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On Some Recent Developments in the Theory and  
Application of Continued Fractions

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## I. Introduction

In this talk I propose firstly to deal with those aspects of the study of continued fractions which are of direct application in the theory of approximation, and secondly to sketch some developments which are of recent origin and as yet incompletely worked out. It has taken more than two hundred years for the theory of continued fractions to attain its present condition, and I do not propose to speak for quite that length of time. For this reason I shall confine myself to sketching the strategic principles involved in the development of the subject, without going into too much tactical detail.

## II. Sequences of Bilinear Transformations

For the purposes of this talk we shall regard a continued fraction as resulting from a sequence of bilinear transformations of the form

$$f_0 = b_0 + f_1, \quad f_r = \frac{a_r}{b_r + f_{r+1}} \quad (r = 1, 2, \dots) \quad (1)$$

As is well known,  $f_0$  can be expressed as a bilinear function of  $f_{n+1}$ , thus

$$f_0 = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \dots \frac{a_n}{b_n + f_{n+1}} \quad (2)$$

$$= \frac{A_n + f_{n+1} A_{n-1}}{B_n + f_{n+1} B_{n-1}} \quad (n = 1, 2, \dots) \quad (3)$$

where the functions  $A_n, B_n$  satisfy the twin recursions

$$A_n = b_n A_{n-1} + a_n A_{n-2} \quad (n = 1, 2, \dots), \quad A_{-1} = 1, \quad A_0 = b_0; \quad (4)$$

$$B_n = b_n B_{n-1} + a_n B_{n-2} \quad (n = 1, 2, \dots), \quad B_{-1} = 0, \quad B_0 = 1. \quad (5)$$

Continued fractions can therefore be brought into play wherever sequences of bilinear transformations occur.

### The Riccati Equation

In order to indicate how bilinear transformations arise and may be exploited we first consider the general Riccati equation

$$f_0' + \alpha_0 f_0^2 + \beta_0 f_0 + \gamma_0 = 0 \quad (6)$$



where  $f_0$  is the dependent variable, and  $\alpha_0, \beta_0, \gamma_0$  are functions of  $x$ . This equation has the property that if the dependent variable  $f_0$  is replaced by means of the bilinear substitutions (1), then the functions  $f_r$  ( $r=1,2,\dots$ ) also satisfy Riccati equations

$$f_r' + \alpha_r f_r^2 + \beta_r f_r + \gamma_r = 0 \quad (7)$$

where very simple recursions exist between the coefficients

$$\alpha_{r-1}, \beta_{r-1}, \gamma_{r-1}; \alpha_r, \beta_r, \gamma_r; a_{r-1}, b_{r-1}. \quad (8)$$

This implies that the solution to a Riccati equation can be directly expanded in the form of a continued fraction. The coefficients in this continued fraction can be chosen in such a way that the resulting expansion has certain asymptotic properties with regard to the variable  $x$ , or they may be chosen in such a way as to simplify the recursions between the members of the set (8), and so on.

A little reflection will reveal that many elementary functions (e.g.  $\exp(x)$ ,  $\ln(1+x)$ ,  $\tan(x)$ ,  $\arctan(x)$ ,  $\arcsin(x)$ ,  $\arccos(x)$ , the error function, the exponential integral, and so on) satisfy differential equations of Riccati form. For example,  $\tan(x)$  satisfies the equation

$$y' = 1+y^2. \quad (9)$$

This means that such functions have continued fraction expansions whose coefficients have a particularly simple form. For example

$$\tan(x) = \frac{x}{1-} \frac{x^2}{3-} \frac{x^2}{5-} \dots \quad (10)$$

In his recent text book A.N. Khovanskii [1] uses this principle to give a unified treatment of the continued fraction expansions of elementary functions.

Such functions as  $\sin(x)$  and  $\cos(x)$  do not satisfy Riccati equations; their continued fraction expansions do not have simple coefficients.



The Riccati equation has other points of interest besides the fact that it is satisfied by many elementary functions. As is well known many differential equations can be derived from the consideration that their solutions maximise or minimise certain functionals. Bellman [2] and Calogero [3] recently proved that a solution of the Riccati equation maximises a certain functional which the latter shows (by use of the Schrödinger equation) to occur in problems in scattering theory. The mechanism of the continued fraction enables us to obtain direct expansions for the solutions of such problems.

#### Nevanlinna Continued Fractions

R. Nevanlinna [4] investigates the following problem: to determine whether the function  $f_0(z)$  obeys the inequality

$$|f_0(z)| < 1 \quad (11)$$

or the unit disc. He shows that this is true if and only if the sequence of constants  $f_s(0)$  ( $s=0,1,\dots$ ) where

$$f_{s+1}(z) = \frac{f_s(z) - f_s(0)}{1 - f_s(0)f_s(z)} \cdot \frac{1}{z}, \quad (s=0,1,\dots) \quad (12)$$

satisfy the inequalities

$$|f_s(0)| < 1. \quad (s=0,1,\dots) \quad (13)$$

Clearly relationship (12) can be solved to give an expression for  $f_s(z)$  in terms of  $f_{s+1}(z)$  similar to (1). This yields a continued fraction of the form

$$f_0(z) = \frac{1}{f_0(0)} + \frac{c_0}{1+d_0z} + \frac{c_1z}{1+d_1z} \dots + \frac{c_rz}{1+d_rz} \dots \quad (14)$$

If the coefficients of this continued fraction have a certain form then  $f_0(z)$  satisfies the inequality (11) upon the unit disc. Conversely, if  $f_0(z)$  is known to satisfy (11) then the coefficients in (14) have a certain form and the convergence behaviour of (14) can be investigated.



Functions whose modulus has a known upper bound on the unit disc occur in numerous physical problems. Nevanlinna continued fractions can be used to compute the solutions to such problems.

Richards - Goldberg Continued Fractions

Recently Richards [5] and Goldberg [6] have investigated the class of analytic functions which map the right half-plane into itself and take the real axis into the real axis. Such functions, referred to as positive real functions, find application in the synthesis of two-terminal passive networks [7], [8], [9]. A function  $f_0(z)$  is positive real if it is single valued and analytic in the open right hand plane, if  $\text{Re}\{f_0(z)\}$  is positive for  $z$  in the open right hand plane, and if  $f_0(z)$  is real for real  $z$ . It can be shown that if  $f_r(z)$  is positive real and neither  $a_r f_r(z) - z f_r(a_r)$  nor  $a_r f_r(a_r) - z f_r(z)$  vanish identically for arbitrary choice of the positive constant  $a_r$ , then the function  $f_{r+1}(z)$  given by

$$f_{r+1}(z) = \frac{a_r f_r(z) - z f_r(a_r)}{a_r f_r(a_r) - z f_r(z)} \quad (15)$$

is also positive real. Clearly relationship (15) can be solved to give an expression for  $f_r(z)$  in terms of  $f_{r+1}(z)$  similar to (1). This yields a continued fraction of the form

$$f_0(z) = \frac{d_0}{z} + \frac{e_1 - z^2}{d_1 z} - \frac{e_2 - z^2}{d_2 z} \dots \quad (16)$$

which can be used for the computation of a positive real function.

### III. Continued Fractions Derived from Power Series

We have just reviewed certain methods of deriving continued fractions based on function-theoretic considerations. We now proceed to what is perhaps the most extensively studied class of continued fractions, namely those continued fractions which are derived by the transformation of power series.

We consider the formal power series

$$f_0(z) \sim \sum_{s=0}^{\infty} c_s z^{-s-1} \quad (17)$$



Clearly the formal power series for  $f_1(z)$  derived from

$$f_0(z) = \frac{c_0}{z - c_1 c_0^{-1} - f_1(z)} \quad (18)$$

is similar to that for  $f_0(z)$  (i.e. it begins with a term in  $z^{-1}$ ), thus substitutions of the form (18) can be repeated and we derive the continued fraction expression

$$\begin{aligned} f_0(z) &= \frac{c_0}{z - \alpha_0} - \frac{\beta_0}{z - \alpha_1} - \dots - \frac{\beta_{r-2}}{z - \alpha_{r-1} - f_r(z)} \\ &\sim \frac{c_0}{z - \alpha_0} - \frac{\beta_0}{z - \alpha_1} - \dots - \frac{\beta_{r-2}}{z - \alpha_{r-1}} - \dots \end{aligned} \quad (19)$$

The successive denominators of this continued fraction are polynomials (see e.g. [10] vol II p. 162) of the form

$$p_r(z) = \sum_{s=0}^r k_{r,s} z^s; \quad (20)$$

They satisfy a recursion of the form

$$p_r(z) = (z - \alpha_{r-1}) p_{r-1}(z) - \beta_{r-2} p_{r-2}(z); \quad (21)$$

They are orthogonal with respect to the system of weights  $c_s$  ( $s=0,1,\dots$ ), i.e.

$$\sum_{s=0}^r c_{t+s} k_{r,s} = \begin{cases} 0 & (t=0,1,\dots,r-1) \\ \neq 0 & (t=r) \end{cases} \quad (22)$$

The successive numerators of (19) are the so-called associated orthogonal polynomials of the form

$$o_r(z) = \sum_{s=0}^{r-1} k'_{r,s} z^s \quad (23)$$

It is a consequence of the way in which the continued fraction (19) is derived that that the series expansion of the quotient

$\frac{o_r(z)}{p_r(z)}$  in inverse powers of  $z$  should agree with the power series

$\sum_{s=0}^{\infty} c_s z^{-s-1}$  as far as the term in  $c_{2r-1}$ , i.e. that

$$\frac{o_r(z)}{p_r(z)} \sim \sum_{s=0}^{\infty} c_s z^{-s-1} + \sum_{s=0}^{\infty} d_{2r+s} z^{-2r-s-1} \quad (24)$$

where in general  $d_{2r+s} \neq 0$  ( $s=0,1,\dots$ ). For this reason the continued fraction (19) is said to be associated with the series  $\sum_{s=0}^{\infty} c_s z^{-s-1}$ .

### Convergence Theory

The investigation of the convergence behaviour of continued fractions of interest in function theory usually proceeds in the following way: we first establish that the continued fraction being investigated is a member of certain class, then, by establishing certain criteria which relate either to the coefficients of the continued fraction or those of the power series from which it was derived, we show that it converges.

### Grommer Fractions

The convergence theory of associated continued fractions is particularly easy to discuss if it is known a priori that in (19)  $\beta_s > 0$  ( $s=0,1,\dots$ ). Associated continued fractions of this type are known as Grommer fractions.

If we denote  
the Hankel determinant

$$\begin{vmatrix} c_0 & c_1 & \dots & c_{k-1} \\ c_1 & c_2 & \dots & c_k \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c_{k-1} & c_k & \dots & c_{2k-2} \end{vmatrix} \quad (k > 0) \quad (25)$$

by  $H_k$ , with

$$H_0 = 1$$



then the coefficients  $\beta_s$  in (19) are given ([11] p 325) by

$$\beta_s = \frac{H_{s+2} H_s}{H_{s+1}^2} \quad (s=0,1,\dots) \quad (27)$$

From the determinantal expression (27) it follows that if

$$H_s > 0 \quad (s=0,1,\dots) \quad (28)$$

then  $\beta_s > 0$  ( $s=0,1,\dots$ ).

From the algebraic theory of quadratic forms it is known that (28) is true if the quadratic form

$$\sum_{i,j=0}^{\infty} c_{i+j} x_i x_j \quad (29)$$

is positive definite. In general, however, if the coefficients  $c_s$  ( $s=0,1,\dots$ ) are given, neither of the criteria derived from (28), (29) are easy to use.

For this reason we proceed to consider the behaviour in the large of the function  $f_0(z)$  which, so far, has been no more than formally associated with the power series  $\sum_{s=0}^{\infty} c_s z^{-s-1}$ . We consider a certain class  $H$  of functions, and say that  $f_0(z) \in H$  if it is asymptotically represented by the series  $\sum_{s=0}^{\infty} c_s z^{-s-1}$  in the sector  $\epsilon \leq \arg(z) \leq \pi - \epsilon$ ,  $0 < \epsilon < \frac{\pi}{2}$  where the coefficients  $c_s$  ( $s=0,1,\dots$ ) are real,  $f_0(z)$  is analytic for  $\text{Im}(z) > 0$  and furthermore for  $\text{Im}(z) > 0$ ,

$$\text{Im}\{f_0(z)\} < 0 \quad (30)$$

It is clear that if (30) is to obtain then  $c_0$  is positive. Furthermore it may easily be shown ([27] p. 30) that if  $f_0(z) \in H$ , then the function  $f_1(z)$  given by (18) is also a member of  $H$ . Thus the coefficients  $\beta_s$  ( $s=0,1,\dots$ ) in (19) are all positive and (19) is a Grommer fraction.

If we transcribe relationship (3) into the notation of the continued fraction (19) we find that



$$f_0(z) = \frac{o_n(z) - f_n(z) o_{n-1}(z)}{p_n(z) - f_n(z) p_{n-1}(z)} \quad (31)$$

This is bilinear transformation of the  $f_n$ -plane into the  $f_0$ -plane. So far we have only used the property of bilinear transformations that the result of a sequence of bilinear transformations is a bilinear transformation. Now we use their other well-known property - that they transform straight lines into circles. In particular the real axis in the  $f_n$ -plane is transformed into the circle of radius  $(r_n(z))$

$$\frac{\beta_0 \beta_1 \dots \beta_{n-2}}{-2\text{Im}\{p_{n-1}(z)\overline{p_n(z)}\}} \quad (32)$$

whose centre lies at the point  $(ce_n)$

$$\frac{\text{Im}\{\overline{o_n(z)}p_{n-1}(z) + o_{n-1}(z)\overline{p_n(z)}\}}{2\text{Im}\{p_{n-1}(z)\overline{p_n(z)}\}}, \quad \frac{\text{Re}\{\overline{o_n(z)}p_{n-1}(z) \cdot o_{n-1}(z)\overline{p_n(z)}\}}{2\text{Im}\{p_{n-1}(z)\overline{p_n(z)}\}} \quad (33)$$

in the  $f_0$ -plane. The lower half of the  $f_n$ -plane is transformed into the interior of this circle. Since  $f_n(z) \in H$ , then the value of  $f_n(z)$  lies in the lower half plane if  $\text{Im}(z) > 0$ , thus for these values of  $z$  the value of  $f_0(z)$  lies in the interior of the above circle. Thus the successive convergents of a Grommer fraction provide a sequence of circles within which the value of the continued fraction must lie.

Let us illustrate this phenomenon by means of the continued fraction

$$f_0(z) \sim \frac{1}{z-} \frac{1}{z-} \frac{1}{z-} \frac{2}{z-} \frac{2}{z-} \dots \frac{r}{z-} \frac{r}{z-} \dots \quad (34)$$

which is of the form (19) with

$$\alpha_s = 0 \quad (s=0,1,\dots), \quad \beta_s = [(s+2)/2] \quad (35)$$

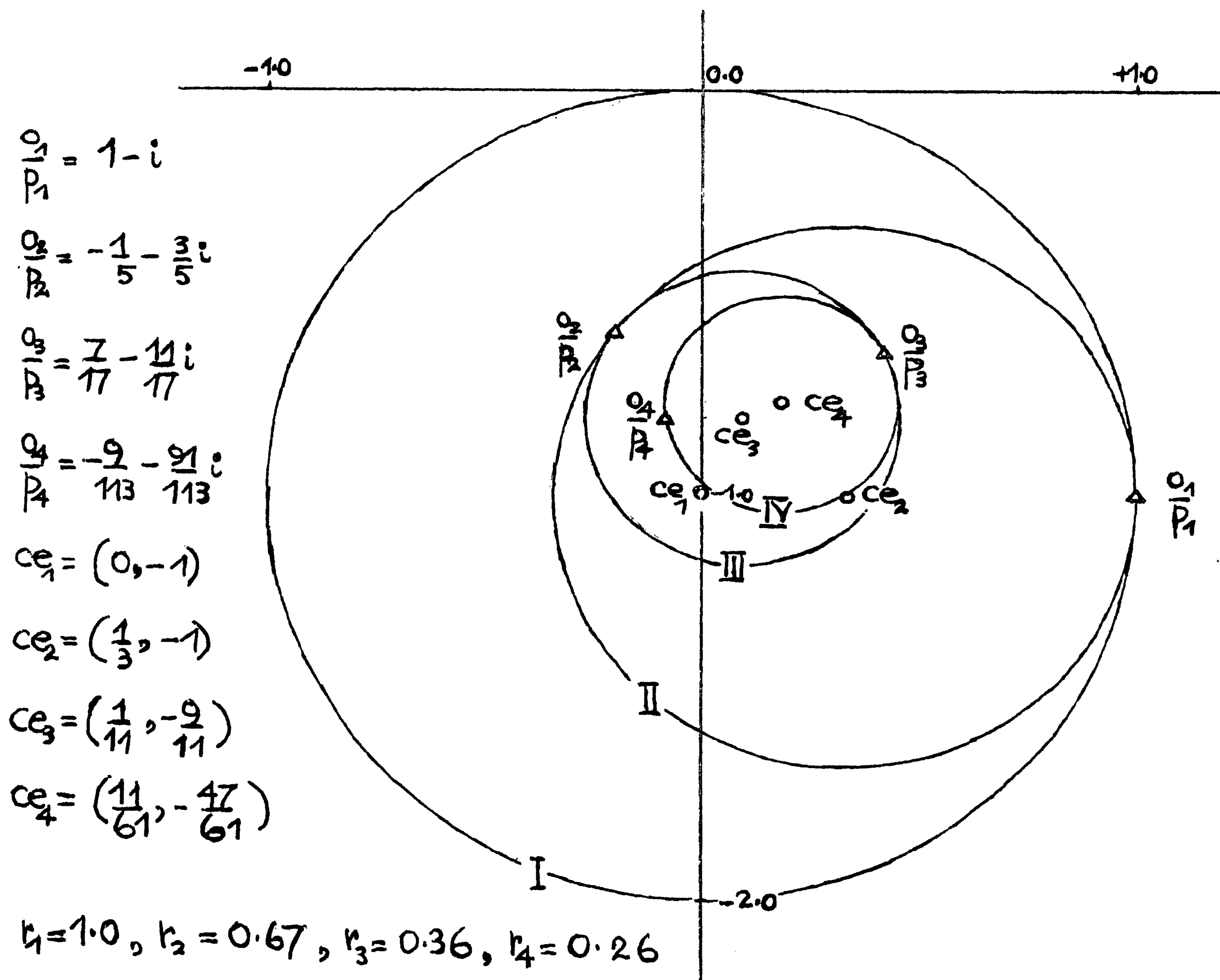
If

$$z' = z^2 \quad (36)$$

then

$$f_0(z) = -ze^{z'} \text{Ei}(-z'). \quad (37)$$

When  $z=0.5+0.5i$  we obtain the following diagram



The first convergent of (34) tells us that the value of  $f_0(z)$  lies within the circle I, the second- within the circle II, the third - within the circle III, and so on.



Clearly, if the value of  $r_n(z)$  tends to zero then  $f_0(z)$  may be determined to any accuracy, and the continued fraction converges: this is the determinate case. If however the value of  $r_n(z)$  tends to a finite non-zero limit, then the successive convergents of the Grommer fraction may tend to a limiting point on a circle of this radius, or they may be distributed in some other fashion on this circle: in any case, without further information we cannot as yet say anything about the convergence of the continued fraction: this is the indeterminate case.

By using the recursion (21) we can derive an expression for  $r_n(z)$  in terms of the partial sum of a certain series. For from (21)

$$\begin{aligned} -\text{Im}\{p_{n-1}(z)\overline{p_n(z)}\} &= \text{Im}(z)|p_{n-1}(z)|^2 - \beta_{n-1}\{p_{n-2}(z)\overline{p_{n-1}(z)}\} \\ &= \text{Im}(z)\left\{|p_{n-1}(z)|^2 + \sum_{s=0}^{n-2} \beta_{n-1}\beta_{n-2}\cdots\beta_{s+1}|p_s(z)|^2\right\} \end{aligned} \quad (38)$$

Thus

$$r_n(z) = \left\{ 2\text{Im}(z) \sum_{s=0}^{n-1} \frac{|p_s(z)|^2}{f_0\beta_1\cdots\beta_s} \right\}^{-1}. \quad (39)$$

The critical problem in the convergence theory of Grommer fractions is to establish the divergence of the series in (39). Now it may be shown that the partial sums of the series (39) satisfy a Volterra sum equation ([13] p. 96), and using this fact it is possible to prove that  $r_n(z)$  and  $r_n(0)$  tend to zero or to a non-zero limit together. This makes it easier to bring the conventional convergence theory of continued fractions into play, and it can be shown [14] that when  $z$  is equal to zero, the series in (39) diverges if  $\sum_{s=0}^{\infty} \beta_s^{1/2}$  diverges. The disadvantage of this result is that it involves the coefficients of the continued fraction, and these are not usually available in simple form. However Carleman, using his result ([15] p. 112) that if  $u_s > 0$  ( $s=1, 2, \dots$ ) and  $\sum_{s=1}^{\infty} u_s$  diverges, then  $\sum_{s=1}^{\infty} (u_1 u_2 \cdots u_s)^{1/s}$  diverges, shows (again by purely algebraic means)



that the series  $\sum_{s=0}^{\infty} \beta_s^{-1/2}$  diverges if the series  $\sum_{s=1}^{\infty} c_{2s}^{-1/2s}$  diverges.

Summarising these results: if  $f_0(z) \in H$  and the series  $\sum_{s=1}^{\infty} c_{2s}^{-1/2s}$  diverges then the the Grommer fraction (49) converges.

The existence and convergence of a Grommer fraction can be versed in terms of the Hamburger moment problem [16]. If it is possible to find a bounded non-decreasing function  $\xi(t)$  such that

$$\int_{-\infty}^{+\infty} t^s d\xi(t) = c_s \quad (s=0,1,\dots), \quad \xi(-\infty) = 0 \quad (40)$$

then the coefficients  $\beta_s$  ( $s=0,1,\dots$ ) in (19) are positive. If it is possible to find only one such function  $\xi(t)$  (i.e. the Hamburger moment problem is determinate) then the Grommer fraction (49) converges, and its value is given by the Stieltjes transform

$$\int_{-\infty}^{\infty} \frac{d\xi(t)}{z-t} \quad (\text{Im}(z) > 0)$$

M Riesz [17] has shown that if the Hamburger moment problem is soluble, then it is determinate if

$$\liminf_{s \rightarrow \infty} \sqrt[s]{\frac{c_{2s}}{(2s)!}} < \infty \quad (41)$$

Thus if  $f_0(z) \in H$  and condition (41) obtains then the associated Grommer fraction is convergent.

R. Nevanlinna has shown [18] that if the Hamburger moment problem is indeterminate then the sequences of convergents of even order and of odd order respectively of the Grommer fraction converge to differing functions.

To conclude this section we mention that there is a Tauberian theorem for Grommer fractions: it was shown by E.V. van Vleck [19] that if a Grommer fraction converges for all  $|z| \geq R$ , then the associated power series (17) also converges for all  $|z| \geq R$ .

Time and space do not allow me to discuss corresponding continued fractions, and their connection with the Stieltjes moment problem



$$\int_0^{\infty} t^s d\xi(t) = c_s \quad (s=0,1,\dots) \quad (42)$$

and the Markoff moment problem

$$\int_0^1 t^s d\xi(t) = c_s \quad (s=0,1,\dots) \quad (43)$$

But the principles involved in establishing the convergence of the continued fraction expansion of a certain function may be summarised as follows. A formal power series expansion is obtained from this power series by ensuring that the power series expansions of the successive convergents agree with the formal power series to an increasing number of terms. From a number of criteria, which may be algebraic or function-theoretic in character, it can be shown that the continued fraction is of a certain type. From this continued ~~fraction~~ <sup>a</sup> sequence of bilinear transformations is obtained which ~~tell~~ us that the value of the given function lies within a succession of circular regions. From a number of further criteria, which again may be algebraic or function-theoretic in character, it can be shown that the radii of these regions tend to zero.

The type of the continued fraction and the question as to whether it converges or not is related to the solution of a certain moment problem. If the coefficients of the continued fraction possess certain properties then the corresponding moment problem has a solution, i.e. a bounded non-decreasing function  $\xi(t)$  can be found satisfying certain conditions. If the continued fraction converges then only one such function can be found. The value of the continued fraction can be expressed by means of an integral transform involving  $\xi(t)$ .

The convergence of the original formal power series has not been mentioned, and is in the first instance irrelevant. For example there exist continued fractions converging in certain bounded domains in the complex plane for the functions  $\exp(x)$ , whose power series converges everywhere in the bounded complex plane; for  $\tan(x)$  whose power series converges for  $|x| < \pi/2$ ; and for  $-e^z \text{Ei}(-z)$  whose series development in inverse powers of  $z$  converges only at one point,  $z^{-1}=0$ .



The  $\epsilon$ -Array

So far we have only considered the series  $\sum_{s=0}^{\infty} c_s z^{-s-1}$ . It is clearly possible also to consider the series  $\sum_{s=0}^{\infty} c_{m+s} z^{-s-1}$ , transform this into an associated continued fraction, and obtain the expansions

$$\sum_{s=0}^{m-1} c_s z^{-s-1} + z^{-m} \left\{ \frac{c_m}{z-\alpha_0^{(m)}} - \frac{\beta_0^{(m)}}{z-\alpha_1^{(m)}} - \dots \right\} \quad (44)$$

If we denote the successive convergents of this continued fraction by  $\epsilon_{2r}^{(m)}$ , then they can be arranged as follows:

$$\begin{array}{cccc} \epsilon_0^{(0)} & & & \\ \epsilon_0^{(1)} & \epsilon_1^{(0)} & & \\ \epsilon_0^{(2)} & \epsilon_2^{(1)} & \epsilon_4^{(0)} & \\ \epsilon_0^{(3)} & \epsilon_2^{(2)} & \epsilon_4^{(1)} & \epsilon_6^{(0)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

this is the even order  $\epsilon$ -array. The successive convergents of (44) lie on a diagonal in this array: the successive partial sums of the series  $\sum_{s=0}^{\infty} c_s z^{-s-1}$  lie in the first column.

This array has a very strong connection with the Padé table [20], indeed if we transpose this array about the diagonal with superscript 1 and change the variable to  $x=z^{-1}$ , then we obtain the upper half of the conventional Padé table.

Determinantal expressions in terms of the coefficients  $c_s$  ( $s=0,1,\dots$ ) for the Padé quotients were given by Frobenius [21], in terms of the partial sums of the series  $\sum_{s=0}^{\infty} c_s z^{-s-1}$  for the quantities  $\epsilon_{2r}^{(m)}$  ( $m,r=0,1,\dots$ ) by Shanks [22].



The types of convergence manifested by the even order  $\epsilon$ -array can in principle be distinguished as follows

- 1) The quantities in all the diagonals converge to the same limit (this corresponds, for example, to the case in which the Stieltjes moment problems for the weights  $c_m, c_{m+1}, c_{m+2}, \dots$  ( $m=0, 1, \dots$ ) are determinate). This is regular convergence
- 2) The quantities in each diagonal converge to differing limits (this corresponds, for example, to a determinate Hamburger moment problem for the weights  $c_m, c_{m+1}, c_{m+2}, \dots$ ). This is irregular convergence.
- 3) The even members and odd members of each diagonal converge to differing limits (this corresponds to indeterminate Hamburger moment problems for the weights  $c_m, c_{m+1}, c_{m+2}, \dots$  ( $m=0, 1, \dots$ )). In deference to the distinguished mathematician who first investigated this behaviour, we shall call this the highly irregular convergence behaviour of R. Nevanlinna.

Here are two examples of regularly convergent even order  $\epsilon$ -arrays. The first (Table I) refers to the transformation of the series  $\sum_{s=0}^{\infty} \{(-1)^s / (s+1)\} z^{-s-1}$  when  $z=-1$ , which is slowly convergent (its value is  $\ln_e 2 = 0.69314\dots$ ). The second refers to the transformation of the series  $\sum_{s=0}^{\infty} (-1)^s s! z^{-s-1}$  when  $z=1$ , which is divergent (this series may be from the exponential integral  $-e^z \text{Ei}(-z)$ , whose value when  $z=1$  is  $0.596\dots$ )

$2s$	0	2	4	6
$m$				
0	0			
1	1.0	0.66667		
2	0.5	0.7	0.69231	
3	0.83333	0.69048	0.69333	0.69312
4	0.58333	0.69444	0.69309	
5	0.78333	0.69242		
6	0.61667			

Table I

m	2s	0	2	4	6
0		0			
1		1.0	0.5		
2		0	0.6	0.571	
3		+2.0	0.5	0.615	0.588
4		-4.0	0.8	0.571	
5		+20.0	0		
6		-100.0			

The members of the even order  $\epsilon$ -array may be constructed in various ways [21]. For later application we mention that the even order  $\epsilon$ -array may be completed by the addition of quantities with odd suffix, thus:

$$\begin{array}{cccc}
 & & \epsilon_0^{(0)} & \\
 \epsilon_{-1}^{(1)} & & & \epsilon_1^{(0)} \\
 & \epsilon_0^{(1)} & & \epsilon_2^{(0)} \\
 \epsilon_{-1}^{(2)} & & \epsilon_1^{(1)} & \epsilon_3^{(0)} \\
 & \epsilon_0^{(2)} & & \epsilon_4^{(1)} \\
 \epsilon_{-1}^{(3)} & & \epsilon_1^{(2)} & \\
 & \epsilon_0^{(3)} & & \\
 \cdot & & \cdot & \\
 \cdot & & \cdot & 
 \end{array}$$

The members of this array then obey the relationships of the  $\epsilon$ -algorithm, viz.

$$\epsilon_{s+1}^{(m)} = \epsilon_{s-1}^{(m+1)} + (\epsilon_s^{(m+1)} - \epsilon_s^{(m)})^{-1}, \quad (m, s = 0, 1, \dots) \quad (45)$$

these concern quantities lying at the vertices of a lozenge in this array. If we take as initial values

$$\epsilon_{-1}^{(m)} = 0, \quad \epsilon_0^{(m)} = \sum_{s=0}^{m-1} c_s z^{-s-1} \quad (46)$$



then the  $\epsilon$ -array may be built up, column by column, from left to right.

#### IV Non-Commutative Continued Fractions

So far we have only considered continued fractions whose coefficients are real or complex numbers. Recently a formal theory has been described of continued fractions whose coefficients obey a non-commutative law of multiplication [23]. We are now dealing with continued fractions of the form

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \dots \quad (47)$$

where  $a_s$  ( $s=1,2,\dots$ ),  $b_s$  ( $s=0,1,\dots$ ) are elements of a ring in which an inverse and unity are defined. With regard to the elements  $A,B,C,\dots$  of  $N$  the following assumptions are made. To every pair  $A,B$  there corresponds an element  $C$ , such that

$$A+B = B+A = C \quad (48)$$

and

$$(A+B)+C = A+(B+C). \quad (49)$$

To every pair  $A,B$  there corresponds an element  $D$  such that

$$AB = D \quad (50)$$

and further

$$(AB)C = A(BC). \quad (51)$$

In general

$$AB \neq BA. \quad (52)$$

There exists an element  $I$  such that

$$AI = IA = A \text{ for all } A \in N \quad (53)$$

To each  $A$  there corresponds an element  $A^{-1}$  such that

$$AA^{-1} = A^{-1}A = I. \quad (54)$$

There is a subset S of N such that for all  $B \in S$  and all  $A \in N$

$$AB = BA. \quad (55)$$

Finally there exists a zero element 0, such that for all  $A \in N$

$$A+0 = A, \quad A0 = 0A = 0 \quad (56)$$

We are dealing, for example, with square matrices.

In the case of scalar continued fractions the  $n^{\text{th}}$  convergent

$$C_n = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_{n-1}}{b_{n-1} +} \frac{a_n}{b_n} \quad (57)$$

may preliminarily be defined by the following rules: divide  $a_n$  by  $b_n$  and add the quotient to  $b_{n-1}$ ; divide  $a_{n-1}$  by the result and add the quotient to  $b_{n-2}$ , and so on. That is, compute  $D_s$  ( $s=0,1,\dots,n$ ) where

$$D_0 = b_n, \quad D_{s+1} = b_{n-s-1} + \frac{a_{n-s}}{D_s} \quad (s=0,1,\dots,n-1) \quad (58)$$

when

$$D_n = C_n. \quad (59)$$

When we come to non-commutative continued fractions whose elements are members of N, we must in addition prescribe the order of the multiplication by the inverse in (58).

The development of the theory of non-commutative continued fractions has been confined to two cases. That in which the convergents are defined by

$$\text{pre} \left\{ b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_n}{b_n} \right\} = D_n \quad (60)$$

where

$$D_0 = b_n, \quad D_{s+1} = b_{n-s-1} + D_s^{-1} a_{n-s} \quad (s=0,1,\dots,n-1) \quad (61)$$



(i.e. premultiplication by the inverse of  $D_s$  is consistently used) and that in which

$$\text{post } \left\{ b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n} \right\} = D_n \quad (62)$$

where

$$D_0 = b_n, \quad D_{s+1} = b_{n-s-1} + a_{n-s} D_s^{-1} \quad (s=0, 1, \dots, n-1) \quad (63)$$

(i.e. where post multiplication by the inverse of  $D_s$  is consistently used).

In order to give a brief indication as to how the theory develops from these definitions we give the derivation of the fundamental recursions. We wish to show that

$$\text{pre } C_n = \text{pre} \left\{ b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n} \right\} = B_n^{-1} A_n \quad (64)$$

where

$$A_n = b_n A_{n-1} + a_n A_{n-2}, \quad A_{-1} = I, \quad A_0 = b_0, \quad (65)$$

$$B_n = b_n B_{n-1} + a_n B_{n-2}, \quad B_{-1} = 0, \quad B_0 = I. \quad (66)$$

The expressions for  $A_0$  and  $B_0$  are clearly correct. The quantity  $\text{pre} C_{n+1}$  is obtained by replacing  $b_n$  in the definition of  $\text{pre} C_n$  by  $b_n + b_{n+1}^{-1} a_{n+1}$ . The right hand side of (65) then becomes

$$(b_n + b_{n+1}^{-1} a_{n+1}) A_{n-1} + a_n A_{n-2} \quad (67)$$

or

$$b_{n+1}^{-1} \{ b_{n+1} (b_n A_{n-1} + a_n A_{n-2}) + a_{n+1} A_{n-1} \} \quad (68)$$

i.e.

$$b_{n+1}^{-1} \{ b_{n+1} A_n + a_{n+1} A_{n-1} \} \quad (69)$$

with a similar expression for the right hand side of (66). The quotient of these two expressions is

$$\left[ b_{n+1}^{-1} \{ b_{n+1} B_n + a_{n+1} B_{n-1} \} \right]^{-1} \left[ b_{n+1}^{-1} \{ b_{n+1} A_n + a_{n+1} A_{n-1} \} \right] \quad (70)$$

i.e.

$$B_{n+1}^{-1} A_{n+1} \quad (71)$$

where  $A_{n+1}$  and  $B_{n+1}$  are given by expressions of the form (65) and (66) in which  $n$  is replaced by  $n+1$ .

Similarly

$$\text{post} \left\{ b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n} \right\} \quad (72)$$

is given by  $A_n B_n^{-1}$ , where

$$A_n = A_{n-1} b_n + A_{n-2} a_n, \quad A_{-1} = I, \quad A_0 = b_0, \quad (73)$$

$$B_n = B_{n-1} b_n + B_{n-2} a_n, \quad B_{-1} = C, \quad B_0 = I. \quad (74)$$

The theory of these two systems of continued fractions has been built up. In particular we have derived orthogonal polynomials  $\text{pre}\{p_n(z)\}$  ( $z \in S$ ) for which

$$\sum_{s=0}^n k_{n,s} c_{r+s} = 0 \quad (r=0, 1, \dots, n-1) \\ \sum_{s=0}^n k_{n,s} c_{r+s} \neq 0 \quad (r=n) \quad (75)$$

and orthogonal polynomials  $\text{post}\{p_n(z)\}$  ( $z \in S$ ) for which

$$\sum_{s=0}^n c_{r+s} k_{n,s} = 0 \quad (r=0, 1, \dots, n-1) \\ \sum_{s=0}^n c_{r+s} k_{n,s} \neq 0 \quad (r=n). \quad (76)$$

Similarly there exist associated polynomials  $\text{pre}\{o_n(z)\}$  and  $\text{post}\{o_n(z)\}$ . An  $\epsilon$ -array such as was described earlier can also be constructed. It will be seen that if  $z \in S$  then the initial conditions (46) for the application of the  $\epsilon$ -algorithm are the same in the pre and in the post systems; furthermore the  $\epsilon$ -algorithm relationships do not involve either pre or post multiplication, i.e.

$$\left[ \text{pre}\{p_n(z)\} \right]^{-1} \left[ \text{pre}\{o_n(z)\} \right] = \left[ \text{post}\{o_n(z)\} \right] \left[ \text{post}\{p_n(z)\} \right]^{-1} \quad (77)$$



Convergence Theory

A somewhat limited convergence theory of non-commutative continued fractions has been constructed by means of the introduction of suitable norms. The coefficients of the continued fraction now become members of a Banach ring with inverse. The norms obey the relationships

$$\|AB\| \leq \|A\| \|B\| \quad (78)$$

and

$$\|A+B\| \leq \|A\| + \|B\|. \quad (79)$$

It is possible to show, for example, that if  $\sum_{s=0}^{\infty} \|r_s\|$  converges, then

$$a_0 + \frac{r_1}{I+r} \frac{a_2}{I+} \dots \quad (80)$$

also converges, and that if  $\sum_{s=0}^{\infty} \|b_s\|$  converges then

$$b_0 + \frac{I}{b_1+} \frac{I}{b_2+} \dots \quad (81)$$

diverges (in general) by oscillation.

The convergence theory which has so far been established is algebraic and depends solely upon the use of inequalities. An appeal to a function theoretic approach, the use of the moment problem, the location of those values of the scalar  $z$  for which the denominator of an associated non-commutative continued fraction becomes singular, and so on, has not been made; though work in this direction presents a number of interesting problems.

V Non-Commutative Determinants

Early work on continued fractions was very much bound up with the theory of determinants. We have seen that determinantal formulae of the coefficients  $c_s$  ( $s=0,1,\dots$ ) of the original power series for



the coefficients  $\beta_s$  ( $s=0,1,\dots$ ) of an associated continued fraction, can be given. Determinantal formulae for the orthogonal polynomials  $p_n(z)$  and the associated polynomials  $\phi_n(z)$  in terms of the  $c_s$  ( $s=0,1,\dots$ ) can also be given.

To give one further simple expression, we have

$$p_n(z) = \begin{vmatrix} z-\alpha_n & \beta_{n-1} & & & 0 \\ -I & z-\alpha_{n-1} & \beta_{n-2} & & 0 \\ 0 & -I & z-\alpha_{n-2} & & \\ & & & z-\alpha_2 & \beta_0 \\ & & 0 & -I & z-\alpha_1 \end{vmatrix} \quad (82)$$

By an appeal to this formula it is possible to show that the roots of the denominators of a Grommer fraction can only be real.

Furthermore Wall establishes his matrix theory of continued fractions ([13] Ch. XII) by studying the expression (82) as  $n$  tends to infinity, and examines the conditions under which the corresponding matrix remains non-singular.

When proceeding to the theory of non-commutative continued fractions we are however confronted by the fundamental difficulty that a non-commutative determinant may be defined in two ways. The first definition derives from a set of linear equations in which the coefficients, the homogeneous terms, and the unknown variables are all linear operators: the second derives from the usual expansion of a determinant with respect to its terms.

In the scalar case these definitions are equivalent: in the non-commutative case they appear to have little to do with one another.

However, if the theory of non-commutative determinants is to be exploited in the theory of non-commutative continued fractions, it may be possible to apply some recent work of Ostrowski [25] in which he considers norms of matrices whose elements  $a_{i,j}$  ( $i,j=1,\dots,m$ ) are



linear operators. He finds that the norm of the matrix depends solely on the norms of the operators (i.e. not individually on the elements of the linear operators).

It is important to bear in mind that Ostrowski is dealing with matrices of linear operators, not with determinants of linear operators (thus if his matrix is an  $m \times m$  matrix of elements which are  $n \times n$  matrices, then the resultant matrix is of dimension  $(m \times n) \times (m \times n)$ , not  $n \times n$ ).

Nevertheless I feel that an attempt to adapt Ostrowski's results to Wall's convergence theory would constitute a very promising line of attack.

#### VI Vector Series

I have also shown [26] that the  $\epsilon$ -algorithm may be used to transform series of the form  $\sum_{s=0}^{\infty} c_s z^{-s-1}$  in which the coefficients  $c_s$  ( $s=0,1,\dots$ ) are vectors. In order to do this use is made (for the inverse which occurs in the  $\epsilon$ -algorithm relationships) of a suggestion of K. Samelson, namely that

$$(y_1, y_2, \dots, y_n)^{-1} = \left( \sum_{r=1}^n y_r \overline{y_r} \right)^{-1} (\overline{y_1}, \overline{y_2}, \dots, \overline{y_n}). \quad (83)$$

This corresponds to more general definitions of the inverse of a rectangular matrix given by Lanczos [27] and Penrose [28].

#### VII Application to Iterative Methods.

Matrix and vector functions have, of course, appeared in the literature; admittedly the equations which such functions satisfy are no more complicated than linear differential equations of the first order with constant coefficients, but all things must have a beginning. To the approximation of such functions the continued fractions which have just been described <sup>have</sup> direct application.

But the most immediate point of application of these continued fractions lies in the acceleration of slowly convergent iterative processes.



To revert to the scalar case for the moment, if we consider the solution of the differential equation

$$y' = x^{-1} \{(1+x)^{-1} - y\}, \quad y(1) = 0.69314 \quad (84)$$

by Picard's method of successive approximations, thus

$$y_{r+1}(x) = 0.69314 + \int_1^x x^{-1} \{(1+x)^{-1} - y_r(x)\} dx \quad (85)$$

then the successive iterates are indeed the successive partial sums of the series

$$x^{-1} \ln(1+x) = \sum_{s=0}^{\infty} \{(-1)^s / (s+1)\} x^s \quad (86)$$

Thus any attempt to accelerate the convergence of the iterative scheme (85) is a direct transformation of the slowly convergent series (86). In this case the correspondence is complete; in the case of a more sophisticated iterative process the story is a little more complicated but the idea behind it is the same.

When we proceed, for example, to the iterative solution of ordinary differential equations with two point boundary conditions, or of integral equations, or of partial differential equations, or of certain multi-parameter minimisation problems (see e.g. [29]), we obtain arrays of higher dimension; i.e. they are, or can easily be mapped upon, vectors or matrices.

To problems of this type the continued fractions of the sort which I have just been describing have direct application. In order to illustrate their use I give a simple example. It relates to the iterative solution of the partial differential equation

$$\phi_{xx} + 2\phi_{xy} + \phi_{yy} = 0 \quad (87)$$

with boundary values given over a square in the x-y plane, by means of the scheme



$$\phi_{xy}^{(r+1)} = \frac{1}{2} \{ \phi_{xx}^{(r)} + \phi_{yy}^{(r)} \}. \quad (88)$$

The problem is discretised and the successive iterates of (88) constitute a sequence of square matrices. With the boundary value chosen (87) has a simple analytic solution  $\phi(x,y)$  and thus if any approximate solution  $\bar{\phi}(x,y)$  to (87) is given then it is easy to compute its distance from the true solution (taken to be

$$\max_{x,y} (|\bar{\phi}(x,y) - \phi(x,y)|). \quad (89)$$

From the initial approximation

$$\phi^{(0)}(x,y) = 0 \quad (90)$$

a sequence of iterates has been computed by means of equation (88). To this sequence the  $\epsilon$ -algorithm (using the Samelson inverse) has been applied. From the resulting even order  $\epsilon$ -array a similar array of distances, computed by means of (89) has been obtained. Table III gives such an array.

s	0	2	4	6	8	10
m						
0	$0.62_{10^{-3}}$					
1	$.77_{10^{-2}}$	$0.41_{10^{-3}}$				
2	$.15_{10^0}$	$.24_{10^{-2}}$	$0.40_{10^{-3}}$			
3	$.29_{10^1}$	$.12_{10^{-1}}$	$.13_{10^{-2}}$	$0.53_{10^{-4}}$		
4	$.56_{10^2}$	$.96_{10^{-1}}$	$.25_{10^{-1}}$	$.11_{10^{-3}}$	$0.90_{10^{-5}}$	
5	$.11_{10^4}$	$.52_{10^0}$	$.56_{10^{-1}}$	$.21_{10^{-3}}$	$.21_{10^{-4}}$	$0.34_{10^{-5}}$
6	$.21_{10^5}$	$.40_{10^1}$	$.13_{10^2}$	$.70_{10^{-3}}$	$0.12_{10^{-3}}$	
7	$.40_{10^6}$	$.23_{10^2}$	$.23_{10^1}$	$0.19_{10^{-2}}$		
8	$.77_{10^7}$	$.17_{10^3}$	$0.37_{10^2}$			
9	$.15_{10^9}$	$0.99_{10^3}$				
10	$0.28_{10^{10}}$					

Table III



It will be seen that the original scheme diverges quite strongly (this is indicated by the *distances* in the first column). However, the convergence of the first diagonal of the  $\epsilon$ -array is quite reasonable.

### VIII Conclusion

In conclusion I wish to make two further points. The first is as follows: I have said that non-commutative continued fractions are of great interest in accelerating the convergence of multi-dimensional iterative schemes in Applied Mathematics; this is undoubtedly true. But conventional continued fractions have played an enormous rôle in Pure Mathematics. To realise this one has only to recall the fluent and ingenious use of continued fractions made by Tschebyscheff and his pupils in their work on the moment problem; the work of Stieltjes, Hamburger, Nevanlinna and many others on the theory of functions of a complex variable; and the part which continued fractions have played in investigations into the arithmetic properties of the continuum. Thus it is eminently reasonable to suggest that non-commutative continued fractions will make a similar contribution to the further development of the theory of linear operators.

The second point (it is rather prosaic) is this: Numerical Analysis is very much an experimental science. For this reason, if a method of computation is proposed and examples are given to illustrate its use, some trouble <sup>should</sup> be taken to indicate exactly how the examples were worked out and how they may be continued. In this instance I have published a number of ALGOL programmes ([30] - [33]) relating to the computational aspects of this talk. From the correspondence which one receives on these occasions it is quite clear that the methods which I have discussed are being incorporated in various computer programme libraries, and that the numerical application of continued fractions is at the present time a subject of active interest.



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