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MATHEMATICS

STRUCTURE OF GROWTH IN LINDENMAYER SYSTEMS

BY

PAUL M. B. VITÁNYI

(Communicated by Prof. A. VAN WIJNGAARDEN at the meeting of January 27, 1973)

SUMMARY

Growth of word length in some rewriting systems (*DOL's*) is investigated by combinatorial arguments concerning the structure of production trees of individual letters. Several growth types are distinguished and algorithms are obtained to classify letters, *DOL's* and semi *DOL's* in these types. It is shown that polynomial growth can not occur without accompanying limited growth. A conceptually easy characterization of the nature of the different growth types is given, yielding expressions for the slowest growth possible in each growth type.

1. INTRODUCTION

Lindenmayer systems or L-Systems are automata theoretic developmental models for filamentous growth arising from biological considerations [2]. An L-System consists of an initial one dimensional array of cells (a filament) symbolized by a word, and the subsequent stages of development are obtained by rewriting every letter of a word simultaneously at each time step. We shall be concerned with the case where the rewriting rules are deterministic and where a cell is not influenced by its neighbors (i.e. zero input). Such systems are called DOL-Systems. With each DOL-System we can associate a growth function f_G, where $f_G(t)$ is the length of the filament produced at time t. Growth functions were studied first by SZILARD [5], later by DOUCET [1], PAZ and SALOMAA [3] and SALOMAA [4]. In [4, section 2] exponential growth is shown to coincide with the occurrence of certain space-time patterns of letters in the sequence of produced words. Previously, in VITÁNYI [6] a similar technique was used to characterize DOL-Systems generating finite languages. Here we improve and extend the study of the structure of growth in DOL-Systems of [4] using the approach of [6].

2. DEFINITIONS

We shall customarily use, with or without indices, i, j, k, m, n, p, r, t to range over the set of natural numbers 0, 1, 2, ...; a, b, c, d, e to range over an alphabet W, and v, w, z to range over W^* i.e. the set of all words over W including the empty word λ. $|Z|$ denotes the cardinality or size of a set Z; $|z|$ the length of a word z, and $|\lambda|=0$. A *semi DOL-System* (semi *DOL*) is an ordered pair $S=\langle W, \delta\rangle$ where W is a finite nonempty *alphabet* and δ a total mapping from W into W^*. A pair $(a, \delta(a))$ is called a *production rule* and is also written as $a \to \delta(a)$. We extend δ to W^* by defining $\delta(\lambda)=\lambda$ and $\delta(a_1a_2 \dots a_m)=\delta(a_1)\delta(a_2) \dots \delta(a_m)$. δ^i is the composition of i copies of δ and is inductively defined by $\delta^0(v)=v$ and $\delta^i(v)= =\delta(\delta^{i-1}(v))$. A *DOL-System* (*DOL*) is a triple $G=\langle W, \delta, w\rangle$ where $S=\langle W, \delta\rangle$ is the underlying semi *DOL* and $w \in W^*\backslash\{\lambda\}$ is the *axiom*. The *DOL language* generated by G is $L(G)=\{\delta^i(w)|i\geqslant 0\}$. The *growth function* of G is defined by $f_G(t)=|\delta^t(w)|$. Clearly, if for a *DOL* $G=\langle W, \delta, w\rangle$ holds $m=\max\{|\delta(a)|\ |a\in W\}$ then $f_G(t)\leqslant m^t|w|$ for all t. Hence the fastest growth possible is exponentially bounded. The growth in a *DOL* G is *exponential* (type 3), *polynomial* (type 2), *limited* (type 1), *terminating* (type 0), if there is no polynomial $p(t)$ such that $f_G(t)\leqslant p(t)$ for all t, the growth is not exponential and there is no constant m such that $f_G(t)\leqslant m$ for all t, there is a constant m such that $0<f_G(t)\leqslant m$ for all t, $f_G(t)=0$ but for a finite number of initial arguments. Previously [3, 4 and 5], exponential and non-exponential growth have been termed, with biological connotations, malignant and normal growth. The presently used adjectives seem more elucidating in a mathematical context. The general form of f_G is given by $f_G(t)=\sum_{i=1}^n p_i(t)c_i^t$ where $p_i(t)$ is a polynomial and c_i ($c_i\neq c_j$ if $i\neq j$) a constant [cf. 1, 3 or 4]. Therefore, if f_G is not exponential, then f_G is indeed a polynomial. It is easily seen that if $G=\langle W, \delta, a_1a_2 \dots a_m\rangle$ then $f_G=f_{G_1}+f_{G_2} \dots +f_{G_m}$ where $G_i=\langle W, \delta, a_i\rangle$ for $1\leqslant i\leqslant m$. If we attribute to a letter a_i the growth type of G_i then the growth type of G is the highest numbered growth type of the letters in its axiom. We designate the growth type of a semi *DOL* $S=\langle W, \delta\rangle$ by $\chi_3\chi_2\chi_1\chi_0$ where $\chi_i=i$ if $G=\langle W, \delta, a\rangle$ is of type i for some $a\in W$ and $\chi_i=\emptyset$ otherwise.

Examples of semi *DOL* types.

type 321 $S_1=\langle\{a, b, c\}, \{a\to a^2b, b\to bc, c\to c\}\rangle$
type 31 $S_2=\langle\{a, b\}, \{a\to a^2b, b\to b\}\rangle$
type 3 $S_3=\langle\{a, b\}, \{a\to b, b\to ab\}\rangle$
type 21 $S_4=\langle\{a, b\}, \{a\to ab, b\to b\}\rangle$
type 1 $S_5=\langle\{a, b\}, \{a\to b, b\to b\}\rangle$
type 0 $S_6=\langle\{d\}, \{d\to\lambda\}\rangle$

We form the types 3210, 310, 30, 210, 10 by adding d and $d\to\lambda$ to the alphabets and production rules of S_1—S_5, respectively. The other possible combinations, i.e. 320, 32, 20, and 2 will be excluded by theorem 9.

3. Algorithms for determining growth types

We present simple algorithms for determining growth types of letters, *DOL*'s and semi *DOL*'s. Lemma 2 and theorem 3 plus corollaries are taken from [6]; theorem 6 is due to Salomaa [4].

Let $S=\langle W,\delta\rangle$ be a semi *DOL*. A letter $a\in W$ is *mortal* ($a\in M$) iff $\delta^i(a)=\lambda$ for some i; *vital* ($a\in V$) iff $a\notin M$; *recursive* ($a\in R$) iff $\delta^i(a)\in W^*\{a\}W^*$ for some $i>0$; *monorecursive* ($a\in MR$) iff $\delta^i(a)\in M^*\{a\}M^*$ for some $i>0$; *expanding* ($a\in E$) iff $\delta^i(a)\in W^*\{a\}W^*\{a\}W^*$ for some i. Clearly, if $a\in M$, R, MR then there is an i as above such that $i\leqslant|M|$, $|R|$, $|MR|$, respectively. A letter $a\in W$ is *accessible* to $G=\langle W,\delta,w\rangle$ iff $\delta^i(w)\in W^*\{a\}W^*$ for some i. We define an order relation $\leqslant$ on W by: $a\leqslant b$ iff there is an $i>0$ such that $\delta^i(b)\in W^*\{a\}W^*$. Clearly, $R=\{a|a\leqslant a\}$. The equivalence relation $\sim$ on R is defined by: $a\sim b$ iff $a\leqslant b$ & $b\leqslant a$. The relation $\sim$ induces a partition on R in equivalence classes and $R/\sim=\{[a_i]|b\in[a_i]$ iff $b\sim a_i\}$.

LEMMA 1. There is an algorithm to determine R and $R/\sim$ for a semi *DOL* $S=\langle W,\delta\rangle$.

PROOF. Define for each $a\in W$ a sequence of nested sets as follows

$$U_1(a)=\{b|\delta(a)\in W^*\{b\}W^*\}$$

$$U_{i+1}(a)=U_i(a)\cup\{b|\delta(c)\in W^*\{b\}W^* \ \&\ c\in U_i(a)\}.$$

By observing

(i) $U_i(a)\subseteq U_{i+1}(a)\subseteq W$ for all $i\geqslant 1$.
(ii) If $U_{k+1}(a)=U_k(a)$ for some k then $U_{k+j}(a)=U_k(a)$ for all j.

We obtain: there is a $k\leqslant|W|$ such that $U_{k+j}(a)=U_k(a)$ for all j. Denote $U_k(a)$ by $U(a)$. Clearly, $U(a)=\{b|b\leqslant a\}$. Since $R=\{a|a\in U(a)\}$ and $[a_i]=\{b|b\sim a_i\}=\{b|b\in U(a_i)\ \&\ a_i\in U(b)\}$ we have $R/\sim=\{[a_i]|a_i\in R\}\square$.

EXAMPLE. $S_7=\langle\{a,b,c,d\},\{a\to cd,\ b\to a^2bc,\ c\to c,\ d\to\lambda\}\rangle$

$U_1(a)=\{c,d\}$	$U_2(a)=U_1(a)$	$U(a)=\{c,d\}$
$U_1(b)=\{a,b,c\}$	$U_2(b)=U_1(b)\cup\{d\}=W$	$U(b)=W$
$U_1(c)=\{c\}$	$U_2(c)=U_1(c)$	$U(c)=\{c\}$
$U_1(d)=\emptyset$		$U(d)=\emptyset$

Hence $R=\{b,c\}$, $[b]=\{b\}$, $[c]=\{c\}$ and $R/\sim=\{\{b\},\{c\}\}$. A sequence $a_0,a_1,\ldots,a_k$, $k>0$, is called a *loop* of a recursive letter a iff $a_0=a_k=a$, a_{i+1} is a subword of $\delta(a_i)$ for $0\leqslant i<k$, and $a_j\neq a$ for $0<j<k$. Clearly, every recursive letter has at least one loop.

LEMMA 2. Let $S=\langle W,\delta\rangle$ be a semi *DOL* and $a\in W$ a monorecursive letter. Then there is exactly one loop $a_0,a_1,\ldots,a_k$ of a. Moreover, $a_i\neq a_j$ for $0\leqslant i<j<k$, $\{a_0,a_1,\ldots,a_{k-1}\}=[a]$ and for all t holds: $\delta^t(a)\in M^*\{a_i\}M^*$ where $i\equiv t \bmod k$.

THEOREM 3. Let $G=\langle W,\delta,w\rangle$ be a *DOL*. $L(G)$ is finite iff

$$\delta^{|W\setminus(R\cup M)|}(w)\in(M\cup MR)^*.$$

COROLLARY. A DOL language is finite iff all recursive letters which are accessible are monorecursive.

COROLLARY. The cardinality of a finite DOL language is determined by: $\text{l.c.m.}(k_1, k_2, \ldots, k_m) \leqslant |L(G)| \leqslant \text{l.c.m.}(k_1, k_2, \ldots, k_m) + |W \backslash R|$ where k_1, $k_2, \ldots, k_m$ are the lengths of the loops of the monorecursive letters in $\delta^{|W \backslash (R \cup M)|}(w)$.

LEMMA 4. A DOL-language $L(G)$ is finite iff the growth of G is limited or terminating.

PROOF. $\rightarrow$. Suppose $|L(G)| \leqslant n$. Then there are $j_1 < j_2 \leqslant n$ such that $\delta^{j_1}(w) = \delta^{j_2}(w)$. Since for all k $\delta^{k+j_1}(w) = \delta^k(\delta^{j_1}(w)) = \delta^k(\delta^{j_2}(w)) = \delta^{k+j_2}(w)$ we have $f_G(t) \leqslant \max\{|\delta^i(w)|\ |0 \leqslant i < j_2\}$. $\leftarrow$. Suppose the maximal word length in $L(G)$ is equal to m, then $|L(G)| \leqslant \sum_{i=0}^{m} |W|^i \square$.

THEOREM 5. There is an algorithm for determining for a semi DOL $S = \langle W, \delta \rangle$ whether $a \in W$ is mortal, recursive or monorecursive.

PROOF. (i) Construct $S' = \langle W, \delta' \rangle$ by, for all $a \in W$, substituting λ for b in $\delta(a)$ iff $\delta(b) = \lambda$. Construct S'' from S' in a similar way. Hence we obtain a sequence $S, S', S'', \ldots, S^{(k)}, S^{(k+1)}, \ldots$. If $S^{(k)} = S^{(k+1)}$, which must happen for some $k \leqslant |W|$ then $M = \{a | \delta^{(k)}(a) = \lambda\}$. Define $\bar{S} = \langle V, \bar{\delta} \rangle$ where $V = W \backslash M$ and $\bar{\delta}(a) = \delta^{(k)}(a)$ for all $a \in V$.

(ii) Determine R and $R/\sim$ by Lemma 1. (Applying lemma 1 to $\bar{S}$ instead of S saves work and gives the same result). By lemma 2 we have $MR = \cup \{[a] |$ if $b \in [a]$ then $|\bar{\delta}(b)| = 1\} \square$.

THEOREM 6. (Salomaa). The growth of a DOL $G = \langle W, \delta, w \rangle$ is exponential iff there is a letter $a \in W$ which is both accessible and expanding.

LEMMA 7. Let $S = \langle W, \delta \rangle$ be a semi DOL and $a \in W$. $a \in E$ iff $\delta^i(a) \in W^*[a]W^*[a]W^*$ for some i.

PROOF. $\leftarrow$. Since there is a j_1 such that $\delta^{j_1}(a) \in W^*\{a\}W^*[a]W^*$ or $\delta^{j_1}(a) \in W^*[a]W^*\{a\}W^*$ there is a j_2 such that $\delta^{j_2}(a)$ contains 3 occurrences of letters from $[a]$. By the same argument there is a j_3 such that $\delta^{j_3}(a)$ contains (at least) $k+1$ occurrences of letters from $[a]$, where $k = |[a]|$, and hence two occurrences of the same letter $b \in [a]$. Then there also exists a j_4 such that $\delta^{j_4}(a) \in W^*\{a\}W^*\{a\}W^*$. $\rightarrow$. Trivially true $\square$.

In [4] an algorithm is given to determine whether $a \in E$. By lemma 7 we can give an improved algorithm.

(i) Determine $R/\sim$ by lemma 1.
(ii) Replace in the production rules all $b \notin [a]$ by λ.
(iii) If there is a production rule $c \rightarrow v$ left such that $c \in W$ and $|v| \geqslant 2$ then $[a] \subseteq E$, and $[a] \cap E = \emptyset$ otherwise.

N.B. The algorithm works for OL-systems as well.

The greatest possible size of $U_k(a)$ is $|W|$ which is also an upper bound on k. (cf. lemma 1). Our construction of nested sets resembles the one used in [4] where, however, the greatest possible size of $U_k(a)$ is $(|W|^2+ +3|W|)/2$. Moreover, the construction of U_{i+1} out of U_i presents considerably more difficulties there.

To determine the growth type of a semi *DOL* $S=\langle W, \delta\rangle$ we now proceed as follows.

(a) By theorem 5 we determine M, R and MR.
(b) Determine E by the algorithm given above.
(c) $RM \overset{\text{def}}{=} R\backslash MR$ and $RME \overset{\text{def}}{=} \{a|a \in RM \ \&\ U(a) \cap E=\emptyset\}$.

From the foregoing it should be clear that $a \in W$ is of growth type 3 iff $U(a) \cap E \neq \emptyset$, of growth type 2 iff $U(a) \cap E=\emptyset \ \&\ U(a) \cap RM \neq \emptyset$, of growth type 1 iff $U(a) \cap (E \cup RME)=\emptyset \ \&\ U(a) \cap MR \neq \emptyset$, of growth type 0 iff $a \in M$, or equivalently, $U(a) \cap R=\emptyset$. We see that the growth type of a letter depends on the kind of accessible recursive letters. Therefore S is of growth type $\chi_3(E)\chi_2(RME)\chi_1(MR)\chi_0(M)$ where $\chi_i(\cdot)=i$ if $\cdot \neq \emptyset$ and $\chi_i(\cdot)=\emptyset$ otherwise.

Theorem 8. There is an algorithm to determine the growth type of a given semi *DOL*, letter of a semi *DOL*, or *DOL*.

Example continued.

$$S_7=\langle\{a, b, c, d\}, \{a \to cd, b \to a^2bc, c \to c, d \to \lambda\}\rangle$$

(a) (i) $S_7'=\langle\{a, b, c, d\}, \{a \to c, b \to a^2bc, c \to c, d \to \lambda\}\rangle=S_7''$.

Hence $M=\{d\}$, $V=\{a, b, c\}$, $\bar{\delta}=\delta'\backslash\{d \to \lambda\}$.

(ii) Previously, we saw $R=\{b, c\}$ and $R/\sim=\{\{b\}, \{c\}\}$. Since $|\bar{\delta}(b)|>1$ and $|\bar{\delta}(c)|=1$: $MR=\{c\}$ and $RM=\{b\}$.

(b) Substituting all letters $e \notin [b]$ by λ in $\bar{\delta}$ leaves a production $b \to b$, i.e. $b \notin E$. Therefore $E=\emptyset$.

(c) $RME \subseteq RM\backslash E=\{b\}$. Since $U(b) \cap E=\emptyset$ we have $RME=\{b\}$. The growth type of S_7 is given by $\chi_3(E)\,\chi_2(RME)\,\chi_1(MR)\,\chi_0(M)=210$.

Theorem 9. If $G=\langle W, \delta, a\rangle$ and $a \in RME$ then there is a letter $a' \in W$ which is both monorecursive and accessible to G.

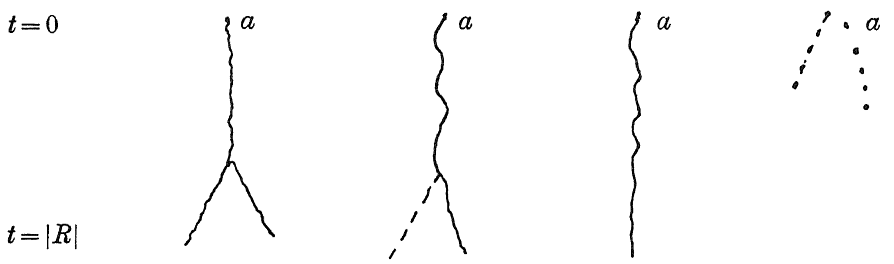
Proof. Suppose $a \in RME$ and no monorecursive letter is accessible to G. There is a $j_1 \leqslant |R|$ and a $b \in V$ such that $\delta^{j_1}(a)=v_1av_2bv_3$ or $\delta^{j_1}(a)= =v_1bv_2av_3$. Since every vital letter produces a recursive letter within $|V\backslash R|$ steps there is a $j_2 \leqslant |V|$, a letter $c \in R$ and a letter $d \in [a]$ such that $\delta^{j_2}(a)$ has c and d as subwords. Because of the assumption $c, d \in RME$. By iteration of the argument we have $|\delta^{n|V|}(a)| \geqslant 2^n$ for all n. But then $f_G(t) \geqslant 2^{\lceil t/|V|\rceil}$, where $\lceil r\rceil$ is the entier of r, which contradicts $a \in RME$ $\square$.

Corollary. If $RME \neq \emptyset$ then $MR \neq \emptyset$ and hence there do not exist semi *DOL*'s of type 320, 32, 20 and 2.

A biological interpretation of what we have just proven is, that we can have exponential (malignant) growth in *DOL*'s with of without accompanying limited or terminal growth, but we can not have polynomial growth occurring in an organism without encountering in the same organism portions with limited growth (in case the organism can be modelled by a *DOL*).

4. Growth type characterization and slow growth

We conclude with a conceptually simple characterization of the necessary and sufficient conditions that determine the growth type of a letter by depicting necessary and sufficient subtrees of the production trees (similar to the production trees of cf grammars) of letters of class E, RME, MR and M.



1a Exponential. 1b Polynomial. 1c Limited. 1d Terminating.

Solid, broken, dotted lines represent sequences of descendants b_i (of a) such that $b_i \in [a]$, $b_i \in V \backslash [a]$ & $U(b_i) \cap E = \emptyset$, $b_i \in M$, respectively. From this characterization it is easy to derive expressions for the slowest growth possible in each of the discussed growth types.

Theorem 10. There are *DOL*'s $G_i = \langle W, \delta_i, a \rangle$, $i = 0, 1, 2, 3$, such that $f_{G_3}(t) = 2^{\lceil t/|W| \rceil}$, $f_{G_2}(t) = \lceil 1 + t/(|W|-1) \rceil$, $f_{G_1}(t) = 1$, $f_{G_0}(0) = 1$ and $f_{G_0}(t) = 0$ for $t \geqslant 1$. For every *DOL* $G = \langle W, \delta, a' \rangle$ holds: if G is of growth type i then $f_G(t) \geqslant f_{G_i}(t)$ for all t, $i = 0, 1, 2, 3$.

Proof. Let $W = \{a_1, a_2, \ldots, a_p\}$ with $a_1 = a$.

$$\delta_3 = \{a_i \to a_{i+1} | 1 \leqslant i < p\} \cup \{a_p \to a_1 a_1\}$$
$$\delta_2 = \{a_i \to a_{i+1} | 1 \leqslant i < p-1\} \cup \{a_{p-1} \to a_1 a_p,\, a_p \to a_p\}$$
$$\delta_1 = \{a_i \to a_{i+1} | 1 \leqslant i < p\} \cup \{a_p \to a_1\}$$
$$\delta_0 = \{a_i \to \lambda | 1 \leqslant i \leqslant p\}$$

(Note that under δ_3 and δ_1 $W = R = [a]$, under δ_2 $W = R = [a] \cup [a_p]$ and under δ_0 $R = \emptyset$).

(i) The growth in G is exponential. According to fig. 1a $f_G(t) \geqslant 2^{\lceil t/k \rceil}$ where $k \leqslant |[a]|$ for some $a \in R$. Clearly, $k \leqslant |W|$.

(ii) The growth in G is polynomial. According to fig. 1b there must be a loop $a_1, a_2, \ldots, a_k$ with $\delta(a_i) = w_1 b w_2 a_{i+1} w_3$, where $b \in V \backslash [a]$, for some $i \in \{1, 2, \ldots, k-1\}$, in the production tree of a. Since $f_G(0) = 1$ we have $f_G(t) \geqslant \lceil 1 + t/k \rceil$ with $k \leqslant |W \backslash \{b\}|$.

(iii) The growth in G is limited or terminating: trivial $\square$.

COROLLARY. Let $G = \langle W, \delta, w \rangle$ be a DOL and let n_3, n_2, n_1, n_0 be the number of occurrences of letters of growth type 3, 2, 1, 0, respectively, in w.

$n_3 2^{\lceil t/|W| \rceil} + n_2 \lceil 1 + t/(|W|-1) \rceil + n_1 + n_0 \varrho \leqslant f_G(t)$ for all t where $\varrho = 1$ for $t = 0$ and $\varrho = 0$ otherwise.

$f_G(t) < n_3 m^t + n_2 2^{\lceil t/|W| \rceil} + n_1 r + n_0 + 1$ for all $t \geqslant t_0$ for some t_0 and r, where $m = \max \{|\delta(b)| \mid b \in W\}$.

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