems the situation is better; almost every input function is a distinguishing input function (cf. [17], see also [15],[1]). From a practical point of view this is of course not a very satisfactory solution. What one really wants is that, as in the linear case, every input function is a distinguishing input. The approach we present here illucidates this difference between linear and nonlinear systems. Building blocks in our treatment are involutive distributions. It is known from differential geometry that these distributions are in fact the nonlinear equivalents of linear subspaces of a given linear space. In that sense our paper mimics the geometric approach to linear multivariable systems ([19]). In the first part, section 2, we will generalize for the system (1.1a,b) with zero input function, the linear observability condition (1.3). This automatically leads us via a nested set of distributions, here called the *unobservable structure*, to a local (nonlinear) canonical form. In section 3 we exactly point out which nonlinear systems of the type (1.1a,b) 'almost' behave locally, as far as observability concerns, as linear systems. Finally section 4 contains a discussion of the results obtained.

## 2. A CANONICAL FORM FOR LOCALLY WEAKLY OBSERVABLE SYSTEMS

Let M be a connected analytic n-dimensional manifold. Consider the affine control system $\sum$ on M, locally defined by

$$(2.1) \qquad \begin{cases} \dot{x}(t) = A(x(t)) + \sum_{i=1}^{m} B_i(x(t))u_i(t) \\ y(t) = C(x(t)) \end{cases}$$

where $A, B, \ldots, B_m$ are analytic vectorfields on M and $C : M \to N$ is an analytic outputmap (N is the analytic output manifold). We will assume that C is a surjective submersion, so that the rank of $C_* : TM \to TN$ is fixed (= dimension of N). Recall the following procedure from HERMANN & KRENER [7]. Define the codistribution G on M as the smallest codistribution that contains $C^*(T^*N)$ and which is closed with respect to Lie differentiating by the vectorfields $A, B_1, \ldots, B_m$. The system $\sum$ satisfies the *observability rank condition* if $\dim G = n$.

The system $\sum$ satisfies the observability rank condition at a point $x_0 \in M$ if $\dim G(x_0) = n$. This observability rank condition is related to the concept of *local weak observability* (cf. [7]). Roughly speaking local weak observability at $x_0$ means that one can instantaneously distinguish $x_0$ from its neighbors (see [7] for a precise definition).

THEOREM 2.1. ([7]) *If* $\sum$ *satisfies the observability rank condition at* $x_0$, *then* $\sum$ *is locally weakly observable at* $x_0$.

From the theorem above we see that if $\sum$ satisfies the observability rank condition then $\sum$ is locally weakly observable. The converse is also almost true:

THEOREM 2.2. ([7]) *If* $\sum$ *is locally weakly observable, then the observability rank condition is satisfied on an open and dense submanifold* $M'$ *of* M.

For deriving a local canonical form we will restrict our attention to the case where there are no inputs (see e.g. also AEYELS [1]). In this case one automatically gets a condition which is analogous to the linear case (compare ISIDORI [9]). So we consider the analytic system

$$(2.2) \qquad \begin{cases} \dot{x}(t) = A(x(t)) \\ y(t) = C(x(t)) \end{cases}$$

where A and C are as in (2.1). The lack of observability can be given in the following way:

DEFINITION 2.3. The *unobservable structure* of (2.2) is defined as a set of distributions on M which are given by

$$(2.3)\qquad \begin{aligned} D_0 &:= \operatorname{Ker} C_* \\ D_i &= \{X \in D_{i-1} \cap V^{\infty}(M) \mid [A,X] \in D_{i-1}\} \end{aligned}$$

(Here $V^{\omega}(M)$ denotes the set of analytic vectorfields on M).

We immediately obtain the following result, which is similar to a result of GAUTHIER & BORNARD ([6]).

PROPOSITION 2.4.

(i) $D_0 \supset D_1 \supset \ldots \supset D_{n-1} = D_n = D_{n+1} = \ldots$

(ii) *Each distribution* $D_i$ $(i = 0,1,2,\ldots)$ *is involutive and has fixed dimension.*

PROOF.

From the fact that $C : M \to N$ is a surjective submersion it follows that $D_0 = \operatorname{Ker} C_*$ is involutive and has fixed dimension. Now suppose that $D_k$ satisfies the properties of the proposition. Then, for $D_{k+1} = \{X \in D_k \cap V^{\omega}(m) \mid [A,X] \in D_k\}$ we have for any pair $X,Y \in D_{k+1}$, by using the Jacobi-identity, that $[A,[X,Y]] = -[X,[Y,A]] - [Y,[A,X]] \in D_k$. Therefore also $[X,Y] \in D_{k+1}$ is involutive. Furthermore let $P_0$ be the analytic codistribution on M defined by $P_0 := C^*(T^*N)$, that is $D_0$ is the *annihilator* of $P_0$ : $X \in D_0 \iff \forall\omega \in P_0$, $\omega(X) \equiv 0$. Then it is easy to see that the analytic codistribution $P_{k+1} := P_0 + L_A P_0 + \ldots + L_A \ldots L_A P_0$ (k+1 times $L_A$) is involutive and has fixed dimension on an open and dense submanifold $M_{k+1}$ of M. For each point in $M \backslash M_{k+1}$ the dimension of $P_{k+1}$ is smaller than the dimension of $P_{k+1}$ on $M_{k+1}$ (lower semi-continuity). Now $D_{k+1}$ is the *analytic* annihilator of $P_{k+1}$ and therefore has fixed dimension (The dimension of an analytic distribution on M is a lower semi-continuous function). □

REMARK. Theorem 2.2 implies that if the system (2.2) is locally weakly observable then it follows that there exists an open and dense submanifold $M' := M_{n-1}$ where the codistribution $P_{n-1}$ has dimension n. In the sequel several times we refer to this submanifold M'. In this paper we will not consider the observability of points on $M \backslash M'$.

Theorem 2.1 reduces to a 'linear' test:

COROLLARY 2.5. *If the system (2.2) is locally weakly observable, then* $D_{n-1} = \underline{0}$ *(= the null-distribution).*

For linear systems corollary 2.5 comes down to the well-known observability test:

$$(2.4) \qquad \bigcap_{j=0}^{n-1} \operatorname{Ker} C A^j = 0$$

and the distribution $D_i$ corresponds to a linear subspace $V_i$ defined by

$$(2.5) \qquad V_i := \bigcap_{j=0}^{i} \operatorname{Ker} C A^j .$$

REMARKS.

(i) The equations (2.5) and (2.4) together with (2.3) show why we call the distributions $D_0 \supset D_1 \supset \ldots \supset D_{n-1}$ the unobservable structure, namely $D_i$ corresponds exactly to those states which we cannot distinguish based on knowledge of $y(t), \dot{y}(t), \ldots, y^{(i-1)}(t)$.

(ii) The analyticity is a necessary condition in the whole procedure given in the definition of the unobservable structure. For smooth systems the dimension of the distribution $D_i$ can change at singular points. This comes from the fact that for an analytic function the infinite jet at a point completely specifies the function (see also [5]).

For example consider the following systems on $\mathbb{R}^4$ :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} x_3 \\ \alpha(x_4) \\ x_4 \\ 0 \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $\alpha(x_4) = \begin{cases} \exp(-1/x_4^2) & x_4 > 0 \\ 0 & x_4 \le 0 \end{cases}$

then $\dim D_0 = 2$ for all $x \in \mathbb{R}^4$

$\dim D_1 = 0$ for $x_4 > 0$

$\phantom{\dim D_1} = 1$ for $x_4 < 0$.

If $D_{n-1} \neq 0$, then the system (2.2) is unobservable; $D_{n-1}$ corresponds to the unobservability ideal of FLIESS ([5]), and the output C is identical for those initial conditions which lie on the same integral manifold of D. Another way of expressing this is given by the *invariantness* condition. Namely the unobservability distribution D of (2.2) is the largest involutive distribution D of fixed dimension in Ker $C_*$ which is invariant under A (see ISIDORI et al. [11]). So D is supremal with respect to

$$(2.6) \qquad \begin{cases} D \subset \text{Ker } C_* \cap V^{\omega}(M) \\ [A,D] \subset D. \end{cases}$$

For deriving a canonical form for the system (2.2) we will restrict our attention to systems that are locally weakly observable. This is a rather usual assumptions, especially for realization purposes.
Based on proposition 2.4 we define the following indices which are the duals of the usual observability indices.

DEFINITION 2.6. The *dual observability indices* $\kappa_i$ $(i = 0,\dots,n-1)$ of the system (2.2) are given by

$$(2.7) \qquad n - \dim D_i =: \kappa_i \qquad i = 0,1,\dots,n-1.$$

REMARK.
(i) $\kappa_0 = p, \quad p = \dim N$
(ii) $\kappa_0 < \kappa_1 \dots \leq \kappa_{n-1} = n$ (by the fact that the system (2.2) is locally weakly observable).

We are now able to derive a canonical form for (2.2) in those points where the observability rank condition is met, i.e. on the open and dense submanifold M' of M given by theorem 2.2. Before we will do this we quote a lemma from JAKUBCZYK & RESPONDEK([12]):

LEMMA 2.7. *Locally around each point* $\in$ M *we can find a coordinate system on* M *such that*

$$D_0 = \text{Span}\left\{ \frac{\partial}{\partial x_{\kappa_0+1}}, \frac{\partial}{\partial x_{\kappa_0+2}}, \dots, \frac{\partial}{\partial x_n} \right\}$$
$$D_i = \text{Span}\left\{ \frac{\partial}{\partial x_{\kappa_i+1}}, \dots, \frac{\partial}{\partial x_n} \right\}$$
$$D_{n-1} = 0.$$

<u>THEOREM 3.8</u>. *For a locally weakly observable system* (2.2) *we can find around each point* p in M' *a coordinate system on* M' *and a coordinate system around* C(p) *in* N *such that there exist functions* $\alpha_i^j$ *such that* (2.2) *has the form*

$$\text{(2.8)} \qquad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{\kappa_1-\kappa_0} \\ x_{\kappa_1-\kappa_0+1} \\ \cdot \\ \cdot \\ x_{\kappa_0} \\ x_{\kappa_0+1} \\ \cdot \\ \cdot \\ x_{\kappa_0+\kappa_2-\kappa_1} \\ x_{\kappa_0+\kappa_2-\kappa_1+1} \\ \cdot \\ \cdot \\ x_{\kappa_1} \\ \cdot \\ \cdot \\ \cdot \\ x_{\kappa_i} \\ x_{\kappa_i+1} \\ \cdot \\ \cdot \\ x_{\kappa_i+\kappa_{i+2}-\kappa_{i+1}} \\ x_{\kappa_i+\kappa_{i+2}-\kappa_{i+1}+1} \\ \cdot \\ \cdot \\ x_{\kappa_{i+1}} \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} x_{\kappa_0+1} \\ x_{\kappa_0+2} \\ \cdot \\ \cdot \\ x_{\kappa_1} \\ \alpha_1^1(x_1,\ldots,x_{\kappa_1}) \\ \cdot \\ \cdot \\ \alpha^1_{2\kappa_0-\kappa_1}(x_1,\ldots x_{\kappa_1}) \\ x_{\kappa_1+1} \\ \cdot \\ \cdot \\ x_{\kappa_2} \\ \alpha_1^2(x_1,\ldots,x_{\kappa_2}) \\ \cdot \\ \cdot \\ \alpha^2_{2\kappa_1-\kappa_0-\kappa_2}(x_1,\ldots,x_{\kappa_2}) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{\kappa_{i+1}+1} \\ \cdot \\ \cdot \\ x_{\kappa_{i+2}} \\ \alpha_1^{\kappa_{i+1}}(x_1,\ldots,x_{\kappa_{i+2}}) \\ \cdot \\ \cdot \\ \alpha^{\kappa_{i+1}}_{2\kappa_{i+1}-\kappa_i-\kappa_{i+2}}(x_1,\ldots,x_{\kappa_{i+2}}) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

$$(2.9) \qquad y = C(x_1,\dots,x_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_{\kappa_0} \end{pmatrix},$$

*to be called the observable canonical form.*

PROOF.
From [16] we know that around each point $p \in M$ (and thus $p \in M'$) and $C(p) \in N$, there exist coordinate systems such that the map $C : M \to N$ takes the form described by equation (2.9). Note that the distribution $D_0$ has the form as given given in lemma 2.7:

$$D_0 = \mathrm{Span}\left\{\frac{\partial}{\partial x_{\kappa_0+1}}, \dots, \frac{\partial}{\partial x_n}\right\}.$$

Denoting the components of the output map (in these local coordinates) by $C_1,\dots,C_{\kappa_0}$ and using lemma 2.7 we obtain that the function

$$(C_1,\dots,C_{\kappa_0}, L_A C_1,\dots,L_A C_{\kappa_0})^T : \mathbb{R} \to \mathbb{R}^{2\kappa_0}$$

has fixed rank (namely this rank equals $\kappa_1$). We may also choose the coordinate system on $M'$ as in lemma 2.7 and we may set (eventually after only a permutation on $y_1,\dots,y_{\kappa_0}$ and shrinking the coordinate neighborhood on $M'$)

$$L_A C_1(x) = x_{\kappa_0+1},\dots,L_A C_{\kappa_1-\kappa_0} = x_{\kappa_1}.$$

Furthermore we have by definition of $D_1$ that $L_A C_j(x)$ only depends on $x_1,\dots,x_{\kappa_1}$ $(j = \kappa_1 - \kappa_0 + 1,\dots,\kappa_0)$. Therefore we define

$$L_A C_j(x) := \alpha^1_{j-\kappa_1-\kappa_0}(x_1,\dots,x_{\kappa_1}) \qquad (j = \kappa_1 - \kappa_0 + 1,\dots,\kappa_0).$$

The above construction exactly yields the first $\kappa_0$ rows of the vector field A as in equation (2.8). Repetition of the above procedure clearly leads to the canonical form given by equations (2.8) and (2.9). □

REMARKS.

(i) For $\kappa_0 = 1$ this canonical form can be found in GAUTHIER & BORNARD [6].

(ii) The observable canonical form derived here is not the usual one that is considered in linear systems theory, see for instance BRUNOVSKY [4], POPOV [13] and related other work on canonical forms. For example for a single output observable linear system $\dot{x} = Ax$, $y = Cx$ one has as canonical form

$$(2.10)\qquad \begin{cases} C = (1 \quad 0 \; . \; . \; . \; . \; . \; 0) \quad \text{and} \\ A = \begin{pmatrix} * & 1 & & & \bigcirc \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ \vdots & \bigcirc & & & 1 \\ * & 0 & & & 0 \end{pmatrix} \end{cases}$$

whereas we obtain

$$(2.11)\qquad \begin{cases} C = (1 \quad 0 \; . \; . \; . \; . \; . \; 0) \\ A = \begin{pmatrix} 0 & 1 & & & \bigcirc \\ & & \ddots & & \\ & & & \ddots & \\ & \bigcirc & & & 1 \\ * & . & . & . & * \end{pmatrix} \end{cases}$$

The main drawback of (2.11) compared to (2.10) is that (2.11) is not very useful for feedback purposes.(output injection!) For linear systems the transformation which brings (2.11) in the form given by (2.10) may be found in the original paper of BRUNOVSKI [3]. At this moment it is unclear if a similar nonlinear transformation for (2.8) can be found.

(iii) We emphasize once more that in the above coordinate system the distributions $D_i$ take the form described by lemma 2.7.

(iv) In a similar way we can also derive a canonical form for the unobservable case. Then, according to corollary 2.5 and also equation (2.6) there exists a distribution $0 \neq D = D_i$ for a certain $i \in \{0,1,\ldots,n-2\}$ such that $[A,D] \subset D$. Analogously we can prove that in a coordinate system as in lemma 2.7 we obtain that

$$D = D_i = \text{Span}\left\{ \frac{\partial}{\partial x_{\kappa_i+1}}, \ldots, \frac{\partial}{\partial x_n} \right\}, \quad A(x) = \begin{pmatrix} A^1(x) \\ A^2(x) \end{pmatrix},$$

where $A^1$ represents the first $\kappa_i$ rows of the vector field A (in this coordinate system) and we have that

$$\begin{pmatrix} A^1(x) \\ A^2(x) \end{pmatrix} = \begin{pmatrix} A^1(x_1,\ldots,x_{\kappa_i}) \\ A^2(x_1,\ldots,x_n) \end{pmatrix}.$$

Furthermore $A^1(x_1,\ldots,x_{\kappa_i})$ is structured as in the above theorem.

## 3. NONLINEAR CONTROL SYSTEMS WITH A UNIFORM UNOBSERVABLE STRUCTURE

In this section we return to the analytic system (2.1):

$$(2.1) \qquad \begin{cases} \dot{x}(t) = A(x(t)) + \sum_{i=1}^{m} B_i(x(t))\, u_i(t) \\ y(t) = C(c(t)) \end{cases}$$

and we will assume that this system is locally weakly observable. It is well known from recent work - see e.g. SONTAG [15], SUSSMANN [17], AEYELS [1] - that local weak observability leads to the existence of a *universal distinguishing* input function, i.e. each two different states give, using this input, different output functions. Furthermore it is known from the references quoted above that the set of universal distinguishing inputs are dense in the set of all analytic inputs (in the $C^{\omega}$ topology). A drawback of this result is that for practical purposes such a generic property is not very useful (constructive). Motivated by this Prodip Sen gives, for bilinear systems, a constructive procedure for a universal distinguishing input (c.f. [14]), which is as close as required to a given input. Another way to overcome this problem is, for single output systems, given in GAUTHIER & BORNARD [6], where each (constant) input function distinguishes different states. Our work is more or less in the spirit of their approach.

Each constant input function $\bar{u} = (\bar{u}_1,\ldots,\bar{u}_m)$ in (2.1) gives rise to a system of the type (2.2), namely

$$(3.1) \qquad \begin{cases} \dot{x} = A(x) + \sum_{i=1}^{m} B_i(x)\bar{u}_i \\ y = C(x). \end{cases}$$

One way to study the observability of the system (2.1) is by investigating the unobservable structure of (3.1) for each constant input function. If for the system (2.1) each constant input function leads to the same unobservable structure then we are in a nice situation; the observability properties of the system can be studied by using the observability properties of the 'free' system (2.2). Therefore we define:

DEFINITION 3.1. The system (2.1) has a *uniform unobservable structure* if for all constant input functions $\bar{u}$ the associated system (3.1) has the same unobservable structure.

REMARKS.

(i) We see that for a system (2.1) with a uniform unobservable structure, the unobservable structure is completely specified by that of the 'free' system (2.2) (see definition 2.3).

(ii) It is easy to see that a linear system $\dot{x} = Ax + Bu$, $u = Cx$ has a uniform unobservable structure, defined by (2.5).

(iii) If a system (2.1) has a uniform unobservable structure then for every analytic time-varying input function we obtain the same unobservable structure.

The following proposition relates the concept of a uniform unobservable structure to recent work in invariant distributions. The notion of an invariant distribution is the generalization of an invariant subspace for linear systems and has been used for solving the nonlinear Disturbance Decoupling (see e.g. ISIDORI et al. [10], HIRSCHORN [8]).

PROPOSITION 3.2. *Suppose that the system* (2.1) *is locally weakly observable. Then the system has a uniform unobservable structure of distributions* $D_0 \supset D_1 \supset \dots \supset D_{n-1} = 0$ *defined according to* (2.3), *if the input vector fields* $B_i$ $(i = 1,\dots,m)$ *satisfy*

$$(3.2) \qquad [B_i, D_j] \subset D_j \qquad \begin{array}{l} i = 1,\dots,m \\ j = 0,\dots,n-1. \end{array}$$

PROOF.
Define the distributions $D_j$ ($j = 0,\dots,n-1$) as in (2.3) and assume that $D_{n-1} = 0$. So the 'free' system (2.2) is locally weakly observable on $M^1$ ($M^1$ defined as in theorem 2.2). Now if the input vector fields $B_i$ satisfy (3.2) then we have for each constant input function $\bar{u} = (\bar{u}_1,\dots,\bar{u}_m)$ that

$$[A + \sum_{i=1}^{m} B_i \bar{u}_i, D_j] \subset [A,D_j] + \sum_{i=1}^{m} [B_i,D_j]\, \bar{u}_i$$

$$\subset D_{j-1} + D_j \subset D_{j-1} \text{ for } j = 1,\dots,n-1.$$

Therefore the system (2.1) has a uniform unobservable structure. □

REMARK. In the coordinate systems of theorem 2.8 the condition $[B_i,D_j] \subset D_j$ $i = 1,\dots,m$, $j = 1,\dots,n-1$ implies that the vector fields $B_i$ have the form:

$$B_i(x_1,\dots,x_n) = (\beta_1^i(x_1,\dots,x_{\kappa_0}),\dots,\beta_{\kappa_0}^i(x_1,\dots,x_{\kappa_0}),\beta_{\kappa_0+1}^i(x_1,\dots,x_{\kappa_1}),\dots$$
$$\dots,\beta_{\kappa_1}^i(x_1,\dots,x_{\kappa_1}),\dots)^T.$$

From the proof of this proposition we see that condition (3.2) is too strong for a uniform unobservable structure. At a first glance already $[B_i,D_j] \subset D_{j-1}$ $j = 1,\dots,n-1$, $i = 1,\dots,m$, is a sufficient condition. Tha this is not true can easily be shown by an example.

EXAMPLE 3.3. Consider the bilinear system on $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = x_2 + ux_2, & y = x_1 \\ \dot{x}_2 = 0. \end{cases}$$

Then we have that $D_0 = \text{span}\{\frac{\partial}{\partial x_2}\}$, $D_1 = \underline{0}$.
Furthermore

$$\left[\begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in D_0,$$

but for $u \equiv -1$ the system is not observable. □

Before we can go to a necessary condition for a uniform unobservable structure we need one more definition.

DEFINITION 3.4. Let $D_0$ and $D_1$ be two analytic involutive distributions of fixed dimension on M with $D_0 \supset D_1$. Then for $A \in V^{\omega}(M)$ define rank $([A,D_1] \bmod D_0)$ in the following way. Choose local coordinates as in lemma 2.7 such that

$$D_0 = \mathrm{Span}\{\frac{\partial}{\partial x_\kappa},\dots,\frac{\partial}{\partial x_\ell},\dots,\frac{\partial}{\partial x_n}\},\ D_1 = \mathrm{Span}\{\frac{\partial}{\partial x_\ell},\dots,\frac{\partial}{\partial x_n}\}.$$

Now

$$\mathrm{rank}([A,D_1] \bmod D_0)(x) := \underset{j=\ell,\dots,k}{\mathrm{rank}} \quad ([A,\frac{\partial}{\partial x_j}] \bmod (\frac{\partial}{\partial x_k},\dots,\frac{\partial}{\partial x_n}))(x)$$

(Note: this definition does not depend on the coordinate system.)

Now we obtain the following result, which is, if we exclude singularities in the rank conditions, on an open and dense submanifold of M a necessary and sufficient condition for a uniform unobservable structure.

THEOREM 3.5. *Suppose there is given an analytic system* (2.1) *and let* $D_0 \supset D \supset \dots \supset D_{n-1}$ *be the unobservable structure of the associated free system* (2.2). *Then the system* (2.1) *has a uniform unobservable structure if*

(3.3) $\qquad [B_i,D_j] \subset D_{j-1}, \qquad i = 1,\dots,m,\ j = 1,\dots,n-1$

*and for every constant input function* $\bar{u} = (\bar{u}_1,\dots,\bar{u}_m)$

(3.4) $\qquad \mathrm{rank}([A+B\bar{u},D_j] \bmod D_{j-1}) = \dim D_{j-1} - \dim D_j, \qquad j = 1,\dots,n-1.$

PROOF. The proof mimics the proof of theorem 2.8. As in theorem 2.8 we have to work on an open and dense submanifold M' of M on which the observability rank condition holds for the free system. Again we will use a coordinate system as in lemma 2.7. First we see that from the output function we can exactly determine (of course locally) the coordinates $x_1,\dots,x_{\kappa_0}$. From $[B_i,D_1] \subset D_0$ and $\mathrm{rank}([A+B\bar{u},D_1] \bmod D_0) = \dim D_0 - \dim D_1 = n-\kappa_0-(n-\kappa_1) = \kappa_0-\kappa_1$ we then exactly can determine the coordinates $x_1,\dots,x_{\kappa_1}$. (In fact we use here information of $\bar{u},y(t),\dot{y}(t)$.) Continuing this procedure we exactly determine the coordinates $x_1,\dots,x_{\kappa_{n-1}}$ (Probably $D_{n-1} \neq \underline{0}$). Finally we note that the above description means that for each constant input function

$\bar{u}$ we have that (on M'):

$$D_i = \{X \in D_{i-1} \cap V^{\omega}(M) \mid [A + B\bar{u}, X] \in D_{i-1}\},$$

namely condition (3.3) implies that

$$(3.5) \qquad [A + B\bar{u}, D_j] \subset D_{j-1} \qquad j = 1,\dots,n-1$$

and from condition (3.4) we see that each of the $D_j$ ($j = 1,\dots,n-1$) is really supremal with respect to the property (3.5). □

From this theorem we obtain the following crucial corollary.

COROLLARY 3.6. *Suppose that the analytic system* (2.1) *is locally weakly observable and that the system has a uniform unobservable structure. Then on an open and dense submanifold* M' *of* M *we have that every analytic input function is a local distinguishing input function.*

REMARK. For a single output system this result reduces to a result of [6]. In fact in a sense the converse as in [6] also holds for the systems of corollary 3.6 (see also remark 4.a).

We want to conclude this paper with some further analysis of locally weakly observable systems, which satisfy proposition 3.2. This is of course a restriction of the systems satisfying theorem 3.5, and clearly of locally weakly observable systems, but it reveals how a further study of observability aspects of nonlinear systems might go. Especially for realization and identification purposes it seems quite natural to do such investigations. It is a surprising fact that a (local) analysis of systems satisfying the conditions of proposition 3.2 almost follows the well-known study of canonical forms of linear systems (see e.g. [4], [13]).

So, let us assume that there is given a nonlinear control system (2.1) which is locally weakly observable and which satisfies the condition $[B_i, D_j] \subset D_j$, $i = 1,\dots,m$, $j = 0,\dots,n-1$, where $D_0 \supset D_1 \supset \dots \supset D_{n-1}$ is the unobservable structure of the associated free system (2.2). Note that $D_{n-1} = 0$. Choose arbitrary local coordinates around $p \in M'$ and $C(p) \in N$.

Then the following procedure is a straightforward extension of a pyramidical basis for linear systems (see e.g. [4]). A *local pyramidical basis* (related to these coordinate systems) consists of a family of function-sets $S_i$ $(i = 0,1,\dots,n-1)$ such that

(i) $S_i$ contains $\kappa_i$ functions from the set

$$\{y_1,\dots,y_{\kappa_0},\ y_1^{(1)},\dots,y_{\kappa_0}^{(1)},\dots,y_1^{(i)},\dots,y_{\kappa_0}^{(i)}\}$$

($y^{(i)}$ is the i-th derivative of the function $y(t)$)

(ii) The kernel of the $\kappa_i$ functions of $S_i$ coincides with the distribution $D_i$ $(i = 0,1,\dots,n-1)$ on the local chart.

(iii) If $y_i^{(k)} \in S_k$ then $y_i^{(\ell)} \in S_k$ $\forall \ell < k$.

So with such a local pyramidical basis is associated a scheme of the form

*output* $y_j$ $(j = 1,\dots,\kappa_0)$

| *derivatives* $y_j^{(i)}$ | 1 | 2 | . | . | . | . | $\kappa_0$ |
|---|---|---|---|---|---|---|---|
| 0 | * | * | * | * | * | * | * |
| 1 | * | | * | * | * | | * |
| 2 | * | | * | * | | | * |
| . | | | * | | | | |
| . | | | * | | | | |
| . | | | * | | | | |

Here we can determine from the first i rows which functions are in the set $S_i$.

REMARK. For each local pyramidical basis $S_0 \subset S_1 \subset \dots \subset S_{n-1}$ we have that the $\kappa_{n-1} = n$ functions in $S_{n-1}$ exactly determine the state of the system (in a local fashion). Note that we also have excluded possible singularities in the functions $y_j^{(i)}$ (by the claim (ii)).

Now we have

THEOREM 3.6. *Suppose there is given a nonlinear control system* (2.1) *which is locally weakly observable and which satisfies the condition* $[B_i,D_j] \subset D_j$, $(i = 1,\dots,m,\ j = 0,\dots,n-1)$ *where* $D_0 \supset D_1 \supset \dots \supset D_{n-1} = 0$ *is the unob-*

*servable structure of* (2.2). *Then every local pyramidical basis on* M' *for the system* (2.2) *is for any (constant) input function* u *a local pyramidical basis for the system* (2.1).

PROOF. Choose a pyramidical basis for the free system (2.2). If we now apply an arbitrary analytic input function u(t) then a function $y_j^{(i)}$ in $S_i$ also will depend on this input function u(t). The surprising fact of this theorem is that the same functions may be used for locally determining the state. In the first step of a local pyramidical basis we see that we can locally determine M(mod $D_0$) by the functions in $S_0$, no matter what input we choose. Now from $[B_i,D_0] \subset D_0$ (i = 1,...,m) we obtain that the vector fields $B_i$ restricted to M(mod $D_0$) only depend on M (mod $D_0$) (see also the remark after proposition 3.2). But this means that the first order derivatives of the function $y_i$ (i = 1,...,$\kappa_0$) only depend on the input function u(t) as u(t) times a function on M (mod $D_0$). So the new state-information in $\dot{y}(t)$, namely the local knowledge of the 'state' M (mod $D_1$)\M(mod $D_0$) is not affected by the input function. Said in another way this means that in the coordinate system of theorem 2.8 we have that y(t) implies knowledge of $(x_1,\ldots,x_{\kappa_0})$ and for $\dot{y}(t) = C_*(A(x(t)) + C_*(B(x(t)))u$ we know that $C_*(B(x(t)))u(t)$ only depends on $x_1,\ldots,x_{\kappa_0}$ and $C_*(A(x(t)))$ implies knowledge of $x_{\kappa_0+1},\ldots,x_{\kappa_1}$, while we already know $x_1,\ldots,x_{\kappa_0}$. So the input function does not influence the *new* state-information we obtain in $\dot{y}(t)$. Analogously we prove that $[B_i,D_j] \subset D_j$ (i = 1,...,m, j = 1,...,n-1) implies that from the functions $y_j^{(i)}$ in $S_i$ we obtain the local state-information of M(mod $D_i$)\M(mod $D_{i-1}$), no matter which input function we apply. Therefore the local pyramidical basis for the system (2.2) can be used for determining the state for each analytic input function. □

The above theorem gives a nice characterization. If we have sufficient information for u ≡ 0 from the output function y(t), namely the n functions in the set $S_{n-1}$, then we can use the same n functions for locally determining the state for each analytic input function. Note that this is not true for the systems satisfying theorem 3.5.

4. DISCUSSION

We conclude this paper with some comments on the results obtained.

a. The whole idea behind local weak observability at the regular points of M, i.e. on M', is that in local coordinates one can determine the initial state $x_0$ as a mapping (see also [18])

$$(u(0),\dot{u}(0),\ldots,u^{(n-1)}(0),\ y(0),\ldots,u^{(n-1)}(0)) \mapsto x_0.$$

In fact, although we will not prove it here, the systems satisfying the conditions (3.3) and (3.4) of theorem 3.5 exactly have this property (called *uniform observability* [6], [18]) on M'. To get some feeling for these conditions, we will give an example where the unobservable structure really changes. Consider the bilinear system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . u\ , \qquad y = (1\ 0\ 0)x$$

Then if follows that for all input functions which satisfy $\dot{u} = -1 + u^2$ we cannot determine the state completely. In other words then the mapping $(u(0),\dot{u}(0),\ddot{u}(0),y(0),\dot{y}(0),\ddot{y}(0)) \mapsto x_0$ is not injective any more. On the other hand we see that for a constant input function $u(t) = \bar{u}$ the unobservable structure is globally defined as

$$D_0 = \mathrm{Span}\{\frac{\partial}{\partial x_2}\ ,\ \frac{\partial}{\partial x_3}\ \},\qquad D_1 = \mathrm{Span}\{\bar{u}\ \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}\ \},\ D_2 = 0.$$

From this representation of the unobservable structure we see that the free system ($\bar{u} = 0$) is observable as well as for any other constant input function $\bar{u}$. But the mapping $(u(0),\dot{u}(0),\ddot{u}(0),y(0),\dot{y}(0),\ddot{y}(0) \mapsto x_0$ is not injective for those input functions that satisfy $\dot{u} = -1 + u^2$. We see that for uniform observability the unobservable structure is not allowed to change. (Note that uniform observability also means that every analytic input function is a distinguishing input function.)

b. The observability indices, introduced in definition 2.6, are a set of invariants for the free system (2.2). The analysis of section 3 deals with those control systems for which these invariants remain unchanged under an arbitrary input function. In that sense the results of section 3 can be interpreted as a dual of [2] and [12]. What one really needs for a complete dual treatment of those papers, is a good understanding of the notion of output injection.

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