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A note on Bernoulli-numbers.

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A NOTE ON BERNOULLI-NUMBERS

1. Introduction.

In the following Bernoulli-numbers are defined by the generating function

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n, \tag{1.1}$$

so that, $B_0 = 0$, $B_1 = -\frac{1}{2}$, $B_2 = 1/6$, $B_3 = B_5 = \ldots = 0$, $B_4 = -1/30$, $B_6 = 1/42$ and so on.
Van Dantzig conjectured that $2(2^{2n}-1)B_{2n}$ is an integer. This will be proved, and moreover it will be shown that for $n > 0$ this integer is odd. A simple recurrence-relation for these integers is derived.

2. First proof.

It is convenient to define

$$b_{2n} = 2(2^{2n}-1)\, B_{2n}. \tag{2.1}$$

The well-known expansion of tanh x then reads

$$\tanh x = \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} (2x)^{2k-1}, \tag{2.2}$$

and as sinh x = cosh x.tanh x it follows that

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)!} x^{2k-1} = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)!} 2^{2k-1} x^{2k-1}.$$

Equating the coefficient of x^{2n-1} on both sides of this equation one finds

$$\frac{1}{(2n-1)!} = \sum_{k=1}^{n} \frac{2^{2k-1} b_{2k}}{(2k)!(2n-2k)!}$$

or after multiplication by $2(2n)!$:

$$\sum_{k=1}^{n} \binom{2n}{2k} 2^{2k} b_{2k} = 4n. \tag{2.3}$$

This is a simple linear recurrence relation for the b_{2k}'s. It can be written as

$$2^{2n} b_{2n} = 4n - \sum_{k=1}^{n-1} \binom{2n}{2k} 2^{2k} b_{2k}. \tag{2.4}$$

If up to $k = n-1$ holds that $2^{2k}\, b_{2k}$ is an integer then it holds apparently also for $k = n$ as then all terms on the right hand side of (2.4) are integers. Hence, it follows by induction that $2^{2n}\, b_{2n}$ is an integer. It has to be proved, however, that already b_{2n} is an integer.

To that end one observes that if $B_{2n} = N_{2n}/D_{2n}$, with $(N_{2n},D_{2n}) = 1$, then D_{2n} contains exactly one factor 2 if $n > 0$. This follows from the theorem of von Staudt, stating that

$$B_{2n} = \text{integer} + \sum p_k^{-1} ,$$

the summation being extended over all primes p_k for which $p_k - 1$ divides $2n$. For each n apparently $p_k = 2$ occurs in the sum, and as all other possible primes are odd it follows that $D_{2n} = \prod p_k$ contains exactly one factor 2. As it has been proved already that D_{2n} divides $2^{2n}.2.(2^{2n}-1)$ it follows that D_{2n} divides $2(2^{2n}-1)$, so that indeed b_{2n} is an integer. Moreover 2^{2n-1} and N_{2n} are odd, whereas the factor 2 in (2.1) is cancelled by the factor 2 in D_{2n}. Hence b_{2n} is odd for $n > 0$.

3. Second proof.

A much simpler proof that does not yield, however, so readily the recurrence-relation (2.3) runs as follows. Starting again with the theorem of von Staudt, it has only to be proved that for odd primes p from $(p-1) | 2n$ follows that $p | (2^{2n}-1)$. This is an immediate consequence of Fermat's theorem. Indeed, from $a^{p-1} \equiv 1 \pmod p$ follows that $a^{2n} \equiv 1 \pmod p$ as $2n \equiv 0 \pmod{p-1}$. For $a = 2$ one has the required result

4. Practical application.

The recurrence-relation (2.4) yields a very easy means tocompute successiv values of b_{2n}, and from (2.1) follows B_{2n} easily enough. In itself the relation (2.4) is much handier than the direct recurrence-relation for B_{2n}, viz.

$$\sum_{k=0}^{2n} \binom{2n}{k} B_{2k} = B_{2n} \qquad (4.1)$$

in sofar that only integers occur in the calculations. However, these integers are unduly large because in reality $D_{2n} \ll 2(2^{2n}-1)$ if $n \gg 1$. So is, for instanc $D_{144} = 2381714790$ a relatively very large denumerator. Indeed, $D_{142} = D_{146} = 6$. But $2(2^{144}-1) = 44601490397061246283071436545296723011960830$.
The practical use seems, therefore, rather doubtful.

A short list of values runs as follows:

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A short list of values runs as follows:

$2n$	b_{2n}	$2(2^{2n}-1)$	N_{2n}	D_{2n}
0	0	0	1	1
2	1	6	1	6
4	- 1	30	- 1	30
6	3	126	1	42
8	- 17	510	- 1	30
10	155	2046	5	66
12	- 2073	8190	- 691	2730
14	38227	32766	7	6
16	- 929569	131070	- 3617	510