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A DIFFERENTIAL GEOMETRIC APPROACH TO OPTIMAL CONTROL

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A differential geometric approach to optimal control*)

by

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ABSTRACT

A general formalism is introduced for the optimal control problem on manifolds. It is based on a general formulation of Lagrange's multiplier theorem and recent definitions of nonlinear control systems. It is shown that we can give Pontryagin's maximum principle in this formalism. We expect that the problem formulation given in this paper is particularly suitable for application of modern results about controllability etc. in nonlinear control systems.

KEY WORDS & PHRASES: *Nonlinear system theory, Optimal control problems on manifolds, Necessary conditions for optimality, First order conditions*

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*) This report will be submitted for publication elsewhere.

INTRODUCTION

In this paper we present a differential geometric formulation of the problem of optimal feedback control of nonlinear time-invariant control systems. Basic to this approach is the definition of control systems as proposed by BROCKETT [1977] and WILLEMS [1981] and worked out by van der SCHAFT [1982] and NIJMEIJER & van der SCHAFT [1982]. The results about the (Lagrange) variational problem as given in sections 2 up to 4 are essentially known, but usually not treated in the way we did or not published in the open literature. Basic references for these sections are CARTAN [1922], CARATHEODORY [1935], HERMANN [1962,1977], SPIVAK [1979,I & II] and unpublished course notes of TAKENS [1978]. Compared with Hermann's work we think that our approach is more geometrical and more suitable for application of modern differential geometric results in system theory. See for instance SUSSMANN & JURDJEVIC [1972], HERMANN & KRENER [1977], ISIDORI et al. [1981], HIRSCHORN [1981], NIJMEIJER [1981], NIJMEIJER & van der SCHAFT [1982] and van der SCHAFT [1982 a,b] . Particularly, the formulation and proof of Lagrange's multiplier theorem on manifolds is new, although the basic ideas are in Takens' course notes. This theorem expresses the equivalence between a variational problem on a manifold with restrictions given by a distribution, and an unrestricted variational problem on the annihilator of this distribution, which is a codistribution on the manifold.

Optimal control concerns itself with finding optimal trajectories of control systems, given a certain optimality criterion. Hence, we search for 1-dimensional immersions in the configuration space which satisfy certain optimality criteria. Results given in this paper may be extendable to more-dimensional immersions, yielding multiple integrals. We did not try to do that. A reference concerning such problems is DEDECKER [1977]. In sections 5 and 6 we actually give the formulation of the optimal control problem on manifolds. The formalism is illustrated by working out the linear-quadratic regulator problem on $\mathbb{R}^n$. This reduces very elegantly to the well-known results expressed by Pontryagin's maximum principle.

We shall close this section by giving some comments about the notational conventions in this paper. The differential geometric notation follows closely that of SPIVAK [1979, I & II]. For instance, if M is a smooth manifold, TM is its tangent bundle (T_xM is the tangent space at $x \in M$) and T^*M is the cotangent bundle. If $f : M \to N$ is a mapping between smooth manifolds M and N then $f_* : TM \to TN$ is its lift to the tangent bundles and for any k-form ω on N, $f^*\omega$ is a k-form on M which is defined by $(f^*\omega)(v) = \omega(f_* v)$ for all $v \in TM$. Some minor deviations from Spivak's notation occur. The set of smooth vector fields on a smooth manifold is denoted by $\mathcal{X}(M)$. Furthermore, given a k-form ω and a vectorfield X on M, we define the *contraction* $\iota_X\omega$ of ω with respect to X, to be the (k-1)-form on M defined by

$$\iota_X(X_1,\ldots,X_{k-1}) = \omega(X,X_1,\ldots,X_{k-1})$$

for

$$X_i \in \mathcal{X}(M) \quad (i=1,\ldots,k-1).$$

Unless stated otherwise all manifolds, mappings, forms and vector fields are assumed to be smooth, i.e. C^∞.

2. THE UNRESTRICTED VARIATIONAL PROBLEM

Let M be a manifold with dim M = m and α a 1-form on M. Let I denote some closed interval, [a,b] say, in $\mathbb{R}$. Then, for C^∞ curves $\phi : I \to M$ we can define the *action* of α along ϕ by

$$(2.1) \qquad J(\phi) = \int_\phi \alpha = \int_I \phi^*\alpha .$$

(In the first integral the integration path is Im ϕ). The *variational problem* on M with respect to α is to find curves which are locally optimal, i.e. which produce an optimal value for the action relative to small variations of the curves. We shall restrict ourselves to *first order necessary conditions*, hence to *stationarity* rather than optimality of the action. The following definition is standard in the calculus of variations.

DEFINITION 2.1. A mapping $\tilde{\phi} : (-\delta,\delta) \times I \to M$ (for some $\delta > 0$) is called a *variation keeping end point fixed* (k.e.p.f) of $\phi : I \to M$ if :

(i) $\tilde{\phi}$ is C^∞ in each variable ;
(ii) $\tilde{\phi}(0,t) = \phi(t)$ for all $t \in I$;
(iii) $\tilde{\phi}(\varepsilon,a) = \phi(a)$, $\tilde{\phi}(\varepsilon,b) = \phi(b)$ for all $\varepsilon \in (-\delta,\delta)$.

We denote for short : $\phi_\varepsilon(t) = \tilde{\phi}(\varepsilon,t)$.
Now, let $X \in \mathcal{X}(M)$ satisfy :

$$(2.2) \qquad X(\phi(a)) = X(\phi(b)) = 0 .$$

Then X defines a variation k.e.p.f. of ϕ by :

$$(2.3) \qquad \tilde{\phi}(\varepsilon,t) = X^\varepsilon(\phi(t)) \qquad , \varepsilon \in (-\delta,\delta), \quad t \in I,$$

for some $\delta > 0$, where X^ε denotes the 1-parameter flow over ε generated by X. Therefore, such an $X \in \mathcal{X}(M)$ satisfying (2.2) is called a *variation vector field* of ϕ k.e.p.f. We denote the set of such fields by $\mathcal{X}_\phi(M)$ or shortly by $\mathcal{X}_\phi$ if no confusion can arise.
If ϕ is an injective map then we can define a *vector field along* ϕ to be a smooth mapping $V : \text{Im}\,\phi \to TM$ such that $V(x) \in T_xM$ for all $x \in \text{Im}\,\phi$. Then, each variation $\tilde{\phi}$ of ϕ defines a vector field along ϕ by :

$$(2.4) \qquad V(\phi(t)) = \tilde{\phi}_*(0,t)\left(\frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0}\right) .$$

This vector field along ϕ can be arbitrarily (smoothly) extended to a variation vector field $X \in \mathcal{X}_\phi$. The set of vector fields along ϕ defined by variations k.e.p.f. is denoted by $\overline{\mathcal{X}}_\phi(M)$ (or $\overline{\mathcal{X}}_\phi$).

Stationary curves for the action are curves which make the first variation of the action vanish. With the above definitions we can make this more precise.

<u>DEFINITION 2.2</u>. A curve ϕ: $I \to M$ is *stationary* with respect to α , if for all variations k.e.p.f. ϕ_ε of ϕ we have

$$(2.5) \qquad \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_I \phi_\varepsilon^* \alpha = 0 .$$

From now on we shall assume that the curves we consider are injective maps. This is a rather natural assumption as curves with double points are usually not optimal, because of occurrence of a loop. In such cases we can formulate the variational problem for piecewise injective curves as a sum of variational problems for each piece (see also SPIVAK [1979,II ch.6.14]). Then, for $X \in \mathcal{X}_\phi$ and ϕ_ε given by (2.3), we have

(2.6) $$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \phi_\varepsilon^* \alpha = \phi^* \left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} (X^\varepsilon)^* \alpha \right) = \phi^* L_X \alpha \ ,$$

where L_X denotes the Lie-derivative with respect to X. Moreover, if $X, \tilde{X} \in \mathcal{X}_\phi$ are both smooth extensions of the same vector field V along ϕ, defined by a variation k.e.p.f. $\tilde{\phi}$ according to (2.4), then

$$\phi^* \left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} (X^\varepsilon)^* \alpha \right) = \phi^* \left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} (\tilde{X}^\varepsilon)^* \alpha \right)$$

or simply $\phi^* L_X \alpha = \phi^* L_{\tilde{X}} \alpha$. Therefore, the following definition is equivalent to 2.2.

<u>DEFINITION 2.2'</u> ϕ is stationary with respect to α on M if for all $X \in \mathcal{X}_\phi$.

(2.7) $$\int_I \phi^* L_X \alpha = 0$$

We shall consider yet another approach to define stationarity of curves. To do so we need some introduction (see SPIVAK [1979,II ch.6]).

A (Koszul) connection on M is a map ∇ :
$\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ which satisfies for $X,Y,Z \in \mathcal{X}(M)$, $f,g \in C^\infty(M)$:

(2.8) $$\nabla_{fX+gY} Z = f.\nabla_X Z + g.\nabla_Y Z ,$$

(2.9) $$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z ,$$

(2.10) $$\nabla_X(fY) = f.\nabla_X Y + X(f).Y.$$

Moreover, we denote $\nabla_X Y(p) = \nabla_{X_p} Y$ and we can consider ∇ also as a map assigning a vector $\nabla_{X_p} Y \in T_pM$ to every $X_p \in T_pM$ and $Y \in \mathcal{X}(M)$.

On M we can choose an arbitrary Riemannian metric (SPIVAK [1979, I ch.9.4]). It is known that, given any metric < ., . > on M, there exists a unique connection ∇, called the *Levi-Civita connection,* which satisfies:

$$(2.11) \qquad X_p < Y,Z > = < \nabla_{X_p} Y, Z_p > + < Y_p, \nabla_{X_p} Z >,$$

$$(2.12) \qquad \nabla_X Y - \nabla_Y X = [X,Y] ,$$

where [,] denotes the Lie-bracket of vector fields. (2.11) expresses *compatibility* of the connection ∇ with the metric < ., . > and (2.12) expresses *symmetry* of ∇. The next step is to define covariant differentiation. Given any connection ∇ and curve ϕ, there is precisely one operation $V \longmapsto \frac{DV}{dt}$, from $\overline{X}_\phi$ to $\overline{X}_\phi$ with the following properties

$$(2.13) \qquad \frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt},$$

$$(2.14) \qquad \frac{D(fV)}{dt} = \frac{df}{dt} V + f \frac{dV}{dt} ,$$

for $V, W \in \overline{X}_\phi$, $f \in C^\infty(M)$ and if $X \in \mathcal{X}(M)$ is an arbitrary extension of V to M then

$$(2.15) \qquad \frac{DV}{dt} = \nabla_{\frac{d\phi}{dt}} X .$$

$\frac{DV}{dt}$ is called the *covariant derivative* of V along ϕ. Note that we can consider V to be a function from I to TM by identifying $V(t) = V(\phi(t))$. Finally, we can uniquely extend the map ∇_X to the algebra of smooth tensor fields on M. Then we have for a k-form ω on M:

$$(2.16) \qquad (\nabla_X\omega)(Y_1,\ldots,Y_k) = \nabla_X(\omega(Y_1,\ldots,Y_k)) - \sum_{i=1}^{k} \omega(Y_1,\ldots,\nabla_X Y_i,\ldots,Y_k).$$

With these notions we can formulate a third equivalent definition of stationarity.

DEFINITION 2.2" ϕ is stationary with respect to α if, for a given Riemannian metric and associated Levi-Civita connection and covariant differentiation as above, we have

$$(2.17) \qquad \int_I \alpha \left(\frac{DV}{dt} \right) dt = 0 \qquad \forall\, V \in \overline{X}_\phi \,,$$

where $t \longrightarrow \phi(t)$ gives a parametrization of ϕ on I.

The equivalence of definitions 2.2" and 2.2' follows from the following lemma.

LEMMA 2.3. *Let* $(M, \langle\ ,\ \rangle)$ *be a Riemannian manifold and* ∇ *the Levi-Civita connection defined by the metric. Let* $\phi : [a,b] \to M$ *be a curve on* M *and* $\frac{D}{dt}$ *denote covariant differentiation along* ϕ *associated with* ∇. *Then, for any* 1-*form* ω *on* M *the following equality holds*:

$$(2.18) \qquad \phi^* L_Y \omega = \omega \left(\frac{DV}{dt}\right) dt$$

for all vector fields Y *defined on a neighbourhood of* ϕ *and vector field* V *along* ϕ *with* $V(\phi(t)) = Y(\phi(t))$.

PROOF. Interpreting $\omega(Y)$ as a function on a neighbourhood of ϕ we have (see SPIVAK [1979,II.6.4])

$$(2.19) \qquad \nabla_{\frac{d\phi}{dt}}(\omega(Y)) = \lim_{h \to 0} \frac{1}{h} \left[\omega(Y)\Big|_{\phi(t+h)} - \omega(Y)\Big|_{\phi(t)} \right] = \frac{d}{dt}(\omega_{\phi(t)}(Y_{\phi(t)})) \,.$$

Choose coordinates $x_1,\dots,x_m$ locally in M and write:

$$Y(x) = \sum_{i=1}^m Y_i(x) \frac{\partial}{\partial x_i}\Big|_x \,; \quad \omega(x) = \sum_{i=1}^m \omega_i(x) dx_i\Big|_x \,; \quad \phi(t) = (\phi_1(t),\dots,\phi_m(t)) .$$

Using the well-known rule :

$$(2.20) \qquad L_Y\, \omega = \iota_Y d\omega + d\, \iota_Y \omega$$

we have

$$(2.21)\qquad (\phi^* L_Y\omega)(\frac{\partial}{\partial t}) = \iota_Y d\omega(\phi_* \frac{\partial}{\partial t}) + (d\iota_Y(\omega))(\phi_* \frac{\partial}{\partial t}) .$$

Now :

$$(2.22)\qquad (d\iota_Y(\omega))(\phi_* \frac{\partial}{\partial t}) = \sum_{i,j} \frac{\partial(\omega_i Y_i)}{\partial x_j} \frac{d\phi_j}{dt} = \nabla_{\frac{d\phi}{dt}}(\omega(Y))$$

according to (2.19). Furthermore, computation shows

$$(2.23)\qquad \iota_Y d\omega(\phi_* \frac{\partial}{\partial t}) = \sum_{i,j} (\frac{\partial \omega_i}{\partial x_j} - \frac{\partial \omega_j}{\partial x_i}) \frac{d\phi_i}{dt} Y_j .$$

The definition of $\nabla\omega$ in coordinates is given by:

$$\nabla\omega = \sum_{i,j} \omega_{j;i} dx_j \wedge dx_i ,$$

with

$$\omega_{k;i} = \frac{\partial \omega_k}{\partial x_i} - \sum_{\mu=1}^{m} \Gamma^{\mu}_{ik} \omega_\mu$$

and Γ^{μ}_{ik} $(\mu,i,k = 1,\ldots,m)$ the so-called Christoffel symbols of the Levi-Civita connection. This last formula yields, by symmetry of the Levi-Civita connection ($\Gamma^{\mu}_{ik} = \Gamma^{\mu}_{ki}$, $i,k,\mu = 1,\ldots,m$) :

$$\frac{\partial \omega_i}{\partial x_j} - \frac{\partial \omega_j}{\partial x_i} = \omega_{i;k} - \omega_{k;i} = \nabla\omega (\frac{\partial}{\partial x_i} , \frac{\partial}{\partial x_k}) .$$

Substituting in (2.23) :

$$(2.24)\qquad \iota_Y d\omega (\phi* \frac{\partial}{\partial t}) = \sum_{i,j} \frac{d\phi_i}{dt} Y_j \nabla\omega (\frac{\partial}{\partial x_i} , \frac{\partial}{\partial x_j}) = \nabla\omega (\frac{d\phi}{dt} , Y).$$

So, (2.21), (2.22), (2.24) yield:

$$(\phi^* L_Y\omega)(\frac{\partial}{\partial t}) = \nabla_{\frac{d\phi}{dt}}(\omega(Y)) - (\nabla_{\frac{d\phi}{dt}} \omega)(Y) = \omega(\nabla_{\frac{d\phi}{dt}} Y) ,$$

where (2.16) is used for the last equality.
The result follows by choosing $V(\phi(t)) = Y(\phi(t))$, $t \in [a,b]$. □

The last characterization of stationarity which is relevant to our theory is expressed in the following theorem.

THEOREM 2.4. *$\phi : I \to M$ is stationary* w.r.t. α *if and only if* $\phi'(t) \in \ker d\alpha$ *for all* $t \in I$, *where* $\ker d\alpha = \{v \in TM \mid d\alpha\,(v,w) = 0 \quad \forall\, w \in T_{\pi(v)}M\}$ *and* π *natural projection on* M.

PROOF. Use of (2.20), Stokes theorem and the fact that $X(\phi(a)) = X(\phi(b)) = 0 \quad \forall\, X \in X_\phi$ yields :

$$\int_I \phi^* L_X \alpha = \int_I \phi^* \iota_X d\alpha + \int_I d\,\phi^* \iota_X \alpha = \int_I \phi^* \iota_X d\,\alpha\ .$$

This proves the theorem. □

Note that $d\alpha$ is an *integral invariant* for the stationary curves of α (cf. CARTAN [1922]). A curve $\phi : I \to M$ satisfying $\phi'(t) \in \ker \omega$, for some 2-form ω , is called a *characteristic curve* of ω .

3. THE RESTRICTED VARIATIONAL PROBLEM

We shall first consider restrictions on curves in M which are defined by a set of C^∞ 1-forms β_i (i=1,...,p) on M. That means: we call a curve $\phi : I \to \mathbb{R}$ *admissable* if $\phi^*\beta_i = 0$ (i=1,...,p) on I. Then we can define stationarity under restrictions by:

DEFINITION 3.1. A curve $\phi : I \to M$ is *stationary* w.r.t. α on M *under the restrictions* β_i (i=1,...,p) if all variations k.e.p.f. ϕ_ε of ϕ which are admissable, satisfy equation (2.5).

This definition may cause serious problems with isolated admissable curves, i.e. admissable curves that do not allow admissable variations. Therefore we shall use a stronger notion suggested by TAKENS [1978] .

DEFINITION 3.2. A curve $\phi : I \to M$ is *formally stationary* w.r.t. α *under restrictions* β_i (i=1,...,p) if ϕ is admissable (i.e. $\phi^*\beta_i = 0$, i = 1,...,p) and, if ϕ_ε is a variation k.e.p.f. of ϕ satisfying

$$(3.1)\qquad \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (\phi_\varepsilon^* \beta_i) = 0 \quad , \quad i = 1,\dots,p \ ,$$

then ϕ_ε satisfies (2.5) .

Clearly, $\phi_\varepsilon^* \beta_i = 0$ implies (3.1), so that formal stationarity implies stationarity. The reverse is not true in general. Conditions which imply equivalence are not worked out in this paper. We shall restrict attention to formal stationarity for restricted problems only.

Similar to definitions 2.2' and 2.2" we can give two more equivalent definitions of formal stationarity.

DEFINITION 3.2'. ϕ is formally stationary w.r.t. α under restrictions β_i (i=1,...,p) if $\phi^*\beta_i = 0$ (i=1,...,p) and for all $X \in X_\phi$:

$$(3.2)\qquad \phi^* L_X \beta_i = 0 \ (i=1,\dots,p) \ \Rightarrow \int_I \phi^* L_X \alpha = 0 \ .$$

And, if (M, < , >) is a Riemannian manifold, ∇ the associated Levi-Civita connection and $\frac{D}{dt}$ the associated covariant differentiation the third definition is given by :

DEFINITION 3.2". ϕ is formally stationary w.r.t. α under restrictions β_i(i=1,...,p) if $\phi^*\beta_i = 0$ (i=1,...,p) and for all $V \in \overline{X}_\phi$:

$$(3.3)\qquad \beta_i\left(\frac{DV}{dt}\right) = 0 \ (i=1,\dots,p) \ \Rightarrow \int_I \alpha\left(\frac{DV}{dt}\right) dt = 0 \ .$$

If the restriction forms are nondegenerate and independent in each point of M, then we can define a (n-p) - dimensional distribution S on M by

$$(3.4)\qquad S_x = \bigcap_{i=1}^{p} \ker \beta_{ix} \quad , \quad \forall\, x \in M \, .$$

Conversely, given a distribution S (of constant dimension n-p) on M, then we can define, at least locally, p nondegenerate independent 1-forms β_i satisfying (3.4) locally. This suggests to consider restrictions defined by distributions. We say that a curve $\phi : I \to M$ is admissable under the

restriction defined by a distribution S on M if $\phi'(t) \in S_{\phi(t)}$, $t \in I$.
We assume that the distribution S has constant dimension n-p. Then we can define its *annihilator* $E(=S^{\perp})$ to be the following submanifold of the cotangent bundle T^*M :

$$(3.5) \qquad E_x = \{\xi \in T_x^* M \mid \xi(s) = 0 \quad \forall\, s \in S_x\}, \quad \forall\, x \in M.$$

E is a p-dimensional codistribution.
Note that $\pi_E : E \to M$, with π_E the natural projection $\pi : T^*M \to M$ restricted to E, is a vector bundle over M. Hence, for all $x \in M$ we can find a neighbourhood U of x in M and 1-forms $\beta_i (i=1,\ldots,p)$ which form a basis for the fibres $\pi_E^{-1}(x) (x \in U)$. So the following definition for formal stationarity under restrictions given by a distribution seems at hand.

DEFINITION 3.3. A curve $\phi : I \to M$ is *formally stationary* under restriction S, with S a distribution on M of constant dimension, if $\phi^*\xi = 0$ for all $\xi \in E = S^{\perp}$ and for all $X \in X_\phi$:

$$(3.6) \qquad \phi^* L_X \xi = 0 \quad \forall\, \xi \in E \;\Rightarrow\; \int_I \phi^* L_X \alpha = 0 \,.$$

This definition is locally consistent with 3.2' as is shown by the following argument. Locally, any $\xi \in E$ can be written as :

$$\xi = \sum_{i=1}^{p} \lambda_i \beta_i$$

for some p (codimension of S) , functions λ_i and 1-forms $\beta_i (i=1,\ldots,p)$ on M. Then

$$\phi^* L_X \xi = \sum_{i=1}^{p} \phi^* L_X(\lambda_i \beta_i) = \sum_{i=1}^{p} \phi^* \lambda_i L_X \beta_i + \sum_{i=1}^{p} \phi^* X(\lambda_i)\, \beta_i =$$

$$= \sum_{i=1}^{p} \phi^* \lambda_i L_X \beta_i + \sum_{i=1}^{p} (X(\lambda_i) \circ \phi)\, \phi^* \beta_i \,.$$

So $\phi^* \xi = 0$ and $\phi^* L_X \xi = 0$ for all $\xi \in U \subset E$ if and only if $\phi^*\beta_i = 0$ and $\phi^* L_X \beta_i = 0$ $(i=1,\ldots,p)$ for p independent nondegenerate 1-forms β_i on U. This shows the local consistency of 3.3 with 3.2'.

REMARK 3.4. Clearly, definition 3.2" can also be translated to distribution constraints. We just have to replace $\beta_i (i=1,\ldots,p)$ by ξ , $\forall\xi \in E$.

With the given formalism we are able to prove equivalence of a restricted variational problem on M w.r.t. α , to an unrestricted variational problem on E with respect to the so-called Cartan form. This equivalence forms the heart of the theory. In fact, it enables us to reduce the restricted variational problem to the problem of finding the characteristic curves of some 2-form on the annihilator bundle E over M. The equivalence theorem is essentially the multiplier theorem of Lagrange, but translated to the variational problem on manifolds. Before stating this theorem some more notions must be defined.

DEFINITION 3.5. Let M be a manifold with cotangent bundle T^*M and natural projection $\pi : T^*M \to M$. Then, the *canonical 1-form* θ on T^*M is defined by

(3.7) $$\theta(\xi) = \pi^* \xi$$

for all $\xi \in T^*M$.

REMARK 3.6. By definition of π^*, (3.7) implies :

(3.8) $$\theta(\xi)(v) = \xi(\pi_* v) ,$$

for all $\xi \in T^*M$, $v \in T_\xi T^*M$ ($\pi_* : TT^*M \to TM$) .
If we choose coordinates $x_1,\ldots,x_m$ in some open neighbourhood in M, then we can define *canonical coordinates* $\overline{x}_i, p_i$, $i = 1,\ldots,m$ by

(3.9) $$\overline{x}_i(\xi) = x_i(\pi\xi) \ ; \ p_i(\xi) = \xi\left(\frac{\partial}{\partial x_i}\bigg|_{\overline{x}(\xi)}\right) .$$

We shall identify x_i and $\overline{x}_i$. In canonical coordinates, the canonical 1-form θ on T^*M is given by

(3.10) $$\theta = \sum_{i=1}^{m} p_i \, dx_i .$$

DEFINITION 3.7. Let M be a manifold with 1-form α and distribution S on M. Let $E = S^{\perp}$ (cf. (3.5)). Then the *Cartan form* θ_α on E, associated with α is defined by :

$$(3.11) \qquad \theta_\alpha = \pi_E^* \alpha + \theta_E ,$$

where π_E is the restriction to E of the natural projection $\pi : T^*M \to M$ and θ_E is the restriction to E of the canonical 1-form on T^*M.

Now we are ready to formulate Lagrange's multiplier theorem.

THEOREM 3.8. *Let* M *be a manifold with 1-form* α *and distribution* S *of constant dimension. Then, an injective curve* ϕ: I $\to$ M *is formally stationary with respect to* α *under restriction* S *if and only if there exists an injective curve* η : I $\longrightarrow$ E *with* $\eta(t) \in \pi_E^{-1}(\phi(t))$ *and* η *stationary in* E *with respect to the Cartan form* θ_α .

PROOF. Choose any Riemannian metric $< \, , \, >_E$ on E with Levi-Civita connection ∇ and covariant differentiation $\frac{D}{dt}$ along an arbitrary curve $\eta : I \longrightarrow E$ which satisfies $\eta(t) \in \pi_E^{-1}(\phi(t))$ for some curve $\phi : I \longrightarrow M$.

Let s: M $\longrightarrow$ E be a smooth section in the bundle π_E: E $\longrightarrow$ M , such that $s \circ \phi = \eta$ on I. Define a metric $< \, , \, >_M$ on M by $< v, w >_M = < s_* v, s_* w >_E$ for all $v,w \in TM$. Let $\overline{\nabla}$ denote the Levi-Civita connection compatible with this metric and let $\overline{D}/dt$ denote the associated covariant differentiation along ϕ .

If $X \in \mathcal{X}_\eta(E)$, then we can split $X(x) = X^T(x) + X^N(x)$, for all $x \in s(M)$ with X^T tangent to s(M) and X^N vertical w.r.t. π_E (i.e. $\pi_{E*}X^N = 0$).

For each vertical variation vector field $X^N \in \mathcal{X}_\eta(E)$ we have

$$\eta^* L_{X^N}(\pi^*\alpha) = \eta^* \, (\iota_{X^N} d(\pi^*\alpha) + d(\iota_{X^N} \pi^*\alpha)) \, .$$

As

$$(\pi^*\alpha)(X^N) = \alpha(\pi_* X^N) = 0$$

and

$$(\iota_{X^N} d\pi^*\alpha)(Y) = (\pi^* d\alpha)(X^N, Y) = 0 \, , \qquad \forall \, Y \in \mathcal{X}\,(E)$$

we conclude, using lemma 2.3, that

$$(3.12) \qquad (\pi^*\alpha)\left(\frac{DV^N}{dt}\right) = 0 ,$$

for all vertical $V^N \in \overline{X}_\eta$ (i.e. $V^N(t) = X^N(\phi(t))$, $X^N \in X_\eta$).

Hence, for arbitrary $V \in \overline{X}_\eta$ we have

$$(3.13) \qquad \int_I \theta_\alpha\left(\frac{DV}{dt}\right)dt = \int_I \theta_\alpha\left(\frac{DV^T}{dt}\right)dt + \int_I \theta_E\left(\frac{DV^N}{dt}\right)dt$$

with V^T and V^N the tangential and vertical component of V.
Now suppose η is stationary in E w.r.t. θ_α .
Then the left hand side of (3.13) vanishes for all $V \in \overline{X}_\eta$. Therefore, an arbitrary vertical variation field V^N satisfies

$$\int_I \theta_E\left(\frac{DV^N}{dt}\right)dt = 0 ,$$

hence $\theta_E\left(\frac{DV}{dt}\right)dt = 0$. Choose, locally, basic forms $\beta_i (i=1,\dots,p)$ for E. Then $\theta_E = \pi^*(\Sigma_{i=1}^p \lambda_i\beta_i)$. The possible choice of V^N such that $L_{X^N}(\pi^*\beta_i) = 0$ ($V^N(t) = X^N(\eta(t))$) yields $\phi^*\beta_i = 0$, which proves feasibility of ϕ .
Now choose $W \in \overline{X}_\phi$ arbitrarily. Then , $V = s_* W \in \overline{X}_\eta$. As s is an isometry by definition of the metric on M we have

$$(3.14) \qquad \frac{D(s_*W)}{dt} = s_*\left(\frac{\overline{D}W}{dt}\right) .$$

Clearly V is tangential and (3.13) yields, with the definition of the Cartan form

$$\int_I \theta_\alpha\left(\frac{DV}{dt}\right)dt = \int_I (\pi_E^*\alpha + \pi_E^*\eta(t))\left(s_*\left(\frac{\overline{D}W}{dt}\right)\right)dt$$

$$= \int_I \alpha\left(\frac{\overline{D}W}{dt}\right)dt + \int_I \eta(t)\left(\frac{\overline{D}W}{dt}\right)dt .$$

As the left hand side equals zero by assumption, we get for all $W \in \overline{X}_\phi$:

$$\eta(t)\,(\frac{\overline{D}W}{dt}) = 0 \quad \Rightarrow \quad \int_I \alpha\,(\frac{\overline{D}W}{dt})\,dt = 0\,,$$

which proves formal stationarity of ϕ (see remark 3.4) given the stationarity of η .

Conversely, let ϕ be formally stationary and $V \in \overline{X}_\eta$ arbitrarily. We shall prove that we can choose a smooth section s on a neighbourhood of ϕ in E such that V(t) is tangent to Im s at $\eta(t) = s\,\phi(t)$, and with $W = \pi_{E*}\,V = \pi_{E*}\,V^T$:

(3.15) $$\int_I \alpha\,(\frac{\overline{D}W}{dt})\,dt + \int_I \eta\,(t)\,(\frac{\overline{D}W}{dt})\,dt = 0$$

Then, it follows from (3.13) that

(3.16) $$\int_I \theta_\alpha\,(\frac{DV}{dt})\,dt = 0$$

hence $\eta = s \circ \phi\colon I \to E$ is stationary w.r.t. θ_α .

For simplicity we first assume that S has codimension 1. Hence, locally we can find a 1-form β on an open neighbourhood $U \subset M$ such that elements of E can be written by $\xi_x = \lambda(x)\,\beta_x$ $(x \in U)$. So a section $s\colon M \to E$ can locally be defined by a choice of β and λ .
Now choose an open covering $\{U_\mu\}$ of ϕ and let $\{f_\nu\}$ be a partition of unity subordinate with it. Then we can write

$$W = \sum_\nu f_\nu\,W \overset{\text{def.}}{=} \sum_\nu W_\nu\,.$$

As $\{f_\nu\}$ is subordinate with $\{U_\mu\}$ we can choose a μ such that W_ν is a variation vector field for $\phi|I_\mu = \phi_\mu$, where I_μ is a subinterval of I which is appropriately chosen with $\operatorname{Im}\phi_\mu \subset U_\mu$ and $W_\nu(\phi_\mu(a_\mu)) = W_\nu(\phi_\mu(b_\mu)) = 0$ $(I_\mu = [a_\mu, b_\mu])$. Moreover,

(3.17) $$\int_I \alpha\,(\frac{\overline{D}W}{dt})\,dt = \sum_\nu \int_{I_\nu} \alpha\,(\frac{\overline{D}W_\nu}{dt})\,dt\,,$$

where on the right hand side covariant differentiation is restricted along ϕ_μ . Clearly, formal stationarity of ϕ implies formal stationarity of ϕ_μ.

So we have broken up the global problem with a distribution constraint into a sum of local problems with a constraint given by a 1-form β_μ. We show that we can find a $\lambda_\mu \in C^\infty(U_\mu)$ such that for all $W_\mu \in \mathcal{X}_{\phi_\mu}(U_\mu)$:

$$(3.18) \qquad \int_{I_\mu} \alpha\left(\frac{\overline{D}W_\mu}{dt}\right) dt = \int_{I_\mu} (\lambda_\mu \beta_\mu)_{\phi(t)}\left(\frac{\overline{D}W_\mu}{dt}\right) dt .$$

To do so, we drop the subscript μ to ease the notational pain and choose β to be such a 1-form, spanning the fibres of E over U.
We choose $Z \in \mathcal{X}(U)$ such that $\beta(Z)_{\phi(t)} = 1$, $t \in I$ and define

$$\mathcal{X}_1 = \{X \in \mathcal{X}(U) \,|\, \phi^* L_X \beta = 0 ,\ X(\phi(a)) = 0\}$$

$$\mathcal{X}_2 = \{X \in \mathcal{X}(U) \,|\, X = \psi Z ,\ \psi \in C^\infty(U)\} .$$

Then, every $X \in \mathcal{X}_\phi(U)$ can be written as a sum $X = X_1 + X_2$ of elements in $\mathcal{X}_1$ and $\mathcal{X}_2$, with $X_2 = \psi Z$ and $\psi(\phi(a)) = 0$, as $X(\phi(a)) = X_1(\phi(a)) = 0$.
Moreover, such a splitting is unique because:

$$(3.19) \qquad \phi^* L_X \beta = \phi^* L_{X_2} \beta = \phi^* \iota_{X_2} d\beta + d(\phi^* \iota_{X_2} \beta)$$
$$= \phi^*(\psi\, \iota_Z\, d\beta) + d(\phi^* \psi) ,$$

and this differential equation for $\overline{\psi} = \psi \circ \phi$ has a unique solution given the initial condition $\overline{\psi}(a) = 0$.
Therefore, X_2 and thus X_1 are uniquely defined by the choice of Z and β.
Then, for $X \in \mathcal{X}_\phi(U)$, $X = X_1 + X_2$ splitted as above, we obtain with Stokes theorem :

$$(3.20) \qquad \int_I \phi^* L_X \alpha = \int_I \phi^* \iota_X d\alpha + \int_I d\, \phi^* \iota_X \alpha = \int_I \phi^* \iota_{X_1} d\alpha + \int_I \phi^* \iota_{X_2}\, d\alpha .$$

Formal stationarity for X_1 and use of Stokes theorem yields:

$$(3.21) \qquad 0 = \int_a^b \phi^* L_{X_1} \alpha = \int_a^b \phi^* \iota_{X_1} d\alpha + (\phi^* \alpha(X_1))(b) .$$

Choose $C_0 \in \mathbb{R}$ such that, with $X_2 = \psi Z$, $\overline{\psi} = \psi \circ \phi$:

$$\alpha(X_1(\phi(b))) = C_0 \overline{\psi}(b) .$$

Note that $X_1(\phi(b)) = - X_2(\phi(b)) = - \overline{\psi}(b) \; Z(\phi(b))$, so that C_0 depends only on the choice of Z and β and not on X. Then (3.20) and (3.21) yield

$$\text{(3.22)} \qquad \int_a^b \phi^* L_X \alpha = \int_a^b \phi^* \iota_{X_2} d\alpha - C_0 \overline{\psi}(b) .$$

Now define Ψ_1 and Ψ_2 on [a,b] (only dependent on $\overline{X}$, β and φ) :

$$\Psi_1 \, dt = \phi^* \iota_Z \, d\beta \quad , \quad \Psi_2 \, dt = \phi^* \iota_Z \, d\alpha$$

and $\qquad \overline{\lambda} : [a,b] \to \mathbb{R}$ by :

$$\Psi_2 = \overline{\lambda}' - \Psi_1 \overline{\lambda} \qquad , \overline{\lambda}(b) = C_0 .$$

Then we have for all $X \in X_\phi(U)$ and $\lambda \in C^\infty(U) (\lambda = \overline{\lambda}\phi^{-1})$ independent of X :

$$\text{(3.23)} \qquad \begin{aligned} - \int_a^b \overline{\lambda}(t) \, \phi^* L_X \beta &\overset{(3.19)}{=} - \int_a^b \overline{\lambda}(t) (\overline{\psi} \, \Psi_1 dt + d\overline{\psi}) \\ &= \int_a^b (\overline{\lambda}(t) \, \Psi_1 \overline{\psi} - \overline{\lambda}'(t) \, \overline{\psi}) dt - \overline{\lambda}(b) \, \overline{\psi}(b) \\ &= \int_a^b \Psi_2 \, \overline{\psi} \, dt + C_0 \overline{\psi}(b) \overset{(3.22)}{=} \int_a^b \phi^* L_X \alpha . \end{aligned}$$

If we introduce again the subscripts denoting the local neighbourhood and extending W_μ to a variation vector field such that :

$$X_\mu (\phi_\mu(t)) = W_\mu (\phi_\mu(t)) \quad , \quad t \in I_\mu \quad ,$$

we obtain (3.18) (with use of lemma 2.3). So, we have a local construction for the section s: M → E specified by λ and β on Im φ , together with the additional condition that V(t) is tangent to Im s. However, this construction is coordinate-free which is easiest seen by choosing another basic form β'. Then (3.23) implies for all X, locally:

$$\int_{I_\mu} \overline{\lambda}'(t)\ \phi^* L_X\ \beta' = \int_{I_\mu} \overline{\lambda}(t)\ \phi^* L_X\ \beta$$

(we omitted subscripts μ in the integrands). Therefore, the integrands are equal on I_μ for all X so that :

$$\lambda'\ \beta' = \lambda\beta \qquad \text{on } \phi(I_\mu).$$

As this product defines the section, we see that the local definition of this section s restricted to Im ϕ is coordinate-free. Therefore we can conclude, using $\eta = s \circ \phi$ and (3.17) that (3.15) and therefore (3.16) are satisfied.

This implies stationarity of η , at least for a distribution S of codimension 1. For codimension $k > 1$ the proof can be given similarly. □

REMARK 3.9. The choice of a metric on E (with Levi-Civita connection and covariant differentiation) does not appear in the formulation of the theorem. It is used only to relate variation vector fields on E with variation vector fields on M. This is not surprising. The only relevant aspects of a variation vector field is its behaviour along the curve under consideration, together with the fact that it is part of a vectorfield on M. Covariant differentiation provides an excellent tool to study just these aspects.

4. THE LAGRANGE PROBLEM

Consider a smooth manifold Q (the *configuration space*) with dim Q = n and a function $L : TQ \times I \to \mathbb{R}$ which is called the *Lagrangian* (I is a closed interval in $\mathbb{R}$ as before). Then we can seek for curves $\psi : I \to Q$ which minimize the action integral

$$(4.1) \qquad J(\psi) = \int_I L(\psi(t), \psi'(t), t)\, dt\ .$$

This is called the *Lagrange problem*. We can formulate this problem according to section 3. To do so, choose a coordinate t on I and let

$$(4.2) \qquad M = TQ \times I \quad , \alpha = L\, dt.$$

Moreover, we can define a mapping ℓ from the set of curves $\psi : I \longrightarrow Q$ to the set of curves $\phi : I \longrightarrow M$ by

$$(4.3) \qquad \ell\,(\psi)(t) = (\psi_* \circ s\,(t),t) \quad ; \quad s(t) = \frac{\partial}{\partial t}\Big|_t \quad ; \quad t \in I$$

(s is a mapping $s : I \longrightarrow TI$). Subsequently, we can define a codistribution $E \subset T^*M$ by

$$(4.4) \qquad E = \{\beta \in T^*M \mid \exists\ \psi : I \to Q \text{ such that } (\ell(\psi))^* \beta = 0\} \ .$$

E is a codistribution as the following coordinate representation shows.

<u>REMARK 4.1.</u> Choosing canonical coordinates $q,\dot{q},t$ on M, it can easily be shown that the fibres of E are spanned by 1-forms β_i (i=1,...,n) locally, given by

$$(4.5) \qquad \beta_i = dq_i - \dot{q}_i dt \ .$$

We have dim E = n and the restriction distribution S for the Lagrange problem is defined as the annihilator $S = E^{\perp}$ ($S = \bigcap_{i=1}^{n} \ker \beta_i$).

The above arguments show that the Lagrange problem can be formulated as the problem of minimizing

$$(4.6) \qquad J\,(\phi) = \int_I \phi^* \alpha$$

over curves in M under restriction S. We restrict ourselves to first order conditions, i.e. stationarity. The restriction distribution in this case has special properties which the following conjecture suggests

CONJECTURE 4.2. *For the above restricted variational problem we have equivalence between stationarity and formal stationarity of curves in* M.

A proof of this conjecture might go along the following lines. First we have the following lemma.

LEMMA 4.3. *Let* S *be a distribution of fixed dimension* p *on* M. *Suppose a curve* $\phi : I \to M$ *is given such that*

(4.7) $$\phi_* \left(\frac{\partial}{\partial t}\Big|_t \right) \in S(\phi(t)) .$$

Let $V \in \overline{X}_\phi$ *be such that* $V(t) \in S(\phi(t))$. *Then there exists a variation of* ϕ k.e.p.f : $\xi : (-\delta,\delta) \times I \to M$ *such that*

(4.8) $$\xi_* \left(\frac{\partial}{\partial \varepsilon}\Big|_{(0,t)} \right) = V(t) \quad ; \quad \xi_* \left(\frac{\partial}{\partial +}\Big|_{(\varepsilon,t)} \right) \in S(\xi(\varepsilon,t)) .$$

PROOF. Choose coordinates $x_1,\dots,x_m$ on a neighbourhood U of a point along ϕ such that for $y \in U$:

$$S(y) = \text{span} \left\{ \frac{\partial}{\partial x_1}\Big|_y , \dots , \frac{\partial}{\partial x_p}\Big|_y \right\} .$$

Let $\phi(t) = (\phi_1(t),\dots,\phi_m(t))$, $V(t) = \sum_{i=1}^{p} V_i(t) \frac{\partial}{\partial x_i}\Big|_{\phi(t)}$ for $t \in \{ s \in I \mid \phi(s) \in U \}$.
It can be checked that ξ defined by:

$$\xi(\varepsilon,t) = (\phi_1(t) + \varepsilon V_1(t) , \dots , \phi_p(t) + \varepsilon V_p(t), \phi_{p+1}(t),\dots,\phi_n(t))$$

satisfies the requirements and is in fact coordinate free. □

A second observation required for the proof of the conjecture is the following. Let ϕ be an admissable stationary curve under restriction S. Choose a metric with Levi-Civita connection and covariant differentiation. If $V(t) \in \overline{X}_\phi$, with $\frac{DV(t)}{dt} \in S(\phi(t))$, can be splitted :

$$V(t) = V_1(t) + V_2(t) ,$$

with $V_1 \not\equiv 0$, $V_1(t) \in S(\phi(t))$, $\frac{DV_2(t)}{dt} = 0$, then $V_1(t)$ defines an admissable variation k.e.p.f. $\phi_\varepsilon(t)$ according to lemma 4.3. So stationarity of ϕ implies :

$$\frac{d}{d\varepsilon} \int_I \phi_\varepsilon^* \alpha = 0 = \int_I \alpha \left(\frac{DV_1}{dt} \right) dt \; .$$

Hence

$$\int_I \alpha \left(\frac{DV}{dt} \right) dt = \int_I \alpha \left(\frac{DV_1}{dt} \right) dt + \int_I \alpha \left(\frac{DV_2}{dt} \right) dt = 0$$

Therefore, stationarity implies formal stationarity if every $V(t) \in \overline{X}_\phi$ with $\frac{DV(t)}{dt} \in S(\phi(t))$ admits such a splitting in a nontrivial part in the distribution and a part which is parallel along ϕ .

The proof of the conjecture can be finished with the following arguments. The distribution S in remark 4.1 is locally given by :

$$S = \text{span} \left\{ \frac{\partial}{\partial \dot{q}_i} , \sum_{j=1}^{n} \dot{q}_j \frac{\partial}{\partial q_j} + \frac{\partial}{\partial t} \right\} \tag{4.9}$$

and its orthogonal complement

$$S^c = \text{span} \left\{ \frac{\partial}{\partial q_i} - \dot{q}_i \frac{\partial}{\partial t} \right\}$$

is integrable. Loosely speaking, that means that TM/S^c is flat in the directions of S^c which makes a choice of a parallel field along ϕ , independent of S possible (i.e. a splitting as above is possible). We shall leave details of the last arguments for future, more general research and leave 4.2 as a conjecture.

A direct consequence of theorem 3.8 and conjecture 4.2 is :

<u>COROLLARY 4.4</u>. *An injective curve $\psi : I \to Q$ is a stationary curve for the Lagrange problem if and only if there exists an injective curve $\eta : I \to E$ (E defined by (4.4)), with $\pi_E \circ \eta = \ell(\psi)$ and η is stationary with respect to the Cartan form*

$$\theta_L = \pi_E^* (L dt) + \theta_E$$

(*see* (3.11) *with* $\alpha = L dt$) .

Note that, by theorem 2.4, $\eta : I \to E$ is stationary with respect to θ_L if and only if $\eta'(t)$ belongs to ker $d\theta_L$. It is illustrative to assume for a moment that we can give global coordinates $q, \mathring{q}, t$ for M, and to work out the consequences of corollary 4.4 in these coordinates. As the fibres of E are spanned by forms β_i $(i=1,\ldots,n)$ (given by (4.5)) we can choose coordinates $q, \mathring{q}, \lambda, t$ on E ($\xi \in E \Rightarrow \xi = \Sigma_{i=1}^n \lambda_i \beta_i$) . We obtain

$$\theta_L = \sum_{i=1}^n \lambda_i \beta_i + L\, dt . \tag{4.10}$$

As variation vector fields for curves in E have , by definition of E, a $\frac{\partial}{\partial t}$ - component unequal to zero we may assume a parametrization of I such that this component equals 1. So we restrict ourselves to vector fields on E of the form

$$X = X_{q_i} \frac{\partial}{\partial q_i} + X_{\mathring{q}_i} \frac{\partial}{\partial \mathring{q}_i} + X_{\lambda_i} \frac{\partial}{\partial \lambda_i} + \frac{\partial}{\partial t} ,$$

where summation over $i = 1,\ldots,n$ is assumed. The condition that X should belong to ker $d\theta_L$, i.e. $\iota_X d\theta_L = 0$ yields:

$$\iota_X \left[\sum_{i=1}^n (d\lambda_i \wedge (dq_i - \mathring{q}_i dt) - \lambda_i d\mathring{q}_i \wedge dt) + dL \wedge dt \right] = 0 .$$

Collecting the terms in $dq_i, d\mathring{q}_i, d\lambda_i$ and dt, respectively, yields the equations $(i=1,\ldots,n)$:

$$X_{\lambda_i} - \frac{\partial L}{\partial q_i} = 0 , \tag{4.11}$$

$$\lambda_i - \frac{\partial L}{\partial \mathring{q}_i} = 0 , \tag{4.12}$$

$$\mathring{q}_i - X_{q_i} = 0 , \tag{4.13}$$

$$(4.14)\qquad -\dot q_i\, X_{\lambda_i} - \lambda_i X_{\dot q_i} + \frac{\partial L}{\partial q_i} X_{q_i} + \frac{\partial L}{\partial \dot q_i} X_{\dot q_i} = 0 \ .$$

It is easily seen that the first three equations imply the fourth, so that we have 3n equations. Clearly (4.12) defines a (2n+1)-dimensional submanifold $N \subset E$. Therefore, stationary curves lie in N and are integral curves of a vector field satisfying (4.11) and (4.13) (on E). In order that X is a vector field on N (i.e. $X_p \in T_pN$ for all $p \in N$) we must have

$$X\Big(\frac{\partial L}{\partial \dot q_i} - \lambda_i\Big) = 0$$

which implies, using (4.11) and (4.13) :

$$(4.15)\qquad \left(\frac{\partial^2 L}{\partial \dot q^2}\right) X_{\dot q} = \frac{\partial L}{\partial q} - \left(\frac{\partial^2 L}{\partial q \partial \dot q}\right)\dot q - \left(\frac{\partial^2 L}{\partial \lambda \partial q}\right)\frac{\partial L}{\partial q} - \frac{\partial^2 L}{\partial t\, \partial \dot q} \ ,$$

where expressions between parenthesis are matrices. So, (4.15) defines $X_{\dot q}$ uniquely if and only if $(\frac{\partial^2 L}{\partial \dot q^2})$ has full rank. In this case, in each point of N, X defines a unique tangent vector to N in ker $d\theta_L$, according to :

$$X = \dot q_i \frac{\partial}{\partial q_i} + X_{\dot q_i} \frac{\partial}{\partial \dot q_i} + \frac{\partial L}{\partial q_i} \frac{\partial}{\partial \lambda_i} + \frac{\partial}{\partial t} \ ,$$

with $X_{\dot q_i}$ defined by (4.15). So stationary curves are integral curves of the thus defined vector field on $N \subset E$.
In this, so-called hyperregular case, we can use coordinates $q, \dot q, t$ as well as q, λ, t on N. Using (4.12) we get for $d\theta_L|N$:

$$d\theta_L = d\lambda_i \wedge dq_i - \dot q_i d\lambda_i \wedge dt + \frac{\partial L}{\partial q_i} dq_i \wedge dt =$$

$$= d\lambda_i \wedge dq_i - dH \wedge dt$$

if we define

$$H(q,\dot q,\lambda,t) = - L\,(q,\dot q,t) + \sum_{i=1}^{n} \dot q_i \lambda_i \ .$$

Using (4.12) to express $\dot{q}$ in terms of q, λ and t and denoting $\tilde{H} = H|N$ in coordinates q,λ and t thus obtained, we have

$$d\tilde{\theta}_L = d\lambda_i \wedge dq_i - d\tilde{H} \wedge dt$$

and

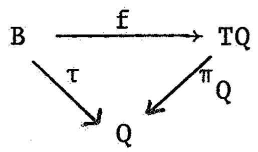
$$\tilde{\theta}_L = \lambda_i dq_i - \tilde{H}\,dt,$$

which gives the Hamiltonian formalism for the Lagrange problem, which is well-known (see ABRAHAM & MARSDEN [1978]).

5. THE LAGRANGE PROBLEM FOR NONLINEAR CONTROL SYSTEMS

We shall first recall the notion of a general nonlinear control system as given by BROCKETT [1977] and WILLEMS [1981] and worked out by van der SCHAFT [1982] and NIJMEIJER & van der SCHAFT [1982] .

DEFINITION 5.1. A *nonlinear (time-invariant) control system* Σ is defined by a smooth manifold Q, a fibre bundle $\tau : B \to Q$ and a smooth map $f : B \to TQ$ such that the following diagram commutes



The system can be seen as a set of trajectories Σ defined by :
$\Sigma \doteq \{q: I \to Q \mid q$ absolutely continuous, $\dot{q}(t) \in f(\tau^{-1}(q(t)))$ almost everywhere$\}$ and is denoted by $\Sigma(Q,B,f)$. I denotes the time interval under consideration. Each trajectory $\phi \in \Sigma$ is associated with an *input map*, represented by a map $\nu : \operatorname{Im}\phi \to B$, such that

(5.1) $$\tau \circ \nu = id_{\operatorname{Im}\phi} \quad ; \quad \phi_*\left(\frac{\partial}{\partial t}\Big|_t\right) = f(\nu(\phi(t))) \;, \quad t \in I \;.$$

In our situation Q represents the configuration space. The fibres of B represent the (state dependent) input spaces. If we denote local coordinates q for Q and (q,u) for B (u local coordinates for the fibres $\tau^{-1}(q)$), then we obtain the familiar condition that the system trajectories satisfy $\dot{q} = f(q,u)$ for some u (with abuse of notation $f: (q,u) \longmapsto (q,f(q,u))$). We shall assume that the dimension of the fibres of B is constant and equal to $m \leq n = \dim Q$. Moreover, we assume that f is injective.

Now suppose a 1-form α on $TB \times I$ is given. Then we can pose the problem of minimizing the action integral

$$(5.2) \qquad \mathcal{J}(\phi) = \int_I \phi^* \alpha$$

over the set of curves $\phi : I \longrightarrow TB \times I$ for which there exists a trajectory $\psi : I \longrightarrow Q$ of $\sum$ with associated input map ν such that

$$(5.3) \qquad \phi = \ell(\nu \circ \psi) ,$$

with ℓ defined by (4.3). If we define $\tilde{\tau}_* : TB \times I \to TQ$ by

$$\tilde{\tau}_*(v,t) = \tau_*(v) \qquad , \forall\, v \in TB \quad ,$$

then we can easily deduce the following commutativity relation:

$$(5.4) \qquad \tilde{\tau}_* \, \ell(\nu \circ \psi)(t) = \tilde{\tau}_*((\nu\psi)_* \, s(t),t) = \tau_* \nu_* \psi_* \, s(t)$$

$$= \psi_* \, s(t) = f(\nu \circ \psi(t)) .$$

In fact, the reasoning of section 4 is applied to B instead of Q and additional restrictions on curves in B are given expressing the condition that they have to be a system trajectory. Denote the set of curves ϕ, for which there exists a ψ,ν such that (5.3) holds, by $\mathcal{C}$. Then, given a $\phi \in \mathcal{C}$, the pair ψ,ν is uniquely defined, by injectivity of f. Furthermore, each curve $\phi = \ell(\nu \circ \psi)$ in $\mathcal{C}$ satisfies

$$f \circ \tilde{\pi}_B(\phi(t)) = f(\nu \circ \psi(t)) = \tilde{\tau}_*(\phi(t)$$

using (5.4), with $\tilde{\pi}_B$:TB × I ⟶ B the natural projection. So, the set

(5.5) $$M = \{(v,t) \in TB \times I \mid f \circ \pi_B(v) = \tau_*(v)\}$$

is a submanifold of TB × I. The curves from C lie in M, so we can restrict our optimization to M. Then, define

(5.6) $$E = \{\beta \mid \beta \in T^*M \ , \ \phi^* \beta \equiv 0 \quad \forall \phi \in C\} ,$$

where C is interpreted as a set of curves in M. So, our minimization problem is in fact a minimization problem in M with restriction distribution $S = E^{\perp}$ on M. That E is a codistribution follows from its coordinate representation.

<u>PROPOSITION 5.2</u>. *Let* q *denote local coordinates in* Q *and* (q,u) *in* B (u *are coordinates for fibres). Let* $(q,u,\dot{q},\dot{u},t)$ *be local canonical coordinates in* TB × I . *Then* M *is locally defined by:*

(5.7) $$\dot{q}_i = f_i(q,u) \qquad i = 1,\ldots,n .$$

Moreover, in local coordinates $(q,u,\dot{u},t)$ *for* M, $E \subset T^*M$ *as defined by* (5.6) *is spanned by* n+m 1-*forms* :

(5.8) $$\beta_i = dq_i - f_i(q,u)dt \qquad i = 1,\ldots,n ,$$
$$\beta_{n+j} = du_j - \dot{u}_j dt \qquad j = 1,\ldots,m .$$

Here $f_i(q,u)$ *denotes the* i-th *coordinate of* f(q,u) *in* $T_{(q,u)}Q$.

<u>PROOF</u>. (5.7) follow easily from (5.5).
Let $\phi(t) = (\phi_q(t), \phi_u(t), \phi_{\dot{u}}(t),t)$ be a curve in M satisfying the restriction. Then (5.3) yields for certain ψ and ν $(\nu(q) = (q,\nu_u(q)))$:

$$\phi_q(t) = \psi(t) ,$$
$$\phi_u(t) = \nu_u(\psi(t)) ,$$
$$\phi_{\dot u}(t) = \frac{d}{dq}(\nu_u(\psi(t))) \; \frac{d\psi(t)}{dt} .$$

Hence, for $i = 1,\ldots, n$:

$$\phi^*\beta_i \left(\frac{\partial}{\partial t}\Big|_t \right) = \frac{d\psi_i(t)}{dt} - f_i(\psi(t), \nu_u(\psi(t))) = 0 ,$$

using (5.1). Furthermore, for $j = 1,\ldots, m$:

$$\phi^*\beta_{n+j}\left(\frac{\partial}{\partial t}\Big|_t \right) = \frac{d\nu_{u_j}(\psi(t))}{dt} - \frac{d}{dq}(\nu_{u_j}(\psi(t))) \frac{d\psi(t)}{dt} = 0 ,$$

using the chain rule. This proves the theorem as the dimension of E_x equals $n + m$ ($x \in M$) due to determination of curves in C by a curve ψ in Q and a map ν in B, satisfying (5.1). □

6. THE NONLINEAR OPTIMAL CONTROL PROBLEM

Let Σ (Q,B,f) be a (time-invariant) nonlinear control system (cf.def. 5.1). Suppose we have given a *cost function* $g:B \to \mathbb{R}$ and two points $q_a, q_b \in Q$. Then, the (time-invariant) optimal control problem, denoted by $(\Sigma(Q,B,f), g)$, is to find system trajectories $\psi : I \; (= [a,b]) \to Q$ and associated input maps ν such that $\psi(a) = q_a$, $\psi(b) = q_b$ and

(6.1) $$\int_I g(\nu \circ \psi(t))\, dt$$

is minimized.

We want to formulate this problem as a Lagrange problem. This can be done if we specify a 1-form α on M (see (5.5)) which satisfies

(6.2) $$\phi^*\alpha = g(\pi_M(\phi(t)))\, dt ,$$

for all curves ϕ with $\phi^*\beta = 0$ $\forall \beta \in E$ (given by (5.6)), where $\pi_M : M \to B$ is the natural projection. As $\pi_M \circ \phi = \nu \circ \psi$ for $\phi = \ell\,(\nu \circ \psi)$ we have the following proposition.

PROPOSITION 6.1. *Let* α *be a 1-form on* M *satisfying* (6.2). *If* ϕ: I $\to$ M *is optimal for* α *under restriction* $S = E^{\perp}$, *given certain end points* $m_a, m_b \in M$. *Then the trajectory* $\psi = \tau \circ \pi_M \circ \phi$ *and associated input defined by* $\nu\psi = \pi_M \phi$, *minimize* (6.1) *provided* $\tau \circ \pi_M(m_a) = q_a$, $\tau \circ \pi_M(m_b) = q_b$. *Conversely, if* ψ,ν *is an optimal solution of the optimal control problem, then* $\ell(\nu \circ \psi)$ *is optimal for* α *under restriction* S *(with* $\ell(\nu \circ \psi)$ *interpreted as curve in* M).

It is clear that any choice of α of the form

$$(6.3) \qquad \alpha = g \circ \pi_M dt + \gamma,$$

with $\gamma \in E$, satisfies (6.2) by definition of E. So we have some freedom in our problem definition which can be used, as will be shown later.

Stationarity of a curve is a necessary condition for optimality. By theorem 3.8, formal stationarity of the restricted problem is equivalent with stationarity of an unrestricted problem with respect to a Cartan form. If stationarity implies formal stationarity, then theorem 3.8 provides first order necessary conditions for optimality of a curve. In the rest of this section we shall use the following technical assumption.

DEFINITION 6.2. The nonlinear optimal control problem $(\Sigma(Q,B,f),g)$ has *property S*, if for the Lagrange problem associated with it as above, stationarity of curves implies formal stationarity of curves.

In general, nonlinear optimal control problems do not have property S. One can easily follow the proof of conjecture 4.2 to see that integrability of the orthogonal complement of the restriction distribution plays a role. We shall not work out here more practical conditions implying property S. This will be a matter of future concern. We assume that all systems concerned have property S. Then we can use theorem 3.8 to obtain the following result.

PROPOSITION 6.3. *Let the nonlinear optimal control problem* $(\Sigma(Q,B,f),g)$ *have property* S. *Let* $\gamma \in E$ *be arbitrary. Then, a trajectory* $\psi : I \to Q$ *with associated input* $\nu : \mathrm{Im}\,\psi \to B$ *is stationary if and only if there exists a curve* $\eta : I \to E$ *which is stationary w.r.t. the Cartan form*

$$(6.4) \qquad \theta_g = \pi_E^* (g \circ \pi_M dt + \gamma) + \theta_E ,$$

(with π_E: E → M *projection,* θ_E *the canonical* 1*-form on* T^*M *restricted to* E) *and which satisfies*

$$(6.5) \qquad \psi = \tau \pi_M \pi_E \eta \quad , \quad \nu\psi = \pi_M \pi_E \eta .$$

A stationary curve can be obtained as a characteristic curve of $d\theta_g$ *(see end of section* 2) *and projection yields the associated trajectory and input.*

We shall illustrate this proposition using local coordinates as in proposition 5.2. For E we use coordinates $(q,u,\dot{u},\lambda,\mu,t)$ for an element $\xi = \Sigma_{i=1}^n \lambda_i \beta_i + \Sigma_{j=1}^m \mu_j \beta_{j+n}$.

We choose

$$(6.6) \qquad \gamma = \sum_{j=1}^m \dot{u}_j (du_j - \dot{u}_j dt) .$$

Then, with summation over $i = 1,\dots,n$, $j = 1,\dots,m$:

$$(6.7) \qquad \theta_g = \lambda_i (dq_i - f_i(q,u)dt) + (u_j + \dot{u}_j)(du_j - \dot{u}_j dt) + g(q,u)dt.$$

As in section 4 we may confine ourselves to variation vector fields with $\frac{\partial}{\partial t}$ - component equal to 1. Then such a vector field on E is given by:

$$X = X_q \frac{\partial}{\partial q} + X_u \frac{\partial}{\partial u} + X_{\dot{u}} \frac{\partial}{\partial \dot{u}} + X_\lambda \frac{\partial}{\partial \lambda} + X_\mu \frac{\partial}{\partial \mu} + \frac{\partial}{\partial t} .$$

Hence $\iota_X d\theta_g = 0$ yields, as in section 4, by equating to zero the coefficients for dq , $d\lambda$, du , $d\mu$ and $d\dot{u}$, respectively:

$$(6.8) \qquad X_{\lambda_i} = \frac{\partial g}{\partial q_i} - \sum_{k=1}^n \lambda_k \frac{\partial f_k}{\partial q_i} ,$$

$$(6.9) \qquad X_{q_i} = f_i (q,u) ,$$

(6.10) $$X_{\mu_j} + X_{\dot{u}_j} = \frac{\partial g}{\partial u_j} - \sum_{k=1}^{n} \lambda_k \frac{\partial f_k}{\partial u_j} ,$$

(6.11) $$X_{u_j} = \dot{u}_j ,$$

(6.12) $$\mu_j = - \dot{u}_j .$$

The equation obtained from the dt-term is satisfied by substituting (6.8) up to (6.12) as in section 4. We have chosen γ such that the last equation is solvable with respect to $\dot{u}_j$. This equation (6.12) determines an $2(n+m) + 1$-dimensional submanifold $\overline{N} \subset E$ and $X_{\dot{u}_j}$ is uniquely defined by the equation

(6.13) $$0 = X(\mu_j + \dot{u}_j) = X_{\mu_j} + X_{\dot{u}_j} .$$

So for all $p \in \overline{N}$, $X_p \in T\overline{N}$, and the problem is reduced to finding integral curves of the vector field $X \mid \overline{N}$, satisfying (6.8) and (6.9) together with:

(6.14) $$0 = \frac{\partial g}{\partial u_j} - \sum_{k=1}^{n} \lambda_k \frac{\partial f_k}{\partial u_j} ,$$

(6.15) $$X_{u_j} = - \mu_j .$$

In the case that

(6.16) $$\det \left(\frac{\partial}{\partial u_i} \left(\frac{\partial g}{\partial u_j} - \sum_{k=1}^{n} \lambda_k \frac{\partial f_k}{\partial u_j} \right)\right)_{ij} \neq 0 ,$$

we can reduce the system even more by solving (6.14) with respect to u. Let this result be $u = F(q,\lambda)$. Then, from $X(u-F(q,\lambda)) = 0$ we obtain

(6.17) $$X_u = \frac{\partial F}{\partial q} X_q + \frac{\partial F}{\partial \lambda} X_\lambda = G(q,\lambda) ,$$

where G is obtained by substituting (6.8) and (6.9). With (6.15) we obtain

(6.18) $$\mu = - G(g,\lambda)$$

and the condition $X(\mu + G(g,\lambda)) = 0$ yields an expression for X_μ :

$$(6.19)\qquad X_\mu = \frac{\partial G}{\partial q} X_q + \frac{\partial G}{\partial \lambda} X_\lambda$$

Summarizing, we may consider the (2n+1)-dimensional submanifold $N \subset E$ defined by

$$(6.20)\qquad \dot{u} = + G(g,\lambda) \ , \quad \mu = - G(q,\lambda) \quad , \quad u = F(g,\lambda),$$

together with the vector field X, which satisfies $X_p \in T_p N$ for all $p \in N$. Hence, formally stationary curves (and if property S holds: stationary curves) of the optimal control problem are integral curves in N of the vector field defined by:

$$(6.21)\qquad \begin{aligned} X_q &= f(q,F(q,\lambda)) \ , \\ X_\lambda &= \frac{\partial g}{\partial q}(q,F(q,\lambda)) - \left[\frac{\partial f}{\partial q}(q, F(q,\lambda))\right] \lambda \ , \end{aligned}$$

provided (6.16) is satisfied. Equation (6.21) expresses first order necessary conditions for optimality of a curve q(t), if property S holds. Note that, with

$$H(q,\lambda,u) = g(q,u) - \lambda^T f(q,u)$$

we have

$$X_q = - \left. \frac{\partial H(q,\lambda,u)}{\partial \lambda} \right|_{u = F(q,\lambda)} \ ,$$

$$X_\lambda = \left. \frac{\partial H(q,\lambda,u)}{\partial g} \right|_{u = F(q,\lambda)} \ .$$

These equations look more familiar. They are also obtained if we apply Pontryagin's maximum principle.

Let us specialize it even more and look what we obtain for the time-invariant linear-quadratic optimal control problem defined by:

$$f(q,u) = Aq + Bu \ ,$$
$$g(q,u) = \tfrac{1}{2}(q^TC^TCq + u^Tu) \ ,$$

then (6.14) yields ,

(6.22) $$u = B^T\lambda \quad (= F(q,\lambda))$$

(the determinant from (6.16) equals 1) . For G we obtain from (6.17):

(6.23) $$G(q,\lambda) = B^T(C^TCq - A^T\lambda).$$

Hence, $\dot{u}$ and μ follow from (6.20) and the vector field (6.21) becomes on N:

(6.24) $$\begin{pmatrix} X_q \\ X_\lambda \end{pmatrix} = \begin{pmatrix} A & BB^T \\ C^TC & -A^T \end{pmatrix} \begin{pmatrix} q \\ \lambda \end{pmatrix} ,$$

which is the well-known equation for the linear Hamiltonian vector field for the linear-quadratic regulator problem (WONHAM [1979]) . We shall not go into details about existence and uniqueness of integral curves of this vector field. Such results can be found for instance in BROCKETT [1970] . Our purpose was to show that our general theory for nonlinear time-invariant optimal control problems easily reduces to the familiar results in the linear-quadratic case.

Proposition 6.3 is the important result in this section. It yields the general formulation of first order necessary conditions for optimality (or necessary and sufficient conditions for stationarity) of curves for the optimal control problem. A particularly nice detail here, is the freedom of choosing γ in E, which can be used to reduce the dimension of the problem by guaranteeing solvability of the equations with respect to certain variables and, moreover, to obtain simple equations, once we have chosen coordinates.

7. CONCLUSION

In this paper we gave a generalization of Lagrange's multiplier rule for restricted variational problems on manifolds. Using a recent formulation for nonlinear control systems on manifolds we are able to give a formulation of optimal control problems on manifolds and to derive first order necessary conditions in this formalism. These are based on finding characteristic curves of a certain 2-form, which is the differential of a specific Cartan form. We expect that the formalism given here is especially suitable for studying nonlinear optimal control problems. In particular, notions like non-linear controllability, stabilizability etc. might be easily introduced in our setting, in order to study existence and uniqueness of solutions in non-linear optimal control problems.

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