STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

DR 26r

On repeated application of the Epsilon algorithm.

(Chiffres, 4 (1961), p 19-22).

P. Wynn.

1961

CHIFFRES - 1 - 1961.

On Repeated Application of the ε-Algorithm

by P. WYNN.

Dans certains cas, la répétition de l'ε-Algorithme accélère efficacement la convergence d'une série lentement convergente. Cette technique est illustrée par un exemple.

In bestimmten Fällen beschleunigt die Wiederholung des ε-Algorithmus wirksam die Konvergenz einer langsam konvergierenden Reihe. Dieses Verfahren wird durch ein Beispiel veranschaulicht.

In certain cases, the repetition of the ε-Algorithm greatly accelerates the convergence of a slowly convergent sequence. This technique is illustrated by an example.

В некоторых случаях, повторение ε-алгорифм существенно ускоряет сходимость медленно сходящихся рядов. Дастся пример этого способа.

The purpose of this note is to draw attention to, and place upon record an example of, a powerful technique for the numerical transformation of slowly convergent sequences. It is mathematically equivalent to a proposal of Shanks [6].

As has been shown (see for example [1] and [2]) the epsilon algorithm [3] :

$$\varepsilon_{s+1}^{(m)} = \varepsilon_{s+1}^{(m+1)} + \frac{1}{\varepsilon_{s}^{(m+1)} - \varepsilon_{s}^{(m)}} \qquad (1)$$

with the initial conditions :

$$\varepsilon_{-1}^{(m)} = 0 \qquad \varepsilon_{0}^{(m)} = S_m \qquad m = 0, 1, \ldots \qquad (2)$$

is, in certain cases, a powerful method for the numerical transformation of a slowly convergent (or divergent) sequence S_m $m = 0, 1, \dots$, in the sense that the quantities $\varepsilon_{2s}^{(0)}$ $s = 0, 1, \dots$ converge more rapidly to a limit with which the sequence S_m may be associated than do the quantities S_m $m = 0, 1, \dots$

The quantities $\varepsilon_{2s}^{(m)}$ produced from the sequence S_m may be referred to as ${}_1\varepsilon_{2s}^{(m)}$; it is then formally possible to apply the relationships (1) to derive sequence of quantities ${}_2\varepsilon_{s}^{(m)}$ $m, s = 0, 1, \dots$ from the initial conditions :

$$ {}_2\varepsilon_{-1}^{(m)} = 0 \qquad {}_2\varepsilon_{0}^{(m)} = {}_1\varepsilon_{2m}^{(0)} \qquad m = 0, 1, \dots; \tag{3} $$

and further to repeat this process. There is a limit to the number of times the process may be repeated. From $2n - 1$ $n = 1, 2, \dots$ members of the sequence $\varepsilon_{0}^{(m)}$ $m = 0, 1, \dots$ there are produced n members of the sequence $\varepsilon_{2m}^{(0)}$ $m = 0, 1, \dots$; the number of starting values is therefore approximately halved at each stage.

The numerical example concerns the transformation of the sequence of partial sums of the series :

$$ -e^z \mathrm{E}i(-z) = e^z \int_z^\infty e^{-t} t^{-1} dt $$

$$ \sim \sum_{n=0}^{\infty} (-1)^n n! z^{-n-1} \tag{4} $$

when $z = 0.1$. The series for this value of z is :

$0 + 10 - 100 + 2{,}000 - 60{,}000 + 2{,}400{,}000 - 120{,}000{,}000 \dots;$ (5)

the partial sums of the series are consecutively

$0;\ 10;\ -90;\ +1{,}910;\ -58{,}090;\ +2{,}341{,}910;\ -117{,}658{,}090;\ \dots$ (6)

Table I gives the quantities ${}_k\varepsilon_{2s}^{(0)}$ $k = 1(1)4$, $s = 0, 1, \dots, 2^{4-k} + 1$

s/k	1	2	3	4
0	0.0000 000	0.0000 000	0.0000 000	0.0000 000
1	0.9090 909	1.5538 362	1.9841 277	2.0146 999
2	1.2863 070	1.8908 117	2.0142 360	
3	1.4966 417	1.9766 953		
4	1.6295 095	2.0018 881		
5	1.7196 014			
6	1.7835 970			
7	1.8305 965			
8	1.8659 999			

TABLE I

The value of $-e^z\mathrm{Ei}\,(-z)$ may be computed for small value of z by use of the expression

$$-\,e^{z}\ \mathrm{E}i(-z) = -\,e^{z}\left(\gamma + \log_e z + \sum_{n=0}^{\infty}\frac{(-z)^n}{n\,!\,n}\right) \tag{7}$$

which gives, when $z = 0.\,1$,

$$-\,e^{0.1}\mathrm{Ei}(-\,0.1) = 2.0146\ 425 \tag{8}$$

It might be thought, in view of the size of the members of the sequence (6) and the relatively small magnitude of the quantities ${}_1\varepsilon_{2s}^{(0)}$, that the process of obtaining the quantities ${}_1\varepsilon_s^{(m)}$ would involve a catastrophic loss of figures due to cancellation. This is not so. Application of the linear error analysis given in [4] shows that the errors tend consistently to cancel each other.

It may be shown [5] that the ε-algorithm transforms the successive partial sums of the series into the successive convergents of the continued fraction

$$\frac{1}{z+1-}\ \frac{1^2}{z+3-}\ \cdots\ \frac{\nu^2}{z+2\nu+1-}\ \cdots \tag{9}$$

and as such the members of the first column of Table I have, without loss accuracy, been computed.

The even order columns of the ε-array relating to the sequence (6), which have been computed using floating point fixed length (30 binary digit) arithmetic, are displayed in Table II.

m\s	0	2	4	6	8	10
0	0.0					
1	+1.0 $\times 10^{1}$	+9.0909 091 $\times 10^{-1}$				
2	−9.0 $\times 10^{1}$	+5.2380 953 $\times 10^{0}$	+1.2863 071 $\times 10^{0}$			
3	+1.91 $\times 10^{3}$	−2.5483 871 $\times 10^{1}$	+3.7972 771 $\times 10^{0}$	+1.4966 421 $\times 10^{0}$		
4	−5.809 $\times 10^{4}$	+4.4658 545 $\times 10^{2}$	−1.0374 696 $\times 10^{1}$	+3.1423 839 $\times 10^{0}$	+1.6295 082	
5	+2.34191 $\times 10^{6}$	−1.1031 172 $\times 10^{4}$	+1.6797 235 $\times 10^{2}$	−4.7103 262 $\times 10^{0}$	+2.7846 563	+1.7195 816
6	−1.1765 809 $\times 10^{8}$	+3.7469 650 $\times 10^{5}$	−3.4921 367 $\times 10^{3}$	+7.9542 832 $\times 10^{1}$	−2.0548 172	
7	+7.0823 420 $\times 10^{9}$	−1.6249 664 $\times 10^{7}$	+1.0279 353 $\times 10^{5}$	−1.4214 087 $\times 10^{3}$		
8	−4.9691 766 $\times 10^{11}$	+8 6012 006 $\times 10^{8}$	−3.9278 420 $\times 10^{6}$			
9	+3.9823 083 $\times 10^{13}$	−5.3840 708 $\times 10^{10}$				
10	−3.5889 770 $\times 10^{15}$					

TABLE II

Comparison of the leading diagonal of Table II and the first column of Table I shows that a certain loss of figures does occur in effecting the ε-algorithm, but catastrophic is it not.

Acknowledgements.

The numerical results contained in Tables I and II were produced upon the Z-22 at Mainz. The author is grateful to the Deutsche Forschungsgemeinschaft for providing him with a grant which has enabled this note to be written.

REFERENCES

1. WYNN (P.). — The Epsilon Algorithm and Operational Formulae of Numerical Analysis (*Maths. of Comp.*, vol. XV, n° 74, 1961, p. 151).
2. WYNN (P.). — A Comparison between the Numerical Performances of the Euler Transformation and the Epsilon Algortihm (*Chiffres*, n° 1, 1961).
3. WYNN (P.). — On a Device for Computing the $e_m(S_n)$ Transformation (*M.T.A.C.*, X, 54, 1956, 91).
4. WYNN (P.). — On the Propagation of Error in Certain Non-linear Algorithms, Numerische Mathematik, 1, (1959), 142.
5. WYNN (P.). — The Rational Approximation of Functions which are Formally defined by a Power Series Expansion (*Maths. of Comp.*, vol. 14, N° 70, 1960, p. 147).
6. SHANKS (D.). — Non-linear Transformations of Divergent and Slowly Convergent Sequences (*J. Math. and Physics*, v. 34, 1955, p. 1).