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On the non-vanishing terms in a product of multivariate polynomials

by

J. van Leeuwen

ABSTRACT. Let terms be expressions of the form $x_0^{i_0} \ldots x_n^{i_n}$, all $i_j \in Z$. With each finite set of terms T a convex polyhedron in Q^{n+1} will be associated and we shall prove that a minimal subset of T spanning the polyhedron must be unique. Extending a technique of KIRKPATRICK we show that the polyhedron of $\{s_0,\ldots,s_\ell\} \times \{t_0,\ldots,t_m\}$ must be identical to the polyhedron spanned by the non-vanishing terms of $(\alpha_0 s_0+\ldots+\alpha_\ell s_\ell) \cdot (\beta_0 t_0+\ldots+\beta_m t_m)$, for non-zero α's and β's. It follows that the elements of the largest set of convexly independent terms in $\{s_0,\ldots,s_\ell\} \times \{t_0,\ldots,t_m\}$ must always appear in $(\alpha_0 s_0+\ldots+\alpha_\ell s_\ell) \cdot (\beta_0 t_0+\ldots+\beta_m t_m)$, no matter how the non-zero α's and β's are choosen.

1. Let $x_0,\ldots,x_n$ be n+1 independent variables, and let terms be expressions of the form $x_0^{i_0} \ldots x_n^{i_n}$, all $i_j \in Z$.

In products like

$$(\alpha x_0+x_1+x_0x_1)(1-x_0+x_0^2) =$$

$$= \alpha x_0^3 - \alpha x_0^2 + \alpha x_0 + x_0^3 x_1 + x_1$$

we call $x_0^3, x_0^2, \ldots$ *appearing* or *non-vanishing* terms and x_0x_1 and $x_0^2x_1$ *dis-appearing* or *vanishing* terms.

In a study of the minimal number of additions and subtractions needed to compute certain functions KIRKPATRICK [1] showed that the maximal number of terms in $\{s_0,\ldots,s_\ell\} \times \{t_0,\ldots,t_m\}$ that are in a well-defined sense rationally independent must be the same as the maximal number of such terms appearing in $(\alpha_0 s_0+\ldots+\alpha_\ell s_\ell) \cdot (\beta_0 t_0+\ldots\beta_m t_m)$, even though in the latter expression many terms may cancel.

In this paper we shall consider some further mathematical aspects of the question how vanishing terms in products of multi-variate polynomials may be characterized and extend KIRKPATRICK's result in finding an explicit maximal set of terms in $\{s_0,\ldots,s_\ell\} \times \{t_0,\ldots,t_m\}$ which must always occur in $(\alpha_0 s_0+\ldots+\alpha_\ell s_\ell) \cdot (\beta_0 t_0+\ldots+\beta_m t_m)$, no matter how non-zero α's and β's are choosen.

It appears that in products $(\alpha_0 s_0+\ldots+\alpha_\ell s_\ell)(\beta_0 t_0+\ldots+\beta_m t_m)$ the convex hull of points $\langle i_0,\ldots,i_n\rangle$ such that $x_0^{i_0} \ldots x_n^{i_n}$ is an appearing term is invariant and equal to the corresponding convex polyhedron associated with the *full* set $T = \{s_0,\ldots,s_\ell\} \times \{t_0,\ldots,t_m\}$. We shall prove that the minimal subset of T spanning the polyhedron is unique. It follows that all elements of the largest set of convexly independent terms in T must always appear in products $(\alpha_0 s_0+\ldots+\alpha_\ell s_\ell) \cdot (\beta_0 t_0+\ldots+\beta_m t_m)$ for any non-zero α's and β's.

2. Let $T = \{t_0,\ldots,t_m\}$ be a set of terms, $m \geq 1$.

Each term $t = x_0^{i_0} \ldots x_n^{i_n}$ canonically corresponds to a point $\emptyset(t) = \langle i_0,\ldots,i_n\rangle$ in Q^{n+1}. We shall usually wish to identify T and $\emptyset(T) = \{\emptyset(t_0),\ldots,\emptyset(t_m)\}$

The convex hull of $\emptyset(T)$ in Q^{n+1} is the closed, bounded region of all points x which can be written as

$$x = \lambda_0 \emptyset(t_0) + \dots + \lambda_m \emptyset(t_m)$$

with $\lambda_i \geq 0$, $\lambda_i \in Q$, and $\sum_0^m \lambda_i = 1$. We shall sometimes refer to it as "the convex hull of T".

When $t_j \in T$ is in the convex hull of $T - \{t_j\}$, then the convex hull of T and $T - \{t_j\}$ coincide. Continuing in this way eliminating terms from T we find a minimal subset of T spanning the convex polyhedron that we associated with it.

<u>DEFINITION</u> Terms $t_{i_0}, \dots, t_{i_k}$ are called convexly independent when no one of them belongs to the convex hull of the others.

The definition is equivalent to calling t convexly dependent on $t_{i_0}, \dots, t_{i_k}$ when there exist integers $\alpha > 0$ and $\beta_0, \dots \beta_k \geq 0$ such that $t^\alpha = t_{i_0}^{\beta_0} \dots t_{i_k}^{\beta_k}$ and $\alpha = \sum_0^k \beta_i$.

Clearly a minimal subset of T which spans the convex hull of T must consist of convexly independent terms. It is not clear that such a subset will be a simplicial basis and a somewhat more "careful" proof is needed to get the result that the minimal spanning subset of T which we found is in fact unique.

<u>THEOREM 2.1</u>. *Each finite set of terms* T *contains a minimal subset spanning the convex hull of* T, *and this minimal subset is unique.*

PROOF. Let $\{u_0, \dots, u_p\} \in T$ and $\{v_0, \dots, v_q\} \in T$ be distinct minimal spanning subsets. Without loss of generality we may assume that $p \geq q$.

Let $j \leq q$ and the ordering of u's and v's be such that $u_0 = v_0, \ldots, u_j = v_j$ ($j=-1$ permitted) but $u_{j+1}, \ldots, u_p \notin \{v_0, \ldots, v_q\}$ and $v_{j+1}, \ldots, v_q \notin \{u_0, \ldots, u_p\}$.

Since the sets were assumed to be distinct it follows that $j < p$.

Consider u_{j+1}. $\emptyset(u_{j+1})$ must be convexly dependent on $\{v_0, \ldots, v_q\}$ and $\lambda_0, \ldots, \lambda_q$ exist such that

$$\emptyset(u_{j+1}) = \lambda_0 \, \emptyset(v_0) + \ldots + \lambda_q \, \emptyset(v_q)$$

with $\lambda_i \geq 0$, $\lambda_i \in Q$, and $\sum_0^q \lambda_i = 1$. When $j = q$ or $j < q$ and $\lambda_{j+1} = \ldots = \lambda_q = 0$ then $\emptyset(u_{j+1})$ would be convexly dependent on $\emptyset(u_0), \ldots, \emptyset(u_j)$ contradicting the fact that $\{u_0, \ldots, u_p\}$ is minimal. It follows that $j < q$ and at least one of the $\lambda_{j+1}, \ldots, \lambda_q$ must be > 0.

Consider

$$\emptyset(u_{j+1}) = \lambda_0 \, \emptyset(v_0) + \ldots + \lambda_{j+1} \, \emptyset(v_{j+1}) + \ldots + \lambda_q \, \emptyset(v_q)$$

Now recall that $v_{j+1}, \ldots, v_q$ are convexly dependent on $u_0, \ldots, u_p$ and there exist non-negative $u_{j+1\,0}, \ldots, u_{j+1\,p}, \ldots, u_{q\,0}, \ldots, u_{q\,p} \in Q$ such that

$$\emptyset(v_k) = u_{k0} \, \emptyset(u_0) + \ldots + u_{kp} \, \emptyset(u_p)$$

and

$$\sum_{\ell=0}^{p} u_{k\ell} = 1$$

for all $j+1 \leq k \leq q$

It all combines into

$$\emptyset(u_{j+1}) = \theta_0 \, \emptyset(u_0) + \ldots + \theta_p \, \emptyset(u_p)$$

where $\theta_i \geq 0$, $\theta_i \in Q$, and $\sum_0^p \theta_i = 1$ and in particular

$$\theta_k = \lambda_k + \sum_{j+1}^{q} \lambda_\ell \, u_{\ell k} \quad \text{for} \quad 0 \le k \le j$$

and

$$\theta_k = \sum_{j+1}^{q} \lambda_\ell \, u_{\ell k} \qquad \text{for} \quad j+1 \le k \le p$$

Consider θ_{j+1} and assume that $\theta_{j+1} \neq 1$. Then we obtain

$$(1-\theta_{j+1})\, \emptyset(u_{j+1}) = \theta_0\, \emptyset(u_0) + \ldots + \theta_j\, \emptyset(u_j) + \theta_{j+2}\, \emptyset(u_{j+2}) + \ldots + \theta_p\, \emptyset(u_p)$$

and hence

$$\emptyset(u_{j+1}) = \frac{\theta_0}{1-\theta_{j+1}}\, \emptyset(u_0) + \ldots + \frac{\theta_j}{1-\theta_{j+1}}\, \emptyset(u_j) + \frac{\theta_{j+2}}{1-\theta_{j+1}}\, \emptyset(u_{j+2}) + \ldots +$$

$$+ \frac{\theta_p}{1-\theta_{j+1}}\, \emptyset(u_p)$$

Since all "coefficients" are non-negative rational numbers and in particular

$$\sum_{i \neq j+1} \frac{\theta_i}{1-\theta_{j+1}} = \frac{1}{1-\theta_{j+1}} \cdot \sum_{i \neq j+1} \theta_i = 1$$

it follows that u_{j+1} is convexly dependent on $\{u_0,\ldots,u_p\} - \{u_{j+1}\}$, a contradiction.

Therefore $\theta_{j+1_q} = \lambda_{j+1}\, u_{j+1\; j+1} + \ldots + \lambda_q\, u_{q\; j+1}$ must equal 1.

Since $\theta_{j+1} \le \sum_{j+1} \lambda_\ell \cdot \max u_{\ell\; j+1} \le \max u_{\ell\; j+1}$ this can only happen when $\max u_{\ell\; j+1} = 1$.

Hence there is a $j+1 \le \ell \le q$ such that $u_{\ell\; j+1} = 1$ (and therefore all $u_{\ell i}=0$ for $i \neq j+1$), and $\emptyset(v_\ell) = \emptyset(u_{j+1})$. This is again a contradiction since we assumed that $u_{j+1} \notin \{v_0,\ldots,v_q\}$.

We conclude that $\{u_0,\ldots\}$ and $\{v_0,\ldots\}$ cannot be distinct.

It is not hard to re-state 2.1. in the more classical terms of combinatorial topology to show that any finite set of points spanning a polyhedron contains a unique subset which are the extreme points of the polyhedron. The theory of convex polyhedra and their extreme points was much further developed in the past.

COROLLARY 2.2. *Each finite set of terms* $T = \{t_0,\ldots,t_m\}$ $(m \geq 1)$ *contains a unique, largest subset of convexly independent terms.*

3. Let both $S = \{s_0,\ldots,s_\ell\}$ and $T = \{t_0,\ldots,t_m\}$ $(\ell,m \geq 1)$ be finite sets of terms.

One may very well determine non-zero coefficients α_i and β_j such that in the expression for $(\alpha_0 s_0+\ldots+\alpha_\ell s_\ell) \cdot (\beta_0 t_0+\ldots+\beta_m t_m)$ many terms cancel. Note that "vanishing" terms are by definition those elements of $S \times T$ which do not show up anymore in the product-expression.

We claim that vanishing terms must always belong to the convex polyhedron that is associated with the non-vanishing terms in $(\alpha_0 s_0+\ldots+\alpha_\ell s_\ell)$ $(\beta_0 t_0+\ldots+\beta_m t_m)$, no matter what the α's and β's are.

As an example let us again consider

$$(\alpha x_0+x_1+x_0x_1)(1-x_0+x_0^2) =$$
$$= \alpha x_0^3 - \alpha x_0^2 + \alpha x_0 + x_0^3x_1 + x_1$$

Observe that for the vanishing terms x_0x_1 and $x_0^2x_1$ we indeed have:

$$\emptyset(x_0x_1) = \tfrac{1}{3}\,\emptyset(x_0^3x_1) + \tfrac{2}{3}\,\emptyset(x_1)$$
$$\emptyset(x_0^2x_1) = \tfrac{2}{3}\,\emptyset(x_0^3x_1) + \tfrac{1}{3}\,\emptyset(x_1)$$

The proof that such a fact holds in general follows a technique which is exhibited in a related, but weaker result of KIRKPATRICK [1]. Since we need different termination-conditions in the construction and no completely satisfactory proof of the non-degeneracy of the technique has yet appeared we describe the process in detail.

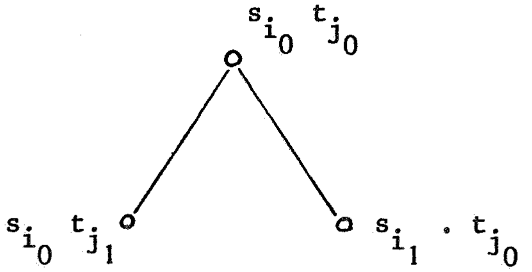
<u>THEOREM 3.1</u>. *All vanishing terms in a product* $(\alpha_0 s_0+\ldots+\alpha_\ell s_\ell)\cdot$ $\cdot(\beta_0\gamma_0+\ldots+\beta_m t_m)$ *are convexly dependent on the non-vanishing terms.*

PROOF. It will be convenient to call a term t convexly expressible (rather than just convexly dependent) in $u_0,\ldots,u_p$ when there exist $\lambda_i > 0$, $\lambda_i \in Q$, such that $\emptyset(t) = \lambda_0\,\emptyset(u_0) + \ldots + \lambda_p\,\emptyset(u_p)$ and $\sum_0^p \lambda_i = 1$ (i.e., all u's are really used).

Let $s_{i_0}\,t_{j_0}$ be a vanishing term in the product. The idea is to start from $s_{i_0}\,t_{j_0}$ and generate more and more "other" product-terms until we hit some which are non-vanishing in the expression.

When $s_{i_0}\,t_{j_0}$ is vanishing there must be another term $s_{i_1}\,t_{j_1}$ (with $i_0 \neq i_1$, and $j_0 \neq j_1$) such that $s_{i_0}\,t_{j_0} = s_{i_1}\,t_{j_1}$. (There may be several with varying real coefficient in the expression, but this is the only condition under which they eventually cancel).

Consider the terms $s_{i_0}\cdot t_{j_1}$ and $s_{i_1}\cdot t_{j_0}$, and observe that it are indeed new terms, i.e., distinct from $s_{i_0}\cdot t_{j_0}$ and distinct from each other. Let us graphically represent the "splitting" of $s_{i_0}\,t_{j_0}$ as the beginning of a tree:



and observe that $\emptyset(s_{i_0} t_{j_0}) = \frac{1}{2} \emptyset(s_{i_0} t_{j_1}) + \emptyset(s_{i_1} t_{j_0})$.

The idea is to continue in a similar manner with the sons of $s_{i_0} \cdot t_{j_0}$ and to grow a big tree according to specific rules (in fact here we're going to follow a somewhat different approach then KIRKPATRICK, although the technique is the same).

Certain nodes at the frontier ("bottom") of the tree will be turned into leaves ("terminal nodes") while remaining nodes in the frontier are called *blossoms*. Only blossoms can be split further and the tree only grows at its frontier. The following algorithm makes precise how the tree is generated and when blossoms are to be split further or turned into leaves. The algorithm starts with

$s_{i_0} t_{j_0}$ 

as a single node (which at the same time is the only blossom).

Step 1. if there are no blossoms anymore then goto step 5 else let $s_i t_j$ be a blossom.

Step 2. if $s_i \cdot t_j$ is a term appearing in the product then turn it into a leaf and go back to step 1

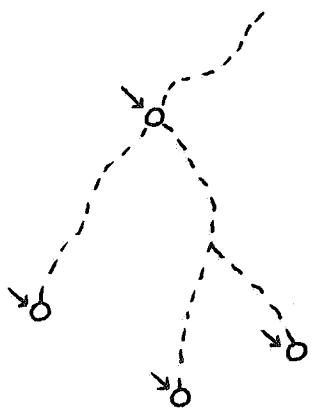
Step 3. if $s_i \cdot t_j$ as algebraic expression is appearing before on the path from the current blossom back to the root then turn it into a leaf and go back to step 1 else $s_i t_j$ is a vanishing term and one can find a term $s_k \cdot t_\ell$ $(k \neq i$ and $\ell \neq j)$ such that $s_i t_j = s_k t_\ell$

Step 4. Split the node $s_i t_j$ (which is then no blossom anymore) into blossoms $s_i t_\ell$ and $s_k t_j$, and return to step 1.

Step 5. halt

Because of the precautions in step 3 for immediately pruning at repeated nodes there can be no path of length > $\#(S\times T) + 1$. The tree is therefore finite and the algorithm terminates.

When a blossom is turned into a leaf because it is a repeated term, then it is easily seen that there is precisely one internal node on the path back to the root where the term occurs. Any internal node which is "distinguished" in this way will be called a *repeater*. Note that it may very well happen that the term at a repeater occurs at various leaves along different downward paths in the subtree:

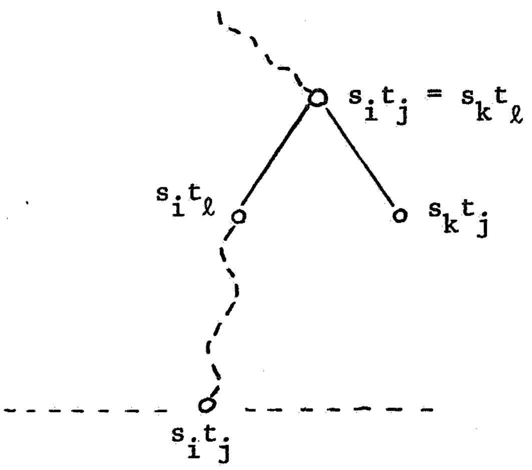


When one repeater is an ancestor of the other, they necessarily carry distinct terms.

Observe that each (term at a) node is convexly expressible in the direct sons, and by an inductive argument it follows that the root of the tree $(s_{i_0} \cdot t_{j_0})$ must be convexly expressible in the leaves of the final tree. Recall that all leaves are used!

We shall prove that the tree must have leaves that are appearing terms and then argue how all leaves that are "repeated" (and therefore vanishing) terms may all be convexly expressed in such appearing terms.

Let us first search for a "lowest" repeater, say it is carrying the term $s_i t_j$:



$s_i t_j$ must have been split at least once, and repeated occurrences of the term appear at a non-trivial distance from the node. $s_i t_j$ is convexly expressible in the leaves of its sub-tree $v_0, \ldots, v_q$ and integers $\alpha, \gamma_0, \ldots, \gamma_q > 0$ exist such that

$$(s_i t_j)^\alpha = (s_i t_j)^\beta \; v_{i_0}^{\gamma_{i_0}} \ldots v_{i_r}^{\gamma_{i_r}}$$

$$\text{all } v_{i_j} \neq s_i t_j$$

with $r \geq -1$ and $\beta > 0$, $\alpha = \beta + \gamma_{i_0} + \ldots + \gamma_{i_r}$.

Suppose $\alpha = \beta$. Then <u>all</u> leaves in the subtree would have been equal to $s_i t_j$, and in particular a direct son like $s_k t_j$ would be convexly expressible in $s_i t_j$ (and thus $s_k t_j = s_i t_j$), a contradiction.

Hence $\alpha > \beta$ and $r \geq 0$ and there is a non-degenerate expression

$$(s_i t_j)^{\alpha-\beta} = v_{i_0}^{\gamma_{i_0}} \ldots v_{i_r}^{\gamma_{i_r}}$$

showing that in the subtree there are other leaves than just repeated occurrences of $s_i t_j$, and that $s_i t_j$ is convexly expressible in these remaining leaves.

Leaves that carry $s_i t_j$ (as algebraic expression) may therefore very well be colored, and we can say that any term appearing in the tree is convexly expressible in non-colored leaves.

In exactly the same manner we can continue coloring leaves, first (as we did) the leaves corresponding to lowest repeaters, then the leaves corresponding to next-to-lowest, second-to-lowest repeaters and so on. At each level the same argument as given above applies to show that in the subtree of the next-level repeater there must be un-colored leaves which are distinct from the repeater-term, and the further coloring of leaves can never degenerate. When the coloring-procedure ends all "repeated term" leaves have been colored, but when considering the last coloring-step it follows that there are still uncolored leaves left!

These leaves have to carry non-repeated, and therefore non-vanishing terms (otherwise they would have been split), and all colored leaves are by induction convexly expressible in non-vanishing terms.

In particular it follows that the root $s_{i_0} t_{j_0}$ must be convexly expressible un-colored leaves and it is therefore convexly dependent on the non-vanishing terms in the product-expression.

From 3.1 we conclude that the convex polyhedron spanned by the non-vanishing terms in a product $(\alpha_0 s_0+\ldots+\alpha_\ell s_\ell)(\beta_0 t_0+\ldots+\beta_m t_m)$ is invariant when $\alpha_0,\ldots,\alpha_\ell$ and $\beta_0,\ldots,\beta_m$ range overall non-zero scalar values, and is always identical to the convex polyhedron of $S \times T$.

<u>THEOREM 3.2</u>. *The elements of the largest set of convexly independent terms in* $\{s_0,\ldots,s_\ell\} \times \{t_0,\ldots,t_m\}$ *always appear among the non-vanishing terms of a product* $(\alpha_0 s_0+\ldots+\alpha_\ell s_\ell)(\beta_0 t_0+\ldots+\beta_m t_m)$, *with non-zero* α's *and* β's

PROOF. Consider a product $(\alpha_0 s_0 + \ldots + \alpha_\ell s_\ell)(\beta_0 t_0 + \ldots + \beta_m t_m)$. In 3.1 we showed that all vanishing terms are convexly dependent on non-vanishing terms. The collection of non-vanishing terms in the product therefore contains a minimal spanning subset for the convex polyhedron of $S \times T$ which subset was shown to be unique in 2.1 and equal to the (likewise unique) largest set of convexly independent terms from $S \times T$ in 2.2.

We note that one may not always be able to determine α's and β's in 3.2 such that only the convexly independent product-terms remain. $S = \{x_0, x_1, x_0x_1\}$ and $T = \{1, x_0, x_0^2\}$ is an easy example.

4. REFERENCES.

[1] KIRKPATRICK, D., *On the additions necessary to compute certain functions*, M.Sc. Thesis, University of Toronto, Toronto (1972).