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ON THE TABULATION OF INDEFINITE INTEGRALS.

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Abstract.
A number of efficient methods have been developed for the numerical solution of linear second order differential equations from which the term involving the first derivative is absent. The function $\theta(z)\int_a^{g(z)} \phi(t)dt+h(z)$ satisfies such an equation. This fact is of material assistance in the tabulation of indefinite integrals.

The tabulation of an auxiliary function which is related to an indefinite integral of the form $\int_\alpha^x \phi(t)dt$ may often be most economically carried out by the numerical integration of a differential equation which is satisfied by the auxiliary function. The choice of auxiliary function and auxiliary variable z where $x = g(z)$ is in the main determined by facilities for interpolation in the resulting table, and is logically that which provides the greatest information about the required indefinite integral in the smallest possible space. Of necessity the auxiliary function must be simply related to the indefinite integral, so that the latter may easily be extracted from the former, the final choice of auxiliary function and auxiliary variable being resolved from the conflict of economy of expression on the one hand and ease of definition on the other. In general it is true to say that the considerations which lead to the choice of auxiliary function and variable are precisely those which make the numerical solution of the differential equation which the function satisfies most pleasant. If an auxiliary function has been chosen which is easy to interpolate, then the appropriate differential equation is in general correspondingly easy to integrate numerically, for the implication of the first statement is that the successive differences of the function decrease smoothly, which is a condition for the second. A form of differential equation which particularly lends itself to numerical integration[1] is that which is linear and of the second order and in which the term in the first derivative is absent. In view of this fact it is of interest to derive the following elementary

THEOREM

If $$y(z) = \theta(z) \int_{\alpha}^{g(z)} \phi(t)dt + h(z) \qquad (1)$$

then $y(z)$ satisfies a linear differential equation of the second order in which the term involving $y'(z)$ is absent. For, from (1)

$$y'(z) = \theta'(z) \int_{\alpha}^{g(z)} \phi(t)dt + \theta(z)g'(z)\phi(g(z)) + h'(z)$$

and hence $$\theta(z)y'(z) - \theta'(z)y(z) = \theta^2(z)g'(z)\phi(g(z)) + \theta(z)h'(z) - \theta'(z)h(z)$$

and by differentiating again

$$\theta(z)y''(z) - \theta''(z)y(z) = \left[\theta^2(z)g'(z)\phi(g(z))\right]' + \theta(z)h''(z) - \theta''(z)h(z) \qquad (2)$$

where primes denote differentiation with respect to z.

Since the computationally unpleasant behaviour of the indefinite integral $\int_{\alpha}^{z}\phi(t)dt$ may well arise from the behaviour of $\phi(x)$, one obvious choice for $\theta(z)$ is

$$\theta(z) = \left[\phi(g(z))\right]^{-1}$$

In this case with $h(z) = 0$ and $x = g(z) = z$ equation (2) becomes

$$\theta(z)y''(z) - \theta''(z)y(z) = \theta'(z) \qquad (3)$$

Another choice which would apply when $\phi(x)$ is dominated by a term of the form x^n, i.e. when

$$\phi(x) = kx^n + \epsilon(x)$$

where $\epsilon(x)$ is relatively small throughout the range of integration, is

$$\theta(z) = [x\phi(x) - \alpha\phi(\alpha)]^{-1} \qquad (4)$$

For if $\phi(x) = kx^n$, then $y(z)$ is constant and little difficulty is to be expected in its tabulation, whilst if $\phi(x) \doteq kx^n$ the substitution (4) is at least a reasonable step which can be made in the hope of easing tabulation. In this case, when $h(z) = 0$ it is necessary to eliminate $x, g(z)$ and $\phi(x)$ from

$$x = g(z) = z \; ; \; \theta(z)y''(z) - \theta''(z)y(z) = [\theta^2(z)g'(z)\phi(g(z))]'$$

and equation (4). Thus the required version of equation (2) is

$$\theta(z)y''(z) - \theta''(z)y(z) = [(1 + \alpha\phi(\alpha)\theta(z))\theta(z)/z]' \qquad (5)$$

For large values of the argument the simplest auxiliary variable to use is $z = {}^1/x$, i.e. $x = g(z) = {}^1/z$. In this case, with $h(z) = 0$ and $\theta(z) = [\phi(g(z))]^{-1}$, equation (2) becomes

$$\theta(z)y''(z) - \theta''(z)y(z) = -[\theta(z)/z^2]'$$

vis à vis equation (3); and with $h(z) = 0$, $\theta(z)$ again being given by equation (4), equation (2) becomes

$$\theta(z)y''(z) - \theta''(z)y(z) = -[(1 + \alpha\phi(\alpha)\theta(z))\theta(z)/z]'$$

vis à vis equation (5).

An example in which the above theorem may be applied can be taken from the literature. Fox and Miller[2] tabulated the function $T(z)$ where $T(z) = e^{-x}Ei(x) - x^{-1}$, $z = x^{-1}$ and $Ei(x) = \int_{-\infty}^{x} t^{-1}e^{t}dt$, for $z = -0{\cdot}1(0{\cdot}01)0{\cdot}1$, by integrating the equation

$$T'' - (z^{-4} - 2z^{-3})T + z^{-2} = 0$$

What seems to be a slightly better choice of auxiliary function was adopted by Vickers[3] who tabulates $S(z)$ over the same range where

$$S(z) = xe^{-x}\,Ei(x)$$

(This corresponds in the general case to $\theta(z) = [\phi(x)]^{-1}$ and $S(z)$ satisfies the differential equation,

$$z^4 S'' - (2z^2 - 4z + 1)S = 3z - 1$$

though Vickers did not integrate this equation to obtain his values). Results displayed in reference 2 indicate that facilities for interpolation in $S(z)$ are better than for those in $T(z)$.

The mathematics in the above work is of a somewhat naive order but it is felt that a statement of the general case serves to indicate one possible line of attack in the tabulation of indefinite integrals, and may save a little of the time of someone who is engaged upon this problem.

I am indebted to Miss D.B. Catton for useful discussion of the above work.

References.

[1]. D.R. HARTREE, "Numerical Analysis", Oxford University Press, 1952, Ch. VII, p.126.

[2]. FOX L. and MILLER J.C.P., "Table Making for Large Arguments. The Exponential Integral". M.T.A.C. V.No.35, July 1951, p.163-167.

[3]. FOX L. and MILLER J.C.P., loc.cit., p.166.

ALGOL Programming.

CONTRIBUTION NO. 2.
SIMPSON NUMERICAL INTEGRATION WITH VARIABLE LENGTH OF STEP.
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```
procedure   simpson(a,b,h,ut,lt,fct,sum,g);
            value a,b,h,ut,lt;  real a,b,h,ut,lt,sum,g;
            real procedure fct;
comment     simpson computes the approximate value of the
            definite integral of fct(x) from a to b with
            automatic search for the appropriate length of
            step. Input parameters are: a and b, the lower
            and upper boundaries of the quadrature interval
            (a < b), h the proposed length of step between
            used function values, ut and lt, the upper and
            lower tolerance factors, fct the function. Out-
            put parameters are: sum the value of the inte-
            gral, g the average length of step. The proce-
            dure will usually give better accuracy than it
            is asked for but should nevertheless be used
            with discrimination;
begin       integer i; real x,F1,F2,F3,F4,F5,eps,p,q;
            Boolean ready ;
            i:=0 ; sum:=0 ; ready:=false; g:=h ; x:=a ;
            F1:=fct(x);
loopstart:  i:=i+1 ; if x + 4×g ≥ b then begin g:=(b-x)/4;
            ready:=true end;
            F5:=fct(x+4×g); F3:=fct(x+2×g);
innerloop:  F2:=fct(x+g); F4:=fct(x+3×g);
            p:= F1 + 4×F3 + F5; q:=p - 2×F3 + 4×(F2+F4);
            eps:= (2×p-q)×g/45 ;
            if abs(q×g×ut/3) > abs(eps) then go to nottobad;
            ready:=false ; g:=g/2 ; F5:=F3; F3:=F2;
            go to innerloop;
nottobad:   sum:=sum + q×g/3 - eps ;
            if ready then go to final ;
            x:=x + 4×g ;
            if abs(q×g×lt/3) ≥ abs(eps) then g:=2×g;
            F1:=F5 ; go to loopstart;
final:      g:= (b-a)/i/4

end;
```

The Program has been successfully tested using the ALGOL Compiler of FACIT Electronics, Gothenburg.