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OBSERVABILITY OF AUTONOMOUS DISCRETE-TIME NONLINEAR SYSTEMS:
A GEOMETRIC APPROACH

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Observability of autonomous discrete-time nonlinear systems: a geometric approach*)

by

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In this paper observability of autonomous discrete-time systems is studied from a purely differential geometric point of view. Similarly as for continuous-time systems this approach leads to a local canonical form for an observable system. A proposal for the generalization of an invariant subspace is made.

KEY WORDS & PHRASES: *discrete-time systems, observability, differential geometric methods, canonical form, distributions, foliations*

*)

This report will be submitted for publication elsewhere.

1. INTRODUCTION

Consider the autonomous discrete-time nonlinear system

$$(1.1) \quad \begin{cases} x_{k+1} = A(x_k) \\ y_k = C(x_k) \end{cases}, \quad k \in \mathbb{N},$$

where $A : M \rightarrow M$ is an analytic map on the analytic state-manifold M and $C : M \rightarrow N$ is the analytic output-map from the state space to the analytic output-manifold N . We will suppose that C is a surjective submersion. Several notions of observability have been introduced for the system (1.1), see especially SONTAG (1979). Here we will introduce a slightly different definition of observability, which is closer to one given for continuous-time nonlinear (control) systems (see HERMANN & KRENER (1977)). In this way the theory becomes a natural extension of the (differential) geometric approach to nonlinear systems. In fact, this framework is a generalization of the geometric approach to linear systems theory (cf. WONHAM (1979) and is in line with a recently developed approach to continuous-time nonlinear systems (NIJMEIJER (1982)). Similarly as for autonomous continuous-time systems a set of invariants, together with a local canonical form, is derived (section 2). In section 3 a proposal is made for the generalization of the linear concept of an invariant subspace.

2. THE UNOBSERVABLE STRUCTURE

The following set-up is the discrete-time version of the framework of nonlinear systems theory developed in HERMANN & KRENER (1977). With the system (1.1) there is, for each given initial state, associated an output sequence, namely $(C(x), C(A(x)), \dots, C(A^k(x)), \dots)$. A pair of points x and \bar{x} is indistinguishable (denoted $x \bar{I} \bar{x}$) if their output sequences coincide, i.e. $C(A^k(x)) = C(A^k(\bar{x}))$ for all $k \in \mathbb{N}$. Indistinguishability I is an equivalence relation on M . The system is *observable at x* if $I(x) = \{x\}$ and the system is *observable* if $I(x) = \{x\}$ for every $x \in M$. Notice that by definition observability at a point becomes an infinite test. Therefore we will narrow

the definition of observability to a stronger notion. Suppose that the dimension of M equals n . Then the system is *strongly observable*, or *finite-time observable*, at x if for any $\bar{x} \in M$, $C(A^k(\bar{x})) = C(A^k(x))$ for $k = 0, 1, \dots, n-1$, implies $\bar{x} = x$. The system is *strongly observable* if it is strongly observable at every point of M . On the other hand we can weaken the concept of observability; in practice it may suffice to be able to distinguish x only from its neighbors. Therefore we define the system to be *locally observable at x* if there exists a neighborhood U of x such that $I(x) \cap U = \{x\}$, and the system is *locally observable* if it is so at every $x \in M$. Combining the above notions of observability we come to the last definition. The system is *strongly locally observable at $x \in M$* if there exists a neighborhood U of x such that for any $\bar{x} \in U$, $C(A^k(\bar{x})) = C(A^k(x))$ for $k = 0, 1, \dots, n-1$, implies $\bar{x} = x$. The system is *strongly locally observable* if it so at every $x \in M$. The advantage of strong local observability over the other concepts is that it lends itself to a simple algebraic test.

Before we can give this test we will introduce one more concept. Define the map

$O^{n-1}: M \longrightarrow N^n = N \times \dots \times N$ (n copies of N) by
 $O^{n-1}(x) := (C(x), C(A(x)), \dots, C(A^{n-1}(x)))$. The system is said to satisfy the *observability rank condition at x* if the rank of the map O^{n-1} equals n , and the system satisfies the *observability rank condition* if this is true for every $x \in M$.

Now we obtain (see also SONTAG (1979)):

THEOREM 2.1. *If the system (1.1) satisfies the observability rank condition at \bar{x} then it is strongly locally observable at \bar{x} .*

PROOF. By definition of the observability rank condition we see that the map $O^{n-1}: M \longrightarrow N^n$ has rank n in \bar{x} . Therefore locally around \bar{x} this map is injective, which clearly implies that the system is strongly locally observable at \bar{x} . □

COROLLARY 2.2. *If the system (1.1) satisfies the observability rank condition then the system is strongly locally observable.*

The converse is almost true:

THEOREM 2.3. *If the system (1.1) is strongly locally observable then the observability rank condition is satisfied on an open and dense submanifold M' of M .*

PROOF. The analytic map $O^{n-1}: M \rightarrow N^n$ has fixed rank on an open and dense submanifold M' of M . From the fact that the system is strongly locally observable, it follows that this rank equals n on M' . \square

Next we will derive some local canonical forms for the system (1.1). The framework needed for doing this is the same as for the linear case and also the continuous time case (cf. NIJMEIJER (1982)). Besides the map $O^{n-1}: M \rightarrow N^n$ we introduce the maps $O^i: M \rightarrow N^{i+1}$, $i = 0, 1, \dots, n-2$, defined by $O^i(x) := (C(x), C(A(x)), \dots, C(A^{i-1}(x)))$.

DEFINITION 2.4. The *unobservable structure* of (1.1) is defined as a set of distributions D_i ($i=0, \dots, n-1$) on M , such that D_i is the largest analytic distribution on M which is contained in $\text{Ker } O_{*}^i$.

REMARK. In NIJMEIJER (1982) we did not introduce the continuous-time version of the maps O^i . For completeness we will do it here. Namely for the system $\dot{x} = A(x)$, $y = C(x)$ we define $O^0: M \rightarrow N$ as $O^0 = C$, then O^1 is defined as the map $O^1: M \rightarrow TN$ given by $O^1 = C_{*}(A)$ and in general $O^i: M \rightarrow T(T \dots (TN))$ - the i -th tangent space of N - is given by $O^i = (\dots((C_{*}(A))_{*} A)_{*} \dots)_{*}(A)$ (Compare with BROCKETT (1979)).

The following result is obvious; for the proof we refer to NIJMEIJER (1982).

PROPOSITION 2.5.

- (i) $D_0 \supset D_1 \supset \dots \supset D_{n-1} = D_n = D_{n+1} = \dots$
- (ii) Each distribution D_i ($i=0, 1, 2, \dots$) is involutive and has fixed dimension.

COROLLARY 2.6. *If the system (1.1) is strongly locally observable, then $D_{n-1} = 0$.*

Now we restrict attention to systems that are strongly locally observable. In fact we are only dealing with the submanifold M' of M given by

theorem 2.3. Therefore we could have defined the unobservable structure on M' . The distributions D_i have an appealing interpretation. Points of the same integral manifold of D_i cannot be distinguished in i (observation) steps.

REMARK. It is also worth noting that one can derive results on strongly *globally* observability for the system on M' . Clearly for each $i = 0, 1, \dots, n-1$ the map O^i has a fixed value on an integral manifold of D^i . If there do not exist different integral manifolds of D^i which have the same O^i -value, then the system is strongly globally observable on M' .

Based on proposition 2.5 we define the following indices which are the duals of the usual observability indices.

DEFINITION 2.7. The *dual observability indices* κ_i ($i=0, 1, \dots, n-1$) of the strongly locally observable system (1.1) are defined to be

$$(2.1) \quad n - \dim D_i = \kappa_i, \quad i = 0, 1, \dots, n-1.$$

REMARKS

- (i) $\kappa_0 = \dim N$.
- (ii) $\kappa_0 < \kappa_1 < \dots \leq \kappa_{n-1} = n$.

Our first local canonical form deals with the map $O^{n-1}: M \rightarrow N^n$. Without any further assumption we cannot give a general local form for the map A with the map C . This comes from the fact that $A^j(x)$ ($j=1, \dots, n-1$) may belong to a neighborhood on M' which is different from the local chart around x .

THEOREM 2.8. For the strongly locally observable system (1.1), with dual observability indices $\kappa_0, \dots, \kappa_{n-1}$, we can find around each point p in M' a coordinate system on M' and a coordinate system around $O^{n-1}(p)$ in N^n such that there exists functions γ_i^j such that the map O^{n-1} takes the form:

$$(2.2) \quad O^{n-1}(x_1, \dots, x_n) =$$

$$\begin{pmatrix} C(x_1, \dots, x_n) \\ C \circ A(x_1, \dots, x_n) \\ \vdots \\ C \circ A^{n-1}(x_1, \dots, x_n) \end{pmatrix} =$$

$$\begin{pmatrix} (x_1, \dots, x_{\kappa_0}) \\ (x_{\kappa_0+1}, \dots, x_{\kappa_1}, \gamma_1^1(x_1, \dots, x_{\kappa_1}), \dots, \gamma_{2\kappa_0 - \kappa_1}^1(x_1, \dots, x_{\kappa_1})) \\ \vdots \\ (x_{\kappa_{i+1}+1}, \dots, x_{\kappa_{i+2}}, \gamma_1^{\kappa_{i+1}}(x_1, \dots, x_{\kappa_{i+2}}), \dots, \gamma_{2\kappa_{i+1} - \kappa_{i+2}}^{\kappa_{i+1}}(x_1, \dots, x_{\kappa_{i+2}})) \\ \vdots \end{pmatrix}^T$$

where, furthermore the first j rows correspond with coordinatization of O^j .

PROOF. Choose a coordinate system around $O^{n-1}(p)$ in N^n . Then we know from SPIVAK (1970) that we can find local coordinates around $p \in M'$ such that O^0 takes the form

$$O^0(x_1, \dots, x_n) = (x_1, \dots, x_{\kappa_0}) .$$

By definition of the dual observability indices the map O^1 has rank κ_1 . So after a permutation of the coordinate functions on N^2 we can perform a coordinate transformation on the local chart around p such that

$$O^1(x_1, \dots, x_n) = \begin{pmatrix} (x_1, \dots, x_{\kappa_0}) \\ (x_{\kappa_0+1}, \dots, x_{\kappa_1}, \gamma_1^2(x_1, \dots, x_{\kappa_1}), \dots, \gamma_{2\kappa_0 - \kappa_1}^1(x_1, \dots, x_{\kappa_1})) \end{pmatrix}^T$$

This coordinate transformation on M' has been chosen such that the first κ_0 coordinate functions remain invariant. Also the permutation of the coordinate functions on N^2 only involves the second set of κ_0 coordinate functions. Repetition of the above construction exactly yields the local canonical form (2.2) for O^{n-1} . \square

Next we will go to a local canonical form for the maps A and C of the strongly locally observable system (1.1). In that case we need one more assumption, namely we suppose that $A(p) = p$. Clearly such an assumption is necessary because otherwise we need different local charts around p and $A(p)$ (and also $A^2(p), \dots, A^{n-1}(p)$).

THEOREM 2.9. *Consider the strongly locally observable system (1.1), with dual observability indices $\kappa_0, \dots, \kappa_{n-1}$, and let p be a point in M' with $A(p) = p$. Then we can find coordinate systems around $p \in M'$ and $C(p) \in N$ and there exist functions α_i^j such that*

(2.3)

$$\left(\begin{array}{c}
 A^1(x_1, \dots, x_n) \\
 \vdots \\
 A^{\kappa_1 - \kappa_0}(x_1, \dots, x_n) \\
 A^{\kappa_1 - \kappa_0 + 1}(x_1, \dots, x_n) \\
 \vdots \\
 A^{\kappa_0}(x_1, \dots, x_n) \\
 A^{\kappa_0 + 1}(x_1, \dots, x_n) \\
 \vdots \\
 A^{\kappa_i + 1}(x_1, \dots, x_n) \\
 \vdots \\
 A^{\kappa_i + \kappa_{i+2} - \kappa_{i+1}}(x_1, \dots, x_n) \\
 A^{\kappa_i + \kappa_{i+2} - \kappa_{i+1} + 1}(x_1, \dots, x_n) \\
 \vdots \\
 A^{\kappa_{i+1}}(x_1, \dots, x_n) \\
 \vdots \\
 A^n(x_1, \dots, x_n)
 \end{array} \right) = \left(\begin{array}{c}
 x_{\kappa_0 + 1} \\
 \vdots \\
 x_{\kappa_1} \\
 \alpha_1^1(x_1, \dots, x_{\kappa_1}) \\
 \vdots \\
 \alpha_{2\kappa_0 - \kappa_1}^1(x_1, \dots, x_{\kappa_1}) \\
 x_{\kappa_1} \\
 \vdots \\
 \vdots \\
 x_{\kappa_{i+1} + 1} \\
 \vdots \\
 \vdots \\
 x_{\kappa_{i+2}} \\
 \alpha_1^{\kappa_{i+1}}(x_1, \dots, x_{\kappa_{i+2}}) \\
 \vdots \\
 \alpha_{2\kappa_{i+1} - \kappa_i - \kappa_{i+2}}^{\kappa_{i+1}}(\alpha_1, \dots, x_{\kappa_{i+2}}) \\
 \vdots \\
 \vdots
 \end{array} \right)$$

$$(2.4) \quad C(x_1, \dots, x_n) = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_{\kappa_0} \end{pmatrix},$$

to be called the observable canonical form.

PROOF. Choose as in theorem 2.8 coordinate systems $(U(p), x)$ and $(V(C(p)), y)$ around $p \in M'$ and $C(p) \in N$ so the map C takes the form $C(x_1, \dots, x_n) = (x_1, \dots, x_{\kappa_0})^T$. Notice that while $A(p) = p$ there exists a neighborhood $\tilde{U}(p) \subset U(p)$ such that $A^j(\tilde{U}(p)) \subset U(p)$ for $j = 0, 1, \dots, n-1$. Next consider the map $C \circ A : M \rightarrow N$. We have that $C(A(p)) = C(p)$, so we can use $(V(C(p)), V(C(p)), (y, y))$ as a coordinate system on $N \times N$ around $(C(p), C(p))$. It easily follows from theorem 2.8 that the first κ_0 rows of the map A restricted to $\tilde{U}(p)$ are given by

$$\begin{pmatrix} A^1(x_1, \dots, x_n) \\ \vdots \\ A^{\kappa_1 - \kappa_0}(x_1, \dots, x_n) \\ A^{\kappa_1 - \kappa_0 + 1}(x_1, \dots, x_n) \\ \vdots \\ A^{\kappa_0}(x_1, \dots, x_n) \end{pmatrix} = \begin{pmatrix} x_{\kappa_0 + 1} \\ \vdots \\ x_{\kappa_1} \\ \alpha_1^1(x_1, \dots, x_{\kappa_1}) \\ \vdots \\ \alpha_{2\kappa_0 - \kappa_1}^1(x_1, \dots, x_{\kappa_1}) \end{pmatrix}$$

Now repetition of the above procedure exactly yields the desired canonical form given by (2.3) and (2.4) □

REMARKS. (i) The above theorem is the discrete-time analogue of theorem 2.8 of NIJMEIJER (1982). Obviously one can repeat the comments of NIJMEIJER (1982) for the discrete-time case. For example if $\dim N = 1$ we obtain the discrete-time version of GAUTHIER & BORNARD (1981). In that case our canonical form reads

$$A(x_1, \dots, x_n) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ \alpha(x_1, \dots, x_n) \end{pmatrix}, \quad C(x_1, \dots, x_n) = x_1.$$

(ii) In a similar way one can treat the unobservable case. Typically there exists an involutive distribution $D \neq 0$, such that $D = D_i$ for a certain $i \in \{0, 1, \dots, n-2\}$, and $D_{i+1} = D_i$. Analogously we can construct a canonical form as in (2.3) and (2.4) if $A(p) = p$. Suppose that $\dim D_i = n - \kappa_i$ then we obtain for the map A the following form

$$A(x_1, \dots, x_n) = \begin{pmatrix} A^1(x_1, \dots, x_{\kappa_i}) \\ A^2(x_1, \dots, x_n) \end{pmatrix}, \quad \text{where } A^1 \text{ represents}$$

the first κ_i components of the map A . $A^1(x_1, \dots, x_{\kappa_i})$ has the same structure as in the above theorem.

(iii) In principle the observable canonical form can be useful in detecting chaos or strange attractors for the system $x_{k+1} = A(x_k)$. Although the starting point is different TAKENS (1981) gives an exposition of this point of view.

(iv) An interesting case arises when the map A is a diffeomorphism on M arising as the time 1-integral of a vector field X on M . Then the unobservable structure of (1.1) is the same as that for the continuous-time system $\dot{x} = X(x)$, $y = C(x)$ (see NIJMEIJER (1982)).

We conclude this section with an example

EXAMPLE 2.10. Consider the bilinear system on $M = \mathbb{R}^4$

$$\begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix}^{(k+1)} = \begin{pmatrix} -p_1 & 1 & 0 & 0 \\ -p_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix}^{(k)}, \quad y(k) = x_1(k)$$

Such a system arises in the study of the extended Kalman filter for parameter estimation (LJUNG (1979)). For this system we have

$M' = \{x_1, x_2, p_1, p_2\} \in M \mid x_1 x_2 p_2 \neq 0\}$ and the unobservable structure reads

$$D_0 = \left\{ \frac{\partial}{\partial x_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2} \right\}$$

$$D_1 = \left\{ x_1 \frac{\partial}{\partial x_2} - \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2} \right\}$$

$$D_2 = \left\{ x_1 x_2 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial p_1} - x_1 p_2 \frac{\partial}{\partial p_2} \right\}$$

$$D_3 = \underline{0}$$

□

3. INVARIANT STRUCTURES

The geometric approach to linear system theory centers around the basic concept of an invariant subspace (WONHAM (1979)). In the last years there has been considerable interest in continuous-time nonlinear feedback theory and various results on linear systems have now been generalized to nonlinear control systems, see e.g. HIRSCHORN (1981), ISIDORI et al. (1981), NIJMEIJER & van der SCHAFT (1982). As argued in GORI GIORGI et al. (1978) it would be desirable to extend such a theory also to discrete-time nonlinear control systems. In this section we will make a first step to such a generalization. Our starting point is again an autonomous discrete-time system

$$(3.1) \quad x_{k+1} = A(x_k)$$

on an analytic manifold M of dimension n . Recall the following definition (see e.g. HIRSCH et al. (1977)).

DEFINITION 3.1. An analytic *foliation* F of a manifold M (with *leaves* of dimension k) is a disjoint decomposition of M into k -dimensional injectively immersed connected submanifolds - the leaves - such that M is covered by analytic charts $\phi: \mathbb{R}^k \times \mathbb{R}^{n-k} \rightarrow M$ and $\phi(\mathbb{R}^k \times y)$ is contained in the leaf through $\phi(0, y)$.

Now a foliation F will play the role of an invariant structure for the system (3.1).

DEFINITION 3.2. A k -dimensional analytic foliation F of M is invariant for the system (3.1), or shortly A -invariant if for all p in M , A maps the leaf through p into the leaf through $A(p)$. We will write $AF \subset F$.

COROLLARY 3.3. *The distribution D_n in the unobservable structure of section 2 gives rise to an invariant foliation on M' . Namely the set of maximal integral manifolds of the distribution D_n forms a foliation of M' which is invariant under A .*

For the fixed points of A an A -invariant foliation F of dimension k induces a nice decomposition for the map A . Suppose that $A(p) = p$. Then there is a coordinate system around p such that

(i) The leaves of F are locally described by

$$\mathbb{R}^k \times \{x_{k+1}, \dots, x_n\} .$$

(ii) If $A = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix}$, where A^1 represents the first

k components of A and A^2 the last $n-k$ components, we have that

$$A(x_1, \dots, x_n) = \begin{pmatrix} A^1(x_1, \dots, x_n) \\ A^2(x_{k+1}, \dots, x_n) \end{pmatrix} .$$

REMARKS. (i) Such a decomposition is the same as for the continuous-time case, see HIRSCHORN (1981), and plays a key role in the solution of the Disturbance Decoupling Problem.

(ii) If the map A arises as the time 1-integral of a vector field on M then this definition is the discretization of the one used in HIRSCHORN, ISIDORI et al. (1981) and NIJMEIJER & van der SCHAFT (1982).

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