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ON A CERTAIN ASYMPTOTIC EXPANSION

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1. Introduction. This little note is concerned with the asymptotic expansion of a certain function, defined by a series resembling that of the modified Bessel function $I_p(z)$, viz.

$$f(z) = \sum_{k=1}^{\infty} \frac{(z/2)^{2k}}{k!k!k^{1/2}}. \tag{1,1}$$

This function plays a role in a certain physical problem, and values of $f(z)$ for large positive values of the argument had to be determined. In itself the problem is of little general interest but it gives an opportunity to stress again the importance of the use of factorial series and to show the simple means they provide to derive an asymptotic expansion in such a case.

2. The factorial series. The modified Bessel function $I_p(z)$ has the expansion

$$I_p(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+p}}{k!(k+p)!}. \tag{2,1}$$

As $k!\,k^{1/2} \sim (k + 1/2)!$ this series for $p = 1/2$ is, apart from some trivial modifications as multiplication by $(z/2)^{1/2}$ and the addition of a term with $k = 0$, closely related to the series (1, 1).

This relationship is used in the following way. The ratio $(k + 1/2)!/(k!\,k^{1/2})$ can be expanded into a factorial series (cf. Nörlund, Differenzengleichungen):

$$\frac{(k+1/2)!}{k!k^{1/2}} = \sum_{s=0}^{\infty} \frac{(-1)^s \binom{s-1/2}{s} B_s^{(s+1/2)}}{(k+3/2)(k+5/2)\cdots(k+s+1/2)}, \qquad (k > 0), \tag{2,2}$$

where $B_s^{(s+1/2)}$ are generalised Bernoulli numbers. As these alternate in sign with increasing values of s, the terms of the series (2, 2) are all positive. Or,

$$\frac{1}{k!k^{1/2}} = \sum_{s=0}^{\infty} (-)^s \binom{s-1/2}{s} B_s^{(s+1/2)} \frac{1}{(k+s+1/2)!}, \qquad (k > 0), \tag{2,3}$$

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so that after inserting into (1, 1) and interchanging the order of the summations

$$f(z) = \sum_{s=0}^{\infty} (-)^s \binom{s-1/2}{s} B_s^{(s+1/2)} \sum_{k=1}^{\infty} \frac{(z/2)^{2k}}{k!(k+s+1/2)!}$$

$$= \sum_{s=0}^{\infty} (-)^s \binom{s-1/2}{s} B_s^{(s+1/2)} \left\{ \left(\frac{z}{2}\right)^{-s-1/2} \sum_{k=0}^{\infty} \frac{(z/2)^{2k+s+1/2}}{k!(k+s+1/2)!} - \frac{1}{(s+1/2)!} \right\},$$

or with regard to (2, 1):

$$f(z) = \sum_{s=0}^{\infty} (-)^s \binom{s-1/2}{s} B_s^{(s+1/2)} \left\{ \left(\frac{z}{2}\right)^{-s-1/2} I_{s+1/2}(z) - \frac{1}{(s+1/2)!} \right\}. \qquad (2,4)$$

So far, no approximations have been introduced. It should be noted that the two terms between brackets in (2, 4) may not be separated as the latter term should give rise, according to (2, 3) to a divergent series.

The functions $I_{s+1/2}(z)$ are, as well known, elementary functions. For the first few values of s they are respectively:

$$I_{1/2}(z) = (2/\pi z)^{1/2} \sinh z,$$

$$I_{3/2}(z) = (2/\pi z)^{1/2} \left(\cosh z - \frac{1}{z} \sinh z \right),$$

$$I_{5/2}(z) = (2/\pi z)^{1/2} \left(\sinh z - \frac{3}{z} \cosh z + \frac{3}{z^2} \sinh z \right),$$

$$I_{7/2}(z) = (2/\pi z)^{1/2} \left(\cosh z - \frac{6}{z} \sinh z + \frac{15}{z^2} \cosh z - \frac{15}{z^3} \sinh z \right),$$

$$I_{9/2}(z) = (2/\pi z)^{1/2} \left(\sinh z - \frac{10}{z} \cosh z + \frac{45}{z^2} \sinh z - \frac{105}{z^3} \cosh z + \frac{105}{z^4} \sinh z \right),$$

$$I_{11/2}(z) = (2/\pi z)^{1/2} \left(\cosh z - \frac{15}{z} \sinh z + \frac{105}{z^2} \cosh z - \frac{420}{z^3} \sinh z + \frac{945}{z^4} \cosh z - \frac{945}{z^4} \sinh z \right).$$

Moreover, the numerical values of the coefficients in (2, 4) between the sum-sign and the brackets are for the first few values of s respectively:

s	$(-)^s \binom{s-1/2}{s} B_s^{(s+1/2)}$
0	1
1	3/8
2	65/128
3	1225/1024
4	131691/32768
5	4596669/262144

The series (2, 4) is convergent, and moreover for large values of z manageable.

3. The asymptotic series. For practical purposes, it is however advantageous to give up the convergence of the expansion and turn it into an ordinary divergent asymptotic series with simpler terms to be used for $z \gg 1$. To that end, one puts $\sinh z \sim \cosh z \sim e^z/2$, neglects the second term between the brackets in (2, 4), inserts the formulae given for $I_{s+1/2}(z)$ into (2, 4) and rearranges the terms with respect to decreasing powers of z. Then one obtains

$$f(z) \sim e^z(\pi z)^{-1/2}\left(1 + \frac{3}{4}z^{-1} + \frac{41}{32}z^{-2} + \frac{445}{128}z^{-3} + \frac{26571}{2048}z^{-4} + \frac{505029}{8192}z^{-5} \cdots\right). \quad (3,1)$$

The given terms yield e.g. $f(20) \sim 1.42507 \cdot 10^7$; actually $f(20) = 1.42508 \cdot 10^7$.