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Upon the Expression of an Integral as the Limit of a Continued Fraction

by

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I. Introduction

In this paper we shall consider methods of obtaining the integral $\int_0^\infty \phi(a+t)dt$ where a is in general complex valued. It is assumed that the function $\phi(a+t)$ is indefinitely differentiable over the range $0 \leqslant t \leqslant \infty$ of the argument.[+)]

The principle underlying the methods is as follows: The terms $h\phi(a+rh)$ of the Riemann sum $\sum_{r=0}^{\infty} h\phi(a+rh)$ are regarded as coefficients of the power series $\sum_{r=0}^{\infty} h\phi(a+rh)\, z^{-r-1}$; this power series is transformed unto an associated continued fraction, the variable z is made to tend to unity, the interval h is made to tend to zero. Here we examine the conditions under which this continued fraction either terminates or converges.

In certain cases the methods developed provide a way of ascribing a value to the integral $\int_0^\infty \phi(a+t)dt$ when the series $\sum_{r=0}^{\infty} h\phi(a+rh)$ diverges, i.e. when the Riemann definition breaks down. In all cases in which convergence c teria are established it is shown that if the function of z'

$$\int_0^\infty \phi(a+t)e^{-z't}dt$$

is meromorphic in some domain, then the process described yields the value of the Borel integral

$$\lim_{z'\to 0} \int_0^\infty \phi(a+t)e^{-z't}dt.$$

This limiting value may of course be infinite, in which case the continued fraction described above diverges inessentially, i.e. the reciprocals of its successive convergents tend to zero.

It should be emphasised that we are concerned here with an entirely new and constructive definition of an integral; it is shown how the value of the (possibly divergent) integral may be determined. The Borel definition of an integral is formal, it is assumed that F(z') is analytic in a certain domain; the definition does not contain a prescription for computing the value of F(z').

+) In point of fact the formulae of this paper are valid if $\phi(a+t)$ is replaced by $\phi(a+e^{i\theta}t)$ where $0 \leqslant \theta < 2\pi$, but since this is not a generalisation productive of any essentially new results and entails a considerable complication in the formulae used, it will not be taken into account.

II. The Second Confluent Form of the ε-Algorithm.

A formal process for computing the successive convergents of the above continued fraction has already been derived. It was established as follows. We are considering continued fractions of the form

$$\frac{\varepsilon_0}{z-\alpha_0-}\ \frac{\beta_0}{z-\alpha_1-}\ \cdots\ \frac{\beta_{r-1}}{z-\alpha_r-}\ \cdots \qquad (1)$$

Under certain conditions the coefficients α_s; β_s $(s=0,1,\dots)$ occurring in this continued fraction may uniquely be determined by imposing the condition that the power series expansion of the r^{th} convergent of (1) agrees with a given power series as far as the $(2+1)^{th}$ term, cr

$$\frac{c_0}{z-\alpha_0-}\ \frac{\beta_0}{z-\alpha_1-}\ \cdots\ \frac{\beta_{r-2}}{z-\alpha_r} = \sum_{s=0}^{\infty} c_s z^{-s-1} + 0(z^{-2r}). \qquad (2)$$

Now it may be shown [1] that the r^{th} convergent of (1), namely the left hand side of (2), is given in terms of the coefficients c_s $(s=0,1,\dots)$ by means of the determinantal expression

$$\frac{\begin{vmatrix} 0 & \sum_{s=0}^{0} c_s z^{-s-1} & \sum_{s=0}^{1} c_s z^{-s-1} & \cdots & \sum_{s=0}^{r-1} c_s z^{-s-1} \\ c_0 z^{-1} & c_1 z^{-2} & c_2 z^{-3} & \cdots & c_r z^{-r-1} \\ c_1 z^{-2} & c_2 z^{-3} & c_3 z^{-4} & \cdots & c_{r+1} z^{-r-2} \\ \cdot & \cdot & \cdot & & \cdot \\ c_{r-1} z^{-r} & c_r z^{-r-1} & c_{r+1} z^{-r-2} & \cdots & c_{2r-1} z^{-2r} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & & 1 \\ c_0 z^{-1} & c_1 z^{-2} & c_2 z^{-3} & \cdots & c_r z^{-r-1} \\ c_1 z^{-2} & c_2 z^{-3} & c_3 z^{-4} & \cdots & c_{r+1} z^{-r-2} \\ \cdot & \cdot & \cdot & & \cdot \\ c_{r-1} z^{-r} & c_r z^{-r-1} & c_{r+1} z^{-r-2} & \cdots & c_{2r-1} z^{-2r} \end{vmatrix}} \qquad (3)$$

Let us introduce the notation

$$H_0^{(m)}=1,\; H_k^{(m)}=\begin{vmatrix} \phi^{(m)}(a) & \phi^{(m+1)}(a) & \dots & \phi^{(m+k-1)}(a) \\ \phi^{(m+1)}(a) & \phi^{(m+2)}(a) & \dots & \phi^{(m+k)}(a) \\ \dots & \dots & & \dots \\ \phi^{(m+k-1)}(a) & \phi^{(m+k)}(a) & \dots & \phi^{(m+2k-2)}(a) \end{vmatrix}, \quad \begin{matrix}(m=0,1,\dots \\ k=1,2,\dots)\end{matrix} \qquad (4)$$

for the Hankel determinant, where

$$\phi^{(-1)}(a) \equiv 0,$$
$$\phi^{(s)}(a) = \left(\frac{d}{da}\right)^s \phi(a). \quad (s=0,1,\dots) \qquad (5)$$

In the determinantal quotient (3) put

$$c_s = h\phi(a+sh) \quad (s=0,1,\dots)$$

Let z tend to unity, let h tend to zero, and denote the resultant value of (3) by $\varepsilon_{2r}^{*}(a)$. Then it is easy to show that +)

$$\varepsilon_{2r}^{*}(a)=\frac{H_{r+1}^{(-1)}}{H_r^{(1)}}. \quad (r=0,1,\dots) \qquad (6)$$

The sequence of functions $\varepsilon_{2r}^{*}(a)$ $(r=0,1,\dots)$ may be constructed by means of the second confluent form of the ε-algorithm [2] Theorem II.1. From the initial values

$$\varepsilon_{-1}^{*}(a)=\varepsilon_0^{*}(a)=0 \qquad (7)$$

+) The effect of introducing the factor of $e^{i\theta}$ mentioned in the foot-note to Page 1 is merely to replace the expression $\varepsilon_{2r}^{*}(a)$ by $e^{i\theta}\varepsilon_{2r}^{*}(a)$.

the sequence $\varepsilon_s^*(a)$ $(s=0,1,\ldots)$ is determined by means of the non-linear difference-differential relationships

$$\left\{\varepsilon_{2r+2}^*(a)-\varepsilon_{2r}^*(a)\right\}\frac{d}{da}\varepsilon_{2r+1}^*(a)=1 \qquad (8)$$

$$\left\{\varepsilon_{2r+1}^*(a)-\varepsilon_{2r-1}^*(a)\right\}\left\{\frac{d}{da}\varepsilon_{2r}^*(a)+\phi(a)\right\}=1. \qquad (9)$$

We shall now establish a fundamental property of the functions $\varepsilon_{2r}^*(a)$ of which considerable use will be made in this paper. Consider the integral

$$F(a;z')=\int_0^\infty e^{-z't}\phi(a+t)dt$$

This has the formal expansion in inverse powers of z'

$$F(a;z')\sim\sum_{s=0}^{\infty}\phi^{(s)}(a)\,z'^{-s-1}$$

This power series may formally be transformed into an associated continued fraction of the form (1) . viz.

$$F(a;z')\sim\lim_{t\to a}\left\{\frac{\phi(t)}{z'-Q_1(t)-}\;\frac{E_1(t)}{z'-Q_2(t)-}\cdots\frac{E_{r-1}(t)}{z'-Q_r(t)-}\cdots\right\}\cdot \qquad (10)$$

The successive convergents

$$C_r(z')=\lim_{t\to a}\left\{\frac{\phi(t)}{z'-Q_1(a)-}\;\frac{E_1(t)}{z'-Q_2(t)-}\cdots\frac{E_{r-1}(t)}{z'-Q_r(t)}\right\}(r=0,1,\ldots) \qquad (11)$$

of this continued fraction are given by determinantal expressions of the form (3) in which

$$c_s=\phi^{(s)}(a) \qquad (s=0,1,\ldots) \qquad (12)$$

and z is replaced by z'. If we let z' tend to zero in this expression (after suitable manipulations) and compare the result with expressions (4) and (6) we find that

$$\lim_{z'\to 0} C_r(z')=\varepsilon_{2r}^*(a).$$

Thus we obtain

Theorem II.2. In the notation of equation (10)

$$\varepsilon^{*}_{2r}(a) = \lim_{t\to a}\left\{\frac{\phi(t)}{-Q_1(t)-}\ \frac{E_1(t)}{-Q_2(t)-}\ \cdots\ \frac{E_{r-1}(t)}{-Q_r(t)}\right\} \tag{13}$$

Note: In equations (10),(11) and (13) and subsequently we use the designation $\lim_{t\to a}$ rather than replace t by a in the relevant expressions; in this way all convergents of the continued fractions of the rather special class being considered are determinate. In more general cases this is not always true; for example the value of the convergent

$$\frac{1}{1+}\ \frac{0}{1-}\ \frac{1}{1}$$

is indeterminate.

We conclude this section on the formal properties of the second confluent ε-algorithm process by remarking that if we replace $\phi(a+t)$ by $e^{-zt}\phi(a+t)$ in expressions (4) and (6) and compare the result with (3) in which the substitution (12) has been made, then we obtain the following consistency result

Theorem II.3. The successive convergents of the continued fraction (10) may be obtained from the relationships

$$\varepsilon^{*}_{-1}(z|t) = \varepsilon^{*}_{0}(z|t) = 0,$$

$$\left\{\varepsilon^{*}_{2r+2}(z|t) - \varepsilon^{*}_{2r}(z|t)\right\}\frac{d}{dt}\varepsilon^{*}_{2r+1}(z|t) = 1,$$

$$\left\{\varepsilon_{2r+1}(z|t) - \varepsilon_{2r-1}(z|t)\right\}\left\{\frac{d}{dt}\varepsilon^{*}_{2r}(z|t) + e^{-zt}\phi(a+t)\right\} = 1,$$

when

$$C_r(z) = \lim_{t\to a}\varepsilon^{*}_{2r}(z|t).$$

III. Continued Fraction Integration

Definition 1: If the functions $\varepsilon^{*}_{2r}(a)$ $(r=0,1,\ldots)$ produced by means of the relationships (7),(8) and (9) terminate (in the sense that $\varepsilon^{*}_{2r+1}(a)$ becomes infinite for all a), or converge to a finite limit, or diverge properly (in the sense that the sequence $\left\{\varepsilon^{*}_{2r}(a)\right\}^{-1}$ $(r=0,1,\ldots)$

converges to zero) then the function $\phi(a+t)$ is said to be continued fraction integrable (c.f.i.).

A Certain Class of Functions $\phi(a+t)$

The coefficient $E_s(a)$, $Q_s(a)$ of (10) may be constructed in a number of ways [3], [4], [5]. They are given by the determinantal formulae

$$E_r(a)=\frac{H_{r+1}^{(0)}\,H_{r-1}^{(0)}}{H_r^{(0)\,2}},\quad Q_r(a)=\frac{H_{r+1}^{(0)}\,H_r^{(0)}}{H_{r+1}^{(0)}\,H_r^{(1)}}+\frac{H_{r+1}^{(0)}\,H_{r-1}^{(1)}}{H_r^{(0)}\,H_r^{(1)}}.$$

Let us dismiss, once and for all, the case in which the function $\phi(a)$ satisfies a linear differential equation with constant coefficients

$$\sum_{s=0}^{n'} a_s\,\phi^{(s)}(a)=0$$

in which n' cannot be replaced by a smaller integer. We shall agree to call this class of functions N.

Lemma III. 1. The equations

$$H_r^{(0)} \begin{array}{ll} \neq 0 & r=0,1,\dots,n-1 \\ = 0 & r=n \end{array}$$

hold for all values of a if and only if $\phi \in N$.

Lemma III. 2. The continued fraction (10) terminates with its n^{th} convergent for all values of a if and only if $\phi(a)\in N$.

In this case $F(a;z')$ is a rational function of z' whose denominator is a polynomial of the n^{th} degree.

Theorem III. 1. The second confluent ε-algorithm process terminates if and only if $\phi(a)\in N$. If the Riemann integral $\int_0^\infty \phi(a+t)dt$ exists then

$$\overset{*}{\varepsilon}_{2n}(a) = \int_0^\infty \phi(a+t)dt.$$

If $\phi(a+t)$ is not Riemann integrable then the Riemann integral $\int_0^\infty e^{-z't}\phi(a+t)dt$ certainly exists for sufficiently large $Re(z')$ and the integral $\int_0^\infty \phi(a+t)dt$

is derived by analytic continuation of this function of z'; in this case

$$\overset{*}{\mathcal{E}}_{2n}(a) = \lim_{z' \to 0} \int_0^\infty e^{-z't}\,\phi(a+t)\,dt.$$

Corollary. All constants, polynomials, exponential functions, sines, cosines, hyperbolic sines, hyperbolic cosines, and any finite linear combination of products of these functions are c.f.i.

2. Two Examples.

At this point it is instructive to consider how the method of integration works in two simple cases.

When $\phi(a+t) = (e^t)^{-1}$, the continued fraction (10) terminates with its first convergent. In this case a=0, and we have

$$\int_0^\infty (e^t)^{-1}\,dt = \varepsilon_2^*(a)$$

$$= \lim_{t\to 0} \frac{(-e^{-t})}{-(e^{-t})} = 1.$$

When $\phi(a+t) = e^t$ then again the continued fraction (10) terminates with its first convergent, a=0, and we have

$$\int_0^\infty e^t dt = \varepsilon_2^*(a)$$

$$= \lim_{t\to 0} \frac{(e^t)}{-(e^t)} = -1$$

thus the value -1 is assigned to the divergent integral $\int_0^\infty e^t dt$ for which the Riemann definition breaks down.

The Borel definition yields the same values, of course. In the first case

$$\int_0^\infty e^{-t} dt = \lim_{z'\to 0} \int_0^\infty e^{-z't}\, e^{-t} dt$$

$$= \lim_{z'\to 0} \frac{1}{z'+1} = 1;$$

in the second

$$\int_0^\infty e^t dt = \lim_{z'\to 0} \int_0^\infty e^{-z't} e^t dt$$

$$= \lim_{z'\to 0} \frac{1}{z'-1} = -1.$$

3. General Convergence Criteria

Now let us proceed to the case in which $\phi(a) \notin N$. Since we have established that the functions $\varepsilon^{*}_{2r}(a)$ are successive convergents of the continued fraction (13) then we can base the convergence theory of the second confluent ε-algorithm process upon the convergence theory of infinite continued fractions. For example we may use the following result [6] which includes a number of celebrated theorems as special cases:

Theorem III. 2. Write

$$c_1 = -\frac{\phi(a)^2}{\phi'(a)},$$

$$c_r = -\frac{E_{r-1}(a)}{Q_{r-1}(a)Q_r(a)} \,. \qquad (r=2,3,\dots)$$

Then if sequenses g_p, h_p $(p=1,2,\dots)$ can be found such that

$$c_p - \mathrm{Re}\,(c_p e^{i(h_p+h_{p+1}+1)}) \leqslant \frac{2\cos(h_p)\cos(h_{p+1})\,(1-g_{p-1})g_p}{(1+\delta\sec(h_p))(1+\delta\sec(h_{p+1}))} \qquad (p=1,2,\dots)$$

$$\delta > 0,\ 0 \leqslant g_{p-1} \leqslant 1,\ -\frac{\pi}{2} < h_p < \frac{\pi}{2},$$

and the series

$$\sum_{p=1}^{\infty} \left\{ \sqrt{|c_p|\,(1+\delta\sec(h_p))(1+\delta\sec(h_{p+1}))} \right\}^{-1}$$

diverges, then $\phi(a)$ is c.f.i.

The disadvantage of this theorem and of a number similar to it is that in general it is difficult to deduce the behaviour of $\phi(a)$ from the conditions which are imposed upon $\alpha_r(a)$, $\beta_r(a)$ $(r=0,1,\dots)$.

4 . Application of the Problem of Moments

The most natural way in which to relate the convergence of continued fractions of the form (13) to the behaviour of the function $\phi(a+t)$, is by exploiting the function theoretic aspects of the convergence theory of continued fractions.

In order to do this we shall be concerned with the moment problems

$$\int_{-\infty}^{+\infty} t^s d\psi(t) = c_s, \qquad (s=0,1,\dots) \qquad (14)$$

and

$$\int_{\sigma}^{\infty} t^s d\xi(t) = c_s, \qquad (s=0,1,\dots) \qquad (15)$$

that is, of determining certain functions $\psi(t)$, $\xi(t)$ satisfying (14) and (15), when $c_s (s=0,1,\dots)$ are prescribed constants.

Now a considerable theory has been built up concerning the case in which the constants $c_s (s=0,1,\dots)$ are real.

We can make use of this theory immediately by putting

$$c_s = \phi^{(s)}(a) \qquad (s=0,1,\dots)$$

and hence we shall now impose the condition that $\phi(a+t)$ should be real for $0 \leqslant t \leqslant \infty$.+

5. Application of the Hamburger Moment Problem

Definition 2. We say that $\phi(a) \in H$ if $\alpha_r(a)$, $\beta_r(a)$ $(r=0,1,\dots)$ (and therefore $\phi(a)$) are real and $\beta_r(a) > 0$ $(r=0,1,\dots)$.

Lemma III. 3. $\phi(a) \in H$ if the Hamburger moment problem is soluble, i.e. if a positive non-decreasing function $\psi(t)$ with an infinite number of points of increase can be found such that the equation

$$\int_{-\infty}^{+\infty} t^s d\psi(t) = \phi^{(s)}(a) \qquad (s=0,1,\dots)$$

holds [7].

A criterion which may frequently be used to determine whether $\phi(a) \in H$ is provided by [8]:

+ Actually it would be permissible to assume that $\phi(a+t) = A\theta(a+t)$ where A is some complex constant and $\theta(a+t)$ is real for real values of a+t, but this is hardly a generalisation worth dealing with.

Lemma III. 4. If there exists a function $f(z')$ such that $\text{Im}\left\{f(z')\right\} \leq 0$ for $\text{Im}(z') \geq \delta > 0$, and having the asymptotic representation

$$f(z') \sim \sum_{s=0}^{\infty} \phi^{(s)}(a)\, z'^{-s-1}$$

in $\epsilon \leq \arg(z') \leq \pi - \epsilon$, $0 < \epsilon < \frac{\pi}{2}$, then $\phi(a) \in H$.

The connection between the Hamburger moment problem and the continued fraction (13) is provided by

Lemma III. 5 The continued fraction (13) converges if the moment problem is determinate, i.e. if only one function $\psi(t)$ satisfying the conditions of Lemma III. 3 can be found [7].

We are thus interested in obtaining conditions under which the Hamburger moment problem is determinate. One such is provided by a result of Carleman ([9] p. 78)

Lemma III.6. If $\phi(a) \in H$ and the series $\sum_{s=0}^{\infty} \left\{\phi^{(s)}(a)\right\}^{-\frac{1}{2s}}$ is divergent then the Hamburger moment problem is determinate.

Combining Lemmas III.4 and III.6 we have

Theorem III. 3. If the Laplace transform $F(z') = \int_0^\infty e^{-z't}\phi(a+t)\,dt$ satisfies the inequality $\text{Im}\left\{F(z')\right\} \leq 0$ for $\text{Im}(z') > 0$ and has the asymptotic representation

$$F(z') \sim \sum_{s=0}^{\infty} \phi^{(s)}(a) z'^{-s-1}$$

for $\epsilon \leq \arg(z') \leq \pi - \epsilon$ where $0 < \epsilon < \frac{\pi}{2}$, and the series $\sum_{s=0}^{\infty} \left\{\phi^{(s)}(a)\right\}^{-\frac{1}{2s}}$ diverges, then $\phi(a)$ is c.f.i.

A further condition for the Hamburger moment problem to be determinate was given by F. Riesz [10]; it is exploited in the following

Theorem III. 4 If $F(z')$ satisfies the conditions imposed upon it in Theorem III.3 and

$$\liminf \left\{\sqrt{\frac{\phi^{(2s)}(a)}{(2s)!}}\right\} < \infty \tag{16}$$

then $\phi(a)$ is c.f.i.

The inequality (16) may also be deduced from the behaviour of $\phi(a)$ in the large. We are concerned with integral functions, i.e. functions

$f(x)$ such that the power series $f(x) = \sum_{s=0}^{\infty} a_s x^s$ converges for all finite values of x. Denote [11] by $M(r)$ the maximum value of $M(r)$ on the circle $|x|=r$. The relation $M(r) \ll g(r)$ means that there exists some finite R such that for all $r > R$ we have $M(r) < g(r)$. Then the lower limit of k such that $M(r) \ll e^{r^k}$ is the order of $f(x)$. Suppose in addition that $M(r) \ll e^{ar^k}$ $(a > 0)$ and that σ is the lower limit of a for which this relationship is valid. Then $f(x)$ is of minimum type if $\sigma = 0$; $f(x)$ is of normal type σ if σ is finite and non-zero. We have the following

Lemma III. 7. The necessary and sufficient condition that $f(x)$ should belong at most to the minimum type of the order ρ is that

$$\lim_{n \to \infty} n^{1/\rho} \left| \sqrt[n]{a_n} \right| = 0, \tag{17}$$

and that that $f(x)$ should belong to the normal type σ of order ρ is that

$$\overline{\lim}\, n^{1/\rho} \left| \sqrt[n]{a_n} \right| = (\sigma e \rho)^{1/\rho}. \tag{18}$$

Comparing formulae (16), (17) and (18), and bearing in mind the expansion

$$\phi(a+t) \sim \sum_{s=0}^{\infty} \frac{\phi^{(s)}(a)}{s!} t^s$$

we have

Theorem III. 5. If $F(z')$ satisfies the conditions imposed upon it in Theorem III, 3 and $\phi(a)$ is an integral function of any order and of minimum or normal type, then $\phi(a)$ is c.f.i.

Note: Theorem III.5 is in a certain sense a complement to Theorem III.1. Theorem III.1 states that if $\phi(t)$ is an integral function of a very specific sort then the second confluent ε-algorithm process terminates Theorem III. 5 states that if $\phi(t)$ is an integral function of a less restricted type then the second confluent ε-algorithm process converges.

If $\phi(a) \in H$ we may in certain cases show that the Borel integral may be evaluated by means of the analytic continuation of a function for which a power series may be given [12]. We have

Theorem III. 6. If $\phi(a+t)e^{-z't}$ is c.f.i. to a finite function for all $|z'| > R$ then the power series $\sum_{s=0}^{\infty} \phi^{(s)}(a) z^{1-s-1}$ converges for all

$|z'| \geqslant R$ to the value of the integral $\int_0^\infty e^{-z't}\phi(a+t)dt$.

To conclude this section we have from Theorem II.3 and certain consistency theorems concerning the convergence behaviour of J-fractions [7] (i.e. continued fractions of the form (1) in which the β_s, (s=0,1,...) are positive).

Theorem III. 7. If $\phi(a) \in H$ and $\phi(a+t)$ is c.f.i. then $\phi(a+t)e^{-z't}$ is c.f.i. for $\mathrm{Im}(z') \geqslant \delta > 0$.

6. The Application of the Stieltjes Moment Problem

It is formally possible to construct Stieltjes type continued fractions

$$\frac{1}{k_1z+}\;\frac{1}{k_2+}\;\frac{1}{k_3z+}\cdots\frac{1}{k_{2r}+}\;\frac{1}{k_{2r+1}z+}$$

in which the coefficients k_s(s=0,1,...) are uniquely determined by the conditions that

$$\frac{1}{k_1z+}\;\frac{1}{k_2+}\;\frac{1}{k_3z+}\cdots\frac{1}{k_{2r}} \sim \sum_{s=0}^{\infty} c_s z^{-s-1} + 0\,(z^{-2r})$$

and

$$\frac{1}{k_1z+}\;\frac{1}{k_2+}\;\frac{1}{k_3z+}\cdots\frac{1}{k_{2r}+}\;\frac{1}{k_{2r+1}z} \sim \sum_{s=0}^{\infty} c_s z^{-s-1} + 0(z^{-2r-1})$$

where $\sum_{s=0}^{\infty} c_s z^{-s-1}$ is a prescribed formal power series.

Clearly the even part of (19) (i.e. that continued fraction whose successive convergents are the successive even order convergents of (19)) is the continued fraction (1). Now as z tend to zero the continued fraction (19) is clearly divergent, but as is easily shown by induction its even order convergents are the successive partial sums of the series $\sum_{s=1} k_{2s}$.

Thus if we can show that the terms of the series $\sum_{s=1} k_s$ are positive, and that the series itself is convergent, then we may conclude that the sequence $\mathcal{E}^*_{2r}(a)$ is a monotonic convergent sequence.

The first condition is provided by the following

Lemma III.8. The coefficients k_s of the continued fraction (19) are real and positive of the Stieltjes moment problem

$$\int_0^\infty t^s d\alpha(t) = \phi^{(s)}(a) \qquad (s=0,1,\ldots) \qquad (20)$$

is soluble, i.e. if a real positive non-decreasing function $\xi(t)$ can be found such that equation (20) holds [13].

If the Stieltjes moment problem is determinate, i.e. if the function $\xi(t)$ determined by equation is unique, then it is known that the series $\sum_{s=1}^{\infty} k_s$ diverges; this of course tells us nothing about the series $\sum_{s=1}^{\infty} k_{2s}$. But if the Stieltjes moment problem is indeterminate then the series $\sum_{s=0}^{\infty} k_s$ converges, and as we have said the sequence $\varepsilon_{2r}^{*}(a)$ $(r=0,1,\ldots)$ certainly converges. Now examples have been constructed in which the Stieltjes moment problem is indeterminate.

We have

Lemma III. 9. The moment problem

$$\int_0^{\infty} t^s d\xi(t) = \int_0^{\infty} t^s e^{-ut^{\alpha}} dt, \qquad \alpha < \tfrac{1}{2}$$

is indeterminate ([14] p. 22)

Lemma III. 10. The moment problem

$$\int_0^{\infty} t^s d\,\xi(t) = \int_0^{\infty} t^s t^{-\ln(t)} \left\{1+A\sin(2\pi\ln(t))\right\} dt, \quad |A| \leqslant 1$$

is indeterminate [13].

Combining Lemmas III.8 and III. 9, and III. 8 and III. 10 we have

Theorem III. 8. The functions

$$\int_0^{\infty} e^{(a+t)x}\, e^{-ux^{\alpha}} dx, \qquad \alpha < \tfrac{1}{2} \tag{21}$$

$$\int_0^{\infty} e^{-(a+t)x} e^{-ux^{\alpha}} dx, \qquad \alpha < \tfrac{1}{2} \tag{22}$$

are c.f.i. for a=0.

Theorem III. 9. The functions

$$\int_0^{\infty} e^{(a+t)x}\, x^{-\ln(x)} \left\{1+A\,\sin(2\pi\ln(x))\right\} dx, \; |A| \leqslant 1 \tag{23}$$

$$\int_0^{\infty} e^{-(a+t)x}\, x^{-\ln(x)} \left\{1+A\sin\,(2\pi\ln(x))\right\} dx, \; |A| \leqslant 1 \tag{24}$$

are c.f.i. for a=0 and real A.

Note: The Riemann integrals (21) and (23) do not, of course. exists (the values assigned to them are the negative values of (22) and (24) respectively).

To conclude this section we adapt the following result of Carleman ([9] p. 86).

Lemma III. 11. We are given a function $f(x)$ (not identically zero) which is indefinitely differentiable and such that

$$f^{(s)}(0)= f^{(s)}(1) = 0 \quad (s= 0,1,\dots). \tag{25}$$

Put

$$m_s^2 = \int_0^1 \left\{ f^{(s)}(x) \right\}^2 dx.$$

The series $\sum_{s=0}^{\infty} m_s^2 (-1)^s z^{-s-1}$ may formally be developed in a Stieltjes continued fraction $\frac{1}{k_1 z+} \frac{1}{k_2 +} \frac{1}{k_3 z+} \dots$ where $k_1, k_2, \dots$ are positive numbers whose sum is finite.

Thus we have

Theorem III. 10. Suppose that some indefinitely differentiable function $f(x)$ satisfying equations (25) can be found such that

$$\phi^{(s)}(a) = \int_0^1 \left\{ f^{(s)}(x) \right\}^2 dx \qquad (s=0,1,\dots)$$

then $\phi(a+t)$ is c.f.i.

IV Rational Fraction Integration

We now consider a second process of integration based on the non-linear difference-differentialrecursions (7) (9). By way of introducing it we remark that if the ε-algorithm [15] relationships

$$\varepsilon_{s+1}^{(m)} = \varepsilon_{s-1}^{(m+1)} + \frac{1}{\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)}} \qquad (m,s=0,1,\dots) \tag{26}$$

are applied to the initial conditions

$$\varepsilon_{-1}^{(m)} = 0, \quad \varepsilon_{0}^{(m)} = \sum_{s=0}^{m-1} c_s x^s \tag{27}$$

then the sequence

$$\varepsilon_{n}^{(m)} \qquad (m=0,1,\dots)$$

is the sequence of Padé [16] quotients

$$\frac{\sum_{s=0}^{m+n} u_{n,s}^{(m)} x^s}{\sum_{s=0}^{n} v_{n,s}^{(m)} x^s} \qquad (m=0,1,\dots) \tag{28}$$

in the n^{th} row of the Padé table of the power series whose partial sums are given by (27).

The connection between the Padé quotient (28) and the determinantal quotient (3) is this: that if we put $m=-1$, $x=z^{-1}$ in (28) we obtain (3). However in this chapter we deal with expansions in ascending powers of x rather than in descending powers of z, since this is the usual form in which the theory of the Padé table is presented.

We now introduce the following fundamental result in the theory of the ε-algorithm:

Lemma IV.1. If the ε-algorithm relationships (26)are applied to the sequence

$$\varepsilon_{-1}^{(m)} = 0, \qquad \varepsilon_{s}^{(m)} = S_m$$

to produce quantities $\varepsilon_{s}^{(m)}$, and to the quantities

$$\varepsilon_{-1}^{(m)\prime} = 0, \qquad \varepsilon_{0}^{(m)\prime} = A+BS_m$$

to produce quantities $\varepsilon_{s}^{(m)\prime}$ then

$$\varepsilon_{2r}^{(m)\prime} = A+B\,\varepsilon_{2r}^{(m)}. \qquad (m,r=0,1,\dots)$$

Putting

$$A = \sum_{s=0}^{m'-1} c_s x^s, \qquad B = x^{m'}$$

we have

Lemma IV.2 If the ε-algorithm relationships are applied to the sequence

$$\varepsilon_{-1}^{(m)}=0\ , \qquad \varepsilon_0^{(m)} = \sum_{s=0}^{m-1} c_s x^s$$

to produce the quantities $\varepsilon_s^{(m)}$ and to the quantities

$$\varepsilon_{-1}^{(m)'} = 0\ , \qquad \varepsilon_0^{(m)'} = \sum_{s=0}^{m'-1} c_s x^s + x^m \sum_{s=m'}^{m'+m-1} c_s x_s{}^s$$

then

$$\varepsilon_{2r}^{(m+m')} = \sum_{s=0}^{m'-1} c_s x^s + x^{m'}\ \varepsilon_{2r}^{(m)'}. \qquad (m=0,1,\ldots) \qquad (29)$$

Thus to form the quantity $\varepsilon_{2r}^{(m)}$ $(m > 0)$ we first add together the terms in the partial sum $\sum_{s=0}^{m-1} c_s x^s$, apply the ε-algorithm relationships to the partial sums of the series $\sum_{s=0}^{\infty} c_{m'+s} x^s$ and form the linear combination (29).

The equivalent confluent ε-algorithm process is as follows: we evaluate the definite integral $\int_0^{t'} \phi(a+t)dt$, evaluate $\overset{*}{\varepsilon}_{2r}(a+t')$ by use of the relationships

$$\overset{*}{\varepsilon}_{-1}(a+t') = \overset{*}{\varepsilon}_0(a+t') = 0$$

$$\left\{\varepsilon_{2r}(a+t') - \varepsilon_{2r}(a+t')\right\} \frac{d}{dt}\ \varepsilon_{2r}(a+t') = 1$$

$$\left\{_{2r+1}(a+t') - \varepsilon_{2r-1}(a+t')\right\} \left\{\frac{d}{dt}\ \varepsilon_{2r}(a+t') + \phi\ (a+t')\right\} = 1$$

and form the sum

$$\int_0^{t'} \phi\ (a+t')dt + \overset{*}{\varepsilon}_{2r}(a+t'). \qquad (30)$$

We shall give a sufficient condition for the convergence of the function (30) as t' tends to infinity, for some finite r.

With regard to the convergence behaviour of the Padé quotients (28) we have the following result of Montessus de Balloire [17]

Lemma IV. 3 Let the power series $\sum_{s=0}^{\infty} c_s x^s$ represent a function f(x) which is regular for $|x| \leqslant R$ except for m poles within this cirde. Then the sequence (28) converges uniformly to f(x) in any domain bounded by $|x| \leqslant R$ hich does not include a pole of f(x).

Introducing the substitution $x=e^{-z'}$ we have

Theorem IV. 1 If the function

$$F(z') = \int_0^{\infty} e^{-z't} \phi(a+t)dt$$

is Riemann integrable in some strip $\alpha < \mathrm{Im}(z') \leqslant \alpha + 2\pi$; $\ln(R) < \mathrm{Re}(z') \leqslant \infty$, $(1 \leqslant R \leqslant \infty)$ and F(z') is meromorphic in the rectangle $\alpha < \mathrm{Im}(z') \leqslant \alpha + 2\pi$; $0 \leqslant \mathrm{Re}(z') < \ln(R)$ then the expression (30) tends to a finite limit or properly to infinity as t' tends to infinity; this limit has the same value as that of the Borel integral $\lim_{z' \to 0} \int_0^{\infty} e^{-z't} \phi(a+t)dt$.

Definition 3. If the function $\phi(a+t)$ satisfies the conditions of Theorem IV. 1 then we say that $\phi(a+t)$ is rational fraction integrable (r.f.i.).

Theorem IV. 2. If $\phi(a) \in N$ then $\phi(a+t)$ is r.f.i.

Theorem IV. 3. All functions $\phi(a+t)$ which are the subject of the Corollary to Theorem III.1 are r.f.i.

An Example

Consider the integral

$$\int_0^{\infty} \sin(t)dt.$$

Here we have a=0 and

$$\int_0^{\infty} e^{-z't} \sin(t)dt = \frac{1}{z'^2+1}. \qquad (31)$$

In the notation of Theorem IV we may put $\alpha = -\pi$, R=1.

In the notation of equations (6) and (30) we find that

$$\varepsilon_2^*(t') = \cos(t')$$

and since

$$\int_0^{t'} \sin(t)dt = 1 - \cos t'$$

we have

$$\int_0^{t'} \sin(t)dt + \varepsilon_2^*(t') = 1 \qquad (32)$$

and of course the limit of expression (32) as t' tends to infinity is also 1. Furthermore we may evaluate the Borel integral by means of (31) and also have

$$\int_0^{\infty} \sin(t)dt = \lim_{z'\to 0} \int_0^{\infty} e^{-z't} \sin(t)dt$$

$$= \lim_{z'\to 0} \frac{1}{z'^2+1}$$

$$= 1.$$

V The First Confluent Form of the ε-Algorithm

Finally we relate the convergence theory which has been established for the process (7),(8),(9) to that of the first confluent form of the ε-algorithm, [18], [19] by means of which functions $\varepsilon_s(t)$ are constructed using the single recursion

$$\left\{\varepsilon_{s+1}(a) - \varepsilon_{s-1}(a)\right\} \frac{d}{da} \varepsilon_s(a) = 1 \qquad (s=0,1,\ldots)$$

from the initial conditions

$$\varepsilon_{-1}(a)=0, \qquad \varepsilon_0(a) = \phi(a).$$

We now introduce the notation (in contradistinction to (5))

$$\phi^{-1}(a) = \int_0^t \phi(a+t)dt. \qquad (33)$$

The following result has been derived [20]

Lemma V.1 If the first confluent form of the ε-algorithm is applied to the function $f^{(m-1)}(t)$ to produce functions $\varepsilon_s^{(m-1)}(t)$, and the second confluent form the ε-algorithm is applied to the function $f^{(m)}(t)$ to produce functions $\varepsilon_s^{(m)*}(t)$ then the equations

$$\varepsilon_{2s+1}^{(m)*}(t) = \varepsilon_{2s+1}^{(m-1)}(t) \qquad (m,s=0,1,\ldots)$$

$$\varepsilon_{2s}^{(m)*}(t) = \varepsilon_{2s}^{(m-1)} - f^{(m-1)}(t) \qquad (m,s=0,1,\ldots)$$

obtain.

The connection between the convergence of the first and second confluent forms of the ε-algorithm is made apparent by Lemma V.1, for with (33)

$$\int_t^{\infty} f^{(m)}(t)dt = \lim_{t\to\infty} f^{(m-1)}(t) - f^{(m-1)}(t) \quad (m=0,1,\ldots)$$

Thus conditions under which $\varepsilon_{2r}(a)$ either tends to a limit as r tends to infinity, or becomes independent of a for some finite r, may be derived from the theory of Chapter III by replacing $\phi(a)$ by $\int_X^a \phi(t)dt$ where X is some finite constant.

In conclusion we remark that the convergence of the first confluent ε-algorithm process has a very intimate connection with the convergence of J-fractions. The value of a convergent J-fraction may be regarded as the result of an infinite sequence of bilinear transformations. Subsequent to each transformation it may be shown that the value of the J-fraction lies within a certain circular region of radius $r_n(z)$. If $r_n(z)$ tends to zero (the determinate case) the real J-fraction converges. It may be shown that in the case of a real J-fraction $r_n(z)$ and $r_n(0)$ tend to a non-zero limit or to zero together. Now it may be shown ([14] p. 72) that

$$r_n(0) = c\,\frac{H_{n+1}^{(0)}}{H_n^{(2)}} \tag{34}$$

where c does not depend upon n. Furthermore

$$\varepsilon_{2n}(a) = \frac{H_{n+1}^{(0)}}{H_n^{(2)}}\,. \tag{35}$$

Comparing expressions (34) and (35) we have

Theorem V.1. If $\phi(a) \in H$, and the first confluent ε-algorithm process applied to $\phi(a)$ produces a sequence $\varepsilon_{2r}(a)$ $(r=0,1,\ldots)$ which converges to zero, then the continued fraction (10) converges for all z' in the sector $0 < \arg(z) < \pi$ and in particular $\phi(a+t)$ is c.f.i..

VI. Conclusion

So far in this paper we have not considered the regularity of the methods discussed nor the extent to which the methods of integration proposed may be considered the inverse of differentiation. More precisely we have not answered the questions (posed in terms of continued fraction integrability):

1) If $\phi(a)$ is c.f.i., then is $\phi(a')$ c.f.i., and if so does

$$\lim_{r\to\infty} \varepsilon^{*}_{2r}(a) = \lim_{r\to\infty} \varepsilon^{*}_{2r}(a') - \int_a^{a'} \phi(t)dt \qquad (36)$$

hold?

2) If $\phi(a)$ is c.f.i., then is

$$\frac{d}{da} \lim_{r\to\infty} \varepsilon^{*}_{2r}(a) = -\phi(a) \text{ ?}$$

3) If the second confluent ε-algorithm process is applied to the function $\frac{d}{da}\phi(a+t)$ to produce the sequence $\varepsilon^{*\prime}_{2r}(a)$ $(r=0,1,\ldots)$, then is

$$\lim_{r\to\infty} \varepsilon^{*\prime}_{2r}(a) = \phi(a) + \text{constant}.$$

When $\phi(a) \in N$, these questions assume a trivial nature and can be answered in the affirmative.

More generally, since whenever conditions sufficient for $\phi(a)$ to be c.f.i. have been given it has also been shown that $\lim_{r\to\infty} \varepsilon^{*}_{2r}(a)$ is equal either to the Riemann integral or the Borel equivalent, it follows that under these conditions the answer to the last two questions is in the affirmative. Under these conditions the answer to the first part of question 1) has been left open, but if $\phi(a)$ and $\phi(a')$ are both c.f.i. then equation (36), of course, holds.

It is to be expected that a further study of the theory of this paper will produce results which are of significance in the theory of continued fractions. To a certain extent this has already been done; as an example of this we mention the following. It may be shown [19] that

$$\varepsilon^{*}_{2r+1}(a) = \frac{H^{(2)}_{r}}{H^{(0)}_{r+1}} \; . \qquad (37)$$

Now in the determinate case of the Hamburger moment problem the radü of certain circular regions were given by expression (34), which is very similar to (37). If we bear in mind equations (34),(37) and (9) then we see that

$$r_{n+1}(0)^{-1} - r_n(0)^{-1} \rightarrow \infty$$

and this tells us something about the rate of decrease of $r_n(0)$: it tells us for example that $r_n(0) \neq \text{constant}/n$.

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