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# ON A PROBLEM IN THE COLLECTIVE BEHAVIOR OF AUTOMATA*)

by

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ABSTRACT

Varshavsky defines the function L(n) as the maximum finite length of a configuration which can be grown from one activated automaton in a linear cell space of identical automata having n internal states. It is shown that L increases faster than any computable function, even if the flow of information in the linear cell space is restricted to one direction.

KEYWORDS & PHRASES: automata theory, cellular automata, linear cell spaces, stabilizing patterns, decidability.

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Varshavsky (1972) posed the following problem. "The following example of collective behavior of automata is related to problems in the behavior of interacting automata such as the 'French Flag' problem, the 'Firing Squad Synchronization' problem, and so on. A characteristic feature of these problems is that a combination of locally simple automata can solve problems which are in principle beyond the capability of any one of them separately. For example, in the solution of the Firing squad synchronization problem an automaton with eight internal states delays the signal for a time $2^{m+1} - 1$, which is in principle impossible for an isolated automaton for $m > 2$. In an example considered below we shall try to show that there are very great possibilities in collections of comparatively simple automata. For this purpose we turn to cell models of the growth of figures.

Suppose we have a linear cell space; this means that we have an infinite chain of automata which are in a passive state. Each automaton interacts with its neighbors, that is, the numbers of the internal states of its right and left neighbors act as inputs to the automata. If an automaton and its two neighbors are in a passive state, then they remain like this, and a chain all of whose automata are passive will remain passive indefinitely. If an external signal puts one of the automata into an active state then it begins to act on its neighbors. Thus a sequence of automata in active states may arise in the chain. Such a connected sequence will be called a configuration. It is easy to see that one activated automaton may 'grow' a configuration of infinite length. Here we meet the problem of stopping the growth process in the following form: what is the maximum finite length L(n) of a configuration which can be grown from one activated automaton in a linear cell space of identical automata having n internal states? ... By completely enumerating all possible tables of transition rules it has been shown that for $n = 3$, the maximal length $L(3) = 7$. For $n = 4$ transition rules have been found giving $L(n) \geq 45$ but this length has not been shown to be maximal. (We assume here that an automaton which has been activated cannot revert to the passive state, that is, no 'break' in the configuration is allowed during the growth process.) We now turn to a universal procedure which will ensure that the process stops for large n." Vashavsky then proceeds to derive a very fastly increasing computable function which is a lower bound on L. We shall show that even for a restricted version of Varshavsky's problem, where each automaton in the chain receives input from its left neighbor only, there is no computable upper bound on L, that is, L increases faster than any computable function.

Define a one directional linear cell space (1LCS) as a 4 tuple $C = \langle W_C, \delta_C, w_C, ¢\rangle$ where $W_C$ is a finite nonempty alphabet and ¢ is a distinguished letter in $W_C$

called the passive letter; $\delta_C$ is a total mapping from $W_C \times W_C$ into $W_C$ such that $\delta_C(¢, ¢) = ¢$ and $\delta_C(a, b) \neq ¢$ for all $(a, b) \in W_C \times (W_C - \{¢\})$; $w_C \in (W_C - \{¢\})(W_C - \{¢\})^*$ is called the initial configuration.

We imagine $C$ as operating on an infinite string $¢^\infty w_C^{(t)} ¢^\infty$ over $W_C$, all the constituent letters of which are ¢'s except for a finite substring $w_C^{(t)}$ over $W_C - \{¢\}$ called the configuration at time $t$. $C$ produces an infinite sequence of configurations $w_C^{(0)}, w_C^{(1)}, \ldots$ as follows. The string at time $t = 0$ is $¢^\infty w_C^{(0)} ¢^\infty$ where $w_C^{(0)} = w_C$. If $w_C^{(k)} = a_1 a_2 \ldots a_n$ is the configuration at time $t = k$ then $w_C^{(k+1)}$ is the configuration at time $t = k + 1$ where $w_C^{(k+1)}$ is defined by

$$¢^\infty w_C^{(k+1)} ¢^\infty = ¢^\infty \delta_C(¢, a_1) \delta_C(a_1, a_2) \ldots \delta_C(a_{n-1}, a_n) \delta_C(a_n, ¢) ¢^\infty.$$

The next thing we need is the notion of a Tag system. A Tag system $T$ is a 4 tuple $T = \langle W_T, \delta_T, w_T, \beta \rangle$ where $W_T$ is a finite nonempty alphabet; $\delta_T$ is a total mapping from $W_T$ into $W_T^*$, $w_T \in W_T W_T^*$ is the initial string and $\beta$ is a natural number called the deletion number. The operation of a Tag system is inductively defined as follows. The string produced at time $t = 0$ is $w_T^{(0)} = w_T$. If $w_T^{(k)} = a_1 a_2 \ldots a_n$ is the string produced at time $t = k$ then $w_T^{(k+1)} = a_{\beta+1} a_{\beta+2} \ldots a_n \delta_T(a_1)$ is the string produced at time $t = k + 1$.

Lemma 1. (Minsky (1967)). Let $k$ be a natural number. It is undecidable whether or not an arbitrary Tag system with deletion number 2 will ever produce a string of length less than or equal to $k$. In particular this is undecidable for $k = 0$.

We shall now proceed to show that if there is a computable function $f$ such that $L(n) \leq f(n)$ for all $n$ then this contradicts Lemma 1.

Lemma 2. Let $T$ be any Tag system with deletion number 2. There is an algorithm which, given $T$, produces 1LCS $C$ such that there is a time $t_0$ such that $w_C^{(t)} = w_C^{(t_0)}$ for all $t \geq t_0$ iff there is a time $t_0'$ such that $w_T^{(t_0')} = \lambda$.

Proof. Let $T = \langle W_T, \delta_T, w_T, 2\rangle$ and let $m = \max\{lg(\delta_T(a)) \mid a \in W_T\}$ where $lg(v)$ denotes the length of a word $v$. Construct $C = \langle W_C, \delta_C, w_C, ¢\rangle$ as follows.

$$W_C = W_T \cup W_T' \cup (W_T \times \bigcup_{i=0}^{m} W_T^i) \cup \{\$, ¢\}, \quad \text{where}$$

$W_T' = \{\underset{\sim}{a} \mid a \in W_T\}$ and $W_T, W_T', \{\$, ¢\}$ are disjoint;

$\delta_C(¢, a) = \delta_C(\$, a) = \delta_C(\$, (a, v)) = \underset{\sim}{a}$ for all $a \in W_T$ and all $v \in \bigcup_{i=0}^{m} W_T^i$,

$\delta_C(¢, \underset{\sim}{a}) = \delta_C(\$, \underset{\sim}{a}) = \delta_C(¢, \$) = \delta_C(\$, \$) = \$$ for all $\underset{\sim}{a} \in W_T'$,

$\delta_C(a, ¢) = ¢$ for all $a \in W_T \cup \{¢, \$\}$,

$$\delta_C(a,\ b) = \delta_C(\underset{\sim}{a},\ (b,\ v)) = \delta_C(a,\ (b,\ v)) = b \quad \text{for all } a, b \in W_T \text{ and all } v \in \bigcup_{i=0}^{m} W_T^i,$$

$$\delta_C(\underset{\sim}{b},\ c) = \delta_C((a,\ \delta_T(b)),\ c) = (c,\ \delta_T(b)) \quad \text{for all } a,\ b,\ c \in W_T,$$

$$\delta_C(\underset{\sim}{b},\ ¢) = (a_1, a_2a_3\ldots a_n) \quad \text{if } \delta_T(b) = a_1a_2\ldots a_n,\ n > 0,\ \text{for all } b \in W_T,$$

$$= ¢ \quad \text{if } \delta_T(b) = \lambda, \quad \text{for all } b \in W_T,$$

$$\delta_C((a,\ a_1a_2\ldots a_n),\ ¢) = (a_1,\ a_2a_3\ldots a_n) \quad \text{for all } a \in W_T \text{ and all } a_1a_2\ldots a_n \in \bigcup_{i=1}^{m} W_T^i,$$

$$= ¢ \quad \text{for all } a \in W_T \text{ and all } a_1a_2\ldots a_n = \lambda;$$

$w_C = w_T$.

(The arguments for which $\delta_C$ is not defined shall not occur in our operation of C.) A sample derivation is (assuming that $w_T = a_1a_2a_3a_4a_5$ and $\delta_T(a_1) = b_1b_2\ldots b_n$):

| T | C |
|---|---|
| $a_1a_2a_3a_4a_5$ | $\ldots\ ¢a_1a_2a_3a_4a_5¢\ \ldots$ |
| $a_3a_4a_5\delta_T(a_1)$ | $\ldots\ ¢\underset{\sim}{a_1}a_2a_3a_4a_5¢\ \ldots$ |
| $a_5\delta_T(a_1)\delta_T(a_3)$ | $\ldots\ ¢\$(a_2,\ \delta_T(a_1))a_3a_4a_5¢\ \ldots$ |
| $\ldots.$ | $\ldots\ ¢\$\underset{\sim}{a_2}(a_3,\ \delta_T(a_1))a_4a_5¢\ \ldots$ |
| | $\ldots\ ¢\$\$a_3(a_4,\ \delta_T(a_1))a_5¢\ \ldots$ |
| | $\ldots\ ¢\$\$\underset{\sim}{a_3}a_4(a_5,\ \delta_T(a_1))¢\ \ldots$ |
| | $\ldots\ ¢\$\$\$(a_4,\ \delta_T(a_3))a_5(b_1,\ b_2b_3\ldots b_n)¢\ \ldots$ |
| | $\ldots\ ¢\$\$\$\underset{\sim}{a_4}(a_5,\ \delta_T(a_3))b_1(b_2,\ b_3b_4\ldots b_n)¢\ \ldots$ |
| | $\ldots.$ |

In the simulating 1LCS signals depart from the left, with distances of one letter in between, and travel to the right at an equal speed of one letter per time step. Therefore the signals cannot clutter up. It is clear that if the Tag system T derives the empty word at time $t'_0$ then there is a time $t_0$ such that $w_C^{(t_0)} = \$^k$ for some k, and $w_C^{(t)} = w_C^{(t_0)}$ for all times $t \geq t_0$. Conversely, the only way for $w_C^{(t)}$ to stop growing is to produce a configuration of the form $\$^k$, i.e. there is a time $t'_0$ such that $w_T^{(t'_0)} = \lambda$. If the configurations produced by C always contain letters other than \$, i.e. T never produces $\lambda$, then at each second time step there appears a new occurrence of \$ and the configuration grows unbounded. Therefore T derives the empty word iff there is a time $t_0$ such that $w_C^{(t)} = w_C^{(t_0)}$ for all times t such that $t \geq t_0$. ■

Now it is easy to see that if $T = \langle W_T, \delta_T, w_T, 2\rangle$ is a Tag system then $T' = \langle W_T \cup \{s\}, \delta_T \cup \{\delta_{T'}(s) = w_T\}, s, 2\rangle$, $s \notin W_T$, is a Tag system such that $w_{T'}^{(t+1)} = w_T^{(t)}$ for all $t > 0$. Therefore Lemma 1 also holds if we restrict our attention to Tag systems with deletion number 2 and an initial string of one letter and disregard the length of the initial string with respect to k.

*Theorem*. There is no computable function $f$ such that $L(n) \leq f(n)$ for all $n$. I.e. $L(n)$ grows faster than any computable function.

*Proof*. Suppose there were such a function $f$. By Lemma 2 and the subsequent discussion this contradicts Lemma 1, i.e. it would imply the decidability of the halting problem for Tag systems which is known to be undecidable. ■

Finally we might point out that Varshavsky's original problem can be shown to be equivalent to the halting problem for Turing machines by encoding the finite control and the scanned symbol in each cell of the linear cell space.

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