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CONTEXT SENSITIVE TABLE LINDENMAYER LANGUAGES AND A RELATION TO THE LBA PROBLEM

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Context sensitive table Lindenmayer languages and a relation to the lba problem *)

by

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ABSTRACT

Families of languages generated by classes of context sensitive Lindenmayer systems with tables using nonterminals are classified in the Chomsky hierarchy. It is shown that the family of languages generated by deterministic λ-free left context sensitive L systems with two tables using nonterminals coincides with the context sensitive languages. Combined with the fact that the family of languages generated by deterministic λ-free context sensitive L systems (with one table) using nonterminals is equal to the DLBA languages this shows the classic LBA problem to be equivalent to whether or not a trade-off is possible between one sided context with two tables and two sided context with one table for deterministic λ-free L systems using nonterminals. Without the restriction to λ-freeness such a trade-off is possible since the recursively enumerable languages are generated in both cases. By stating the results in their strongest form a complete classification of the considered language families is obtained since the hierarchies induced by the involved parameters (λ-freeness, determinism, number of tables, amount of context, closure under various types of homomorphisms) basically collapse to the recursively enumerable languages, context sensitive languages and DLBA languages.

KEY WORDS & PHRASES: *Formal languages - Lindenmayer systems - Classification of Language families - LBA-problem.*

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1. INTRODUCTION

Lindenmayer systems, or L systems, are parallel rewriting systems originally introduced as automata theoretic models for growth and development of filamentous organisms (LINDENMAYER , 1968). As an alternative to the usual generative grammars, and also because of its elegant mathematical nature caused by the simultaneous rewriting of all letters of a string, a large amount of formal language theoretical work has been done in this area, see e.g. (HERMAN & ROZENBERG, 1975, ROZENBERG & SALOMAA, 1974, LINDENMAYER & ROZENBERG, 1975). If each letter of a given string can be rewritten in but one way according to the rewriting rules the L system is deterministic. From both the biological and formal language theory viewpoints deterministic rewriting systems are relatively important. In (VITÁNYI, 1975) families of languages generated by deterministic context sensitive L systems with various restrictions were investigated. The resulting families were classified with respect to each other and the Chomsky hierarchy. The present paper will extend this research by considering L systems with tables. Table L systems were introduced by Rozenberg and consist of L systems with several sets (tables) of rewriting rules, where at each moment all letters in a string are rewritten simultaneously according to a single table. Whereas in the sequential rewriting of generative grammars this would not constitute any difference, because of the parallel nature of L systems the use of tables can result in an increase of generating power.

We can obtain languages from L systems in various ways. One way is to consider all strings generated from the initial string: the "pure" L language of the system. By dividing the alphabet in a set of terminals and nonterminals we can consider the language consisting of all strings over the terminals in the "pure" L language. Such a language is called an extension language since the terminal-nonterminal mechanism extends the generative power of a class of L systems. Another device is taking an homomorphism of a "pure" L language or extension language. We will treat all families of languages generated by context sensitive L systems with tables using nonterminals accordig to the effects of restrictions like: λ-freeness of production rules, determinism of production rules, number of tables, one-or two sided context, and closure of these families under various types of homomorphisms. Because of the great generative power of already deterministic context sensitive L systems using the terminal-nonterminal mechanism, the partial ordering according to set inclusion of the considered language families basically collapses to the recursively enumerable languages, context sensitive languages and deterministic linear bounded automaton (DLBA) languages. Hence the classification yields an interesting equivalence of the classic LBA problem (is the family of DLBA languages equal to the family of context sensitive languages?) in terms of L systems. In (VITÁNYI, 1975) it was proved that the family of DLBA languages coincides with the family of languages generated by λ-free, deterministic context sensitive L systems (with one table) using nonterminals. Previously, VAN DALEN, 1971, showed that the family of context sensitive languages equals the family of languages generated by λ-free context sensitive L systems (with one table) using nonterminals. Hence the LBA problem can be stated in terms of determinism versus nondeterminism in L systems. By arguments similar to those in (VITÁNYI, 1975) WOOD, 1975, proved that the family of languages generated by λ-free deterministic context sensitive L systems with two tables using nonterminals is equal to the family of context sensitive languages. Here the LBA problem was stated in the form of whether or not two tables can be reduced to one in the case under consideration. We shall demonstrate that the family of context sensitive languages equals the family of languages generated by λ-free deterministic left context sensitive L systems with two tables using nonterminals, thereby molding the LBA problem in the form of whether or not a trade-off is possible between one sided context with two tables and two sided context with one table for λ-free deterministic L systems using nonterminals. From the results it will appear that any further restriction on one of the two participants in the trade-off reduces the generative power to below the DLBA languages. If we relax the restriction of λ-freeness we obtain in both cases the recursively enumerable languages. We should stress, however, that although it seems that the trade-off corresponding to the LBA problem is between two deterministic rewriting systems, nondeterminism creeps in whenever we use more than one table since the choice of the next table to be used is nondeterministic. For a survey of the LBA problem and its reduction to other problems see (HARTMANIS & HUNT, 1974).

2. PRELIMINARIES

We shall use the definitions and terminology of formal language theory as in (SALOMAA, 1974) and terminology and results on L systems as in (HERMAN & ROZENBERG, 1975, VITÁNYI, 1975). $\# Z$ denotes the cardinality of a set Z, $lg(z)$ the length of a *string* or *word* z, and λ is the *empty word*, i.e., $lg(\lambda) = 0$.

A *(m,n)L system* is a triple $G = \langle W,P,w \rangle$ where W is a nonempty finite alphabet, $w \in W^+$ is the *initial string* and $P \subseteq (\bigcup_{i=0}^{m} W^i \times W \times \bigcup_{j=0}^{n} W^j) \times W^*$ is a finite set of *production rules*. We write an element of P as $(u,a,v) \to \alpha$ where $u \in \bigcup_{i=0}^{m} W^i$, $a \in W$, $v \in \bigcup_{j=0}^{n} W^j$ and $\alpha \in W^*$. The *operation* of an (m,n)L system G is defined as follows:

$$a_1 a_2 \ldots a_k \underset{G}{\Rightarrow} \alpha_1 \alpha_2 \ldots \alpha_k, \qquad a_1, a_2, \ldots, a_k \in W \text{ and } \alpha_1, \alpha_2, \ldots, \alpha_k \in W^*,$$

if

$$(a_{i-m}\, a_{i-m+1} \ldots a_{i-1},\ a_i,\ a_{i+1}\, a_{i+2} \ldots a_{i+n}) \to \alpha_i \in P$$

for all i, $1 \leq i \leq k$, where we take $a_j = \lambda$ whenever $j < 1$ or $j > k$.

As usual we define $\overset{*}{\underset{G}{\Rightarrow}}$ and $\overset{+}{\underset{G}{\Rightarrow}}$ as the transitive reflexive and transitive closure of $\underset{G}{\Rightarrow}$, respectively. We dispense with the subscripts if G is understood.

The *language* produced by G is defined as $L(G) = \{v \mid w \overset{*}{\Rightarrow} v\}$. We can squeeze languages out of L systems in various ways. One of these, a favorite in formal language theory, is by the use of nonterminals. The *extension* (language) produced by G with respect to a *terminal alphabet* V_T is defined as $E(G,V_T) = L(G) \cap V_T^*$. We also call the four tuple $G' = \langle W,P,w,V_T \rangle$ where W,P, and w are as before and V_T is a subset of W, an *E(m,n)L system*, and $E(G') = E(G,V_T)$ an *E(m,n)L language*.

Although the "pure" L languages (obtained without additional "squeezing" mechanisms) are not neatly nested in the Chomsky hierarchy and have none of the usual closure properties, families of extensions behave quite nicely in this respect. See, e.g., (HERMAN & ROZENBERG, 1975, VITÁNYI, 1975). Extensions of classes of deterministic (m,0)L systems form an exception (VITÁNYI, 1975). Another squeezing method is to apply homomorphisms of various types to the produced languages (extensions).

L systems are usually classified as follows:

(i) By *context*. $m,n \geq 0$: *IL systems (context sensitive)*; $m = n = 1$: *2L systems (two sided context)*; $m + n = 1$: *1L systems (one sided context)* where for our considerations *left context sensitive* or $(1,0)$L systems suffice; $m = n = 0$: *0L systems (context free)*

(ii) By *determinism*. If for each element $(u,a,v) \to \alpha \in P$ there is no $(u,a,v) \to \beta \in P$ with $\beta \neq \alpha$ then we call the system *deterministic*. The property is indicated by prefixing a D as in D2L system.

(iii) By *λ-freeness*. If $(u,a,v) \to \lambda \notin P$ then the system is called *λ-free* or *propagating*. The property is indicated by prefixing a P as in P2L systems and PD0L systems.

A *table (m,n)L system* is a triple $G = \langle W,\mathcal{P},w\rangle$ with $\mathcal{P} = \{P_1,P_2,\ldots,P_q\}$. such that for each i, $1 \leq i \leq q$, $G_i = \langle W,P_i,w\rangle$ is a (m,n)L system. The *operation* is as follows:

$$a_1a_2\ldots a_k \underset{G}{\Rightarrow} \alpha_1\alpha_2\ldots\alpha_k, \quad a_1,a_2,\ldots,a_k \in W \text{ and } \alpha_1,\alpha_2,\ldots,\alpha_k \in W^*,$$

if there is a *table* P_i in the *set of tables* $\mathcal{P}$ such that

$$a_1a_2\ldots a_k \underset{G_i}{\Rightarrow} \alpha_1\alpha_2\ldots\alpha_k.$$

$\overset{*}{\underset{G}{\Rightarrow}}$ and $\overset{+}{\underset{G}{\Rightarrow}}$ are the transitive reflexive and transitive closures of $\underset{G}{\Rightarrow}$, respectively. The *language* generated by G is defined by $L(G) = \{v \mid w \overset{*}{\underset{G}{\Rightarrow}} v\}$. *Extensions* are defined as before, the properties (i)-(iii) hold for G if they hold for every G_i, $1 \leq i \leq q$, and are indicated by the appropriate capitals. The fact that we are dealing with a table L system using q tables is indicated by prefixing T_q as in $PDT_q(m,n)L$ system. No subscript on T means $q \geq 1$; no T_q means $q = 1$. E.g., $PDT_1$0L-systems are identical to PD0L systems. A *control word* u of G is an element of $\{1,2,\ldots,q\}^*$ and $v \overset{u}{\Rightarrow} v'$, $v, v' \in W^*$ and $u = i_1i_2\ldots i_k \in \{1,2,\ldots,q\}^*$, means that

$$v = v_0 \underset{G_{i_1}}{\Rightarrow} v_1 \underset{G_{i_2}}{\Rightarrow} v_2 \underset{G_{i_3}}{\Rightarrow} \cdots \underset{G_{i_k}}{\Rightarrow} v_k = v' \text{ for some } v_1,v_2,\ldots,v_{k-1} \text{ in } W^*.$$

Languages generated by a class of XL systems constitute the family of XL languages. In the following we are most interested in the EPD2L (=$EPDT_1 2L$) languages and the $EPDT_2 1L$ languages. *REG, CF, INDEX, DLBA, CS and RE* denote the families of the regular languages, context free languages, indexed languages, DLBA languages, context sensitive languages and recursively enumerable languages, respectively. Some of the results of the theory which we shall have occasion to use later on are:

LEMMA 1.(see HERMAN & ROZENBERG, 1975) $CF \subsetneq EOL \subsetneq ET_2OL = ETOL \subsetneq INDEX$

LEMMA 2. (VITANYI, 1975). $H_{1:1}\ ED1L = H_s PD1L = ED2L = E1L = RE$ *where* $H_{1:1}X$ *signifies the closure of language family X under letter-to-letter homomorphisms and* $H_s X$ *signifies the closure of language family X under letter-to-itself-or-letter-to-*λ *homomorphisms. On the other hand* $REG \subsetneq H_\lambda EPD1L$ *where* $H_\lambda X$ *signifies closure of a language family X under nonerasing homomorphisms.*

LEMMA 3. (VITANYI, 1975). EX2L = EXIL

$$EX1L = \bigcup_{\substack{n = 0 \ \& \ m \geq 1 \\ n \geq 1 \ \& \ m = 0}} EX(m,n)L$$

Where X *denotes any of the combinations of* (ii), (iii) *and* $T_{\underline{q}}$, $\underline{q} \geq 1$,

Hence for context sensitive (table) L systems using nonterminals the amount of context is not important with respect to generative power, the only differences lie in two sided, one sided and no context. In the sequel we shall prove our results about EX1L systems only for EX(1,0)L systems, the case of EX(0,1)L system is completely analogous and gives the same results.

LEMMA 4. (VAN DALEN, 1971, WOOD, 1975) $EP2L = EPDT_2 2L = CS$.

LEMMA 5. (VITÁNYI, 1975). EPD2L = DLBA.

3. CLASSIFICATION OF FAMILIES OF ETIL LANGUAGES.

THEOREM. *The families of languages generated by the various subclasses* *of ETIL systems are classified by the diagram of figure 1. Solid arrows imply proper set inclusion of the lower family in the upper family; dotted arrows imply inclusion where properness is not known; if two of the displayed families are not connected by (a sequence of) arrows it means that these families are incomparable, i.e., their intersection contains nontrivial languages and neither family contains the other;* $X \equiv Y$ *mod* λ *means* $L \in X$ *iff* $L - \{\lambda\} \in Y$.

Note that all families of context sensitive table L languages obtained with the use of nonterminals (as distinguished in section 2) are classified by the displayed diagram since the results are stated in their strongest form and cannot be improved (except for the dotted arrow which corresponds to the LBA problem). But for EPD1L and EP1L all families are closed under nonerasing homomorphism. The proof of the theorem proceeds by a number of lemmas.

<u>LEMMA 6</u>. $EDT_2 1L = RE$.

<u>PROOF</u>. By lemma 2 $H_{1:1}$ ED1L = RE. Let $G = \langle W,P,w,V_T\rangle$ be an ED1L system and $h : V_T^* \to V^*$ a letter-to-letter homomorphism. Assume without loss of generality that $W \cap V = \emptyset$. Construct the $EDT_2 1L$ system $G' = \langle W', \{P_1, P_2\}, w, V\rangle$ as follows.

$W' = W \cup V \cup \{F\}$ with $F \notin W \cup V$;

$$P_1 = P \cup \{(x,a) \to F \mid (x,a) \notin (W \cup \{\lambda\}) \times W\}$$

$$P_2 = \{(x,a) \to h(a) \mid (x,a) \in (V_T \cup \{\lambda\}) \times V_T\}$$
$$\cup \{(x,a) \to F \mid (x,a) \notin (V_T \cup \{\lambda\}) \times V_T\}$$

The reader can satisfy himself easily that $E(G') = h(E(G))$. □

By lemma 2 and Lemma 6 it follows that $RE = ED2L = EDT_2 1L = H_{1:1}$ ED1L = $= H_s$PD1L. In (VAN DALEN, 1971) it is proved that EP2L = CS. By the working space theorem (SALOMAA, 1974) or the usual LBA simulation argument it follows that EPTIL = CS. WOOD, 1975, proved that $EPDT_2 2L = CS$. We now come to the main result:

LEMMA 7. $EPDT_2 1L = EP1L = CS$

PROOF. According to PENTTONEN, 1975, left context sensitive grammars (or more restrictedly, generative grammars with production rules of the form $AB \to A\beta$ or $B \to \beta$ where A and B are nonterminals and β is a nonempty string over the terminals and nonterminals) suffice to generate all context sensitive languages.

Claim. $EP1L = CS$

Proof of Claim. Since $EP2L = CS$ we only have to prove $CS \subseteq EP1L$. Let $G = \langle V_N, V_T, P, S\rangle$ be a grammar with nonterminals V_N, terminals V_T, the production rules in P of the form $AB \to A\beta$ or $B \to \beta$ where $A,B \in V_N$ and $\beta \in (V_N \cup V_T)^+$, and starting symbol $S \in V_N$. Construct an EP1L system $G' = \langle W', P', w', V_T\rangle$ as follows.

$W' = V_N \cup \bar{V}_N \cup V_T \cup \{F\}$, $\bar{V}_N = \{\bar{A} \mid A \in V_N\}$ and V_N, $\bar{V}_N$, V_T and $\{F\}$ are pairwise disjoint. $w' = S$ and P' is defined by:

$$\left.\begin{array}{ll}(1) & (x,A) \to A \\ (2) & \to \bar{A}\end{array}\right\} \text{ for all } A \in V_N \text{ and all } x \in W' \cup \{\lambda\}$$

(3) $(A,\bar{B}) \to \beta$ if $AB \to A\beta \in P$ and $A \in V_N$, $\bar{B} \in \bar{V}_N$

(4) $(x,\bar{B}) \to \beta$ if $B \to \beta \in P$ and $x \in (W' \cup \{\lambda\}) - \bar{V}_N$

(5) $(\bar{A},\bar{B}) \to F$ for all $\bar{A},\bar{B} \in \bar{V}_N$

(6) $(x,F) \to F$ for all $x \in W' \cup \{\lambda\}$

(7) $(x,a) \to a$ for all $a \in V_T$ and $x \in W' \cup \{\lambda\}$

(i) Clearly, if $S \overset{*}{\underset{G}{\Rightarrow}} v$ and $v \in V_T^*$ then there is a twice as long derivation $S \overset{*}{\underset{G'}{\Rightarrow}} v$. Therefore $L(G) \subseteq E(G')$.

(ii) Suppose $S \overset{*}{\underset{G'}{\Rightarrow}} v$ and $v \in V_T^*$. Because of (6) at no step of the derivation (5) was used: no adjacent barred nonterminals occurred in a word of the derivation.

Therefore, if $S = v_0 \underset{G'}{\Rightarrow} v_1 \underset{G'}{\Rightarrow} v_2 \underset{G'}{\Rightarrow} \cdots \underset{G'}{\Rightarrow} v_k = v$ then for each derivation step $v_i \underset{G'}{\Rightarrow} v_{i+1}$, $0 \le i < k$, there are $u_{i_1}, u_{i_2}, \ldots, u_{i_\ell} \in (V_N \cup V_T)^*$ such that either $\ell = 1$ or $u_{i_1} \underset{G}{\Rightarrow} u_{i_2} \underset{G}{\Rightarrow} \cdots \underset{G}{\Rightarrow} u_{i_\ell}$ where u_{i_1} and u_{i_ℓ} are equal to v_i and v_{i+1} with all bars removed from the nonterminals, respectively. Hence $S \overset{*}{\underset{G}{\Rightarrow}} v$ and $E(G') \subseteq L(G)$.

By (i) and (ii) $E(G') = L(G)$ and in view of the cited result of PENTTONEN, 1975, this proves the claim.

Above we noted that EPTIL $\subseteq$ CS and by the claim it therefore suffices to prove EP1L $\subseteq$ EPDT$_2$1L to prove the lemma. Let $G = \langle W, P, w, V_T\rangle$ be an EP1L system with $W = \{a_1, a_2, \ldots, a_n\}$ and P defined by:

$$\begin{array}{llll} (\lambda, a_j) & \to \alpha_{0j0} & (a_i, a_j) & \to \alpha_{ij0} \\ & \to \alpha_{0j1} & & \to \alpha_{ij1} \\ & \vdots & & \vdots \\ & \to \alpha_{0jn_{0j}} & & \to \alpha_{ijn_{ij}} \end{array}$$

for $1 \le i, j \le n$.

Construct an EPDT$_2$1L system $G' = \langle W', \{P_1, P_2\}, w, V_T\rangle$ defined by:

$$W' = W \cup W_\lambda \times W \cup (\bar{W}_\lambda \times \bar{W} \cup \bar{\bar{W}}_\lambda \times \bar{\bar{W}} \cup \bar{\bar{\bar{W}}}_\lambda \times \bar{\bar{\bar{W}}}) \times \{0, 1, \ldots, k\} \cup \{F\}$$

where $k = \max\{n_{ij} \mid 0 \le i \le n \text{ and } 0 < j \le n\}$; $X_\lambda = X \cup \{\lambda\}$ and $\bar{X} = \{\bar{a} \mid a \in X\}$ for $X \in \{W, W_\lambda, \bar{W}, \bar{W}_\lambda, \bar{\bar{W}}, \bar{\bar{W}}_\lambda\}$; F is a new letter.

P_1: $(y, (a_{i_1}, a_{i_2})) \to (a_{i_1}, a_{i_2})$

$(y, (\bar{a}_{i_1}, \bar{a}_{i_2}, i)) \to (\bar{\bar{a}}_{i_1}, \bar{\bar{a}}_{i_2}, i)$

$(y, (\bar{\bar{a}}_{i_1}, \bar{\bar{a}}_{i_2}, i)) \to (\bar{\bar{\bar{a}}}_{i_1}, \bar{\bar{\bar{a}}}_{i_2}, i)$

$(y, (\bar{\bar{\bar{a}}}_{i_1}, \bar{\bar{\bar{a}}}_{i_2}, i)) \to (\bar{\bar{a}}_{i_1}, \bar{\bar{a}}_{i_2}, i)$

For all $y \in W' \cup \{\lambda\}$, $a_{i_1} \in W_\lambda$, $a_{i_2} \in W$ and i such that $0 \le i \le k$.

$(.,.) \to F$ if $(.,.)$ is not in the above list.

P_2: $(a_{i_1}, a_{i_2}) \to (a_{i_1}, a_{i_2})$

$(\lambda, (a_{i_1}, a_{i_2})) \to (\bar{a}_{i_1}, \bar{a}_{i_2}, 0)$

$((\bar{\bar{\bar{a}}}_{i_1}, \bar{\bar{\bar{a}}}_{i_2}, i), (a_{i_3}, a_{i_4})) \to (\bar{a}_{i_3}, \bar{a}_{i_4}, 0)$

$(x, (a_{i_1}, a_{i_2})) \to (a_{i_1}, a_{i_2})$

$(y, (\bar{\bar{a}}_{i_1}, \bar{\bar{a}}_{i_2}, i)) \to (\bar{a}_{i_1}, \bar{a}_{i_2}, \text{remainder } ((i+1)/(n_{i_1 i_2}+1)))$

$(y,(\bar{\bar{\bar{a}}}_{i_1}, \bar{\bar{\bar{a}}}_{i_2}, i)) \to (\bar{\bar{\bar{a}}}_{i_1}, \bar{\bar{\bar{a}}}_{i_2}, i)$

$(z,(\bar{\bar{a}}_{i_1}, \bar{\bar{a}}_{i_2}, i)) \to \alpha_{i_1 i_2 i}$

for all $a_{i_1} \in W_\lambda$, a_{i_2}, a_{i_3}, $a_{i_4} \in W$,

$x \in W_\lambda \times W \cup (\bar{W}_\lambda \times \bar{W} \cup \bar{\bar{W}}_\lambda \times \bar{\bar{W}}) \times \{0, 1, \ldots, k\}$,

$y \in \bar{\bar{\bar{W}}}_\lambda \times \bar{\bar{\bar{W}}} \times \{0, 1, \ldots, k\}$, $z \in \bar{\bar{W}}_\lambda \times \bar{\bar{W}} \times \{0, 1, \ldots, k\}$.

$(.,.) \to F$ if F is not in the above list. Suppose

$$a_{i_1} a_{i_2} \ldots a_{i_n} \underset{G}{\Rightarrow} \alpha_{0 i_1 j_1} \alpha_{i_1 i_2 j_2} \ldots \alpha_{i_{n-1} i_n j_n}.$$

Then

$$a_{i_1} a_{i_2} \ldots a_{i_n} \overset{u}{\underset{G'}{\Rightarrow}} \alpha_{0 i_1 j_1} \alpha_{i_1 i_2 j_2} \ldots \alpha_{i_{n-1} i_n j_n}$$

under the controlword

$$u = 2\ 2^{j_1+1}\ 11\ 2^{j_2+1}\ 11 \ldots 11\ 2^{j_n+1}\ 12.$$

Hence

$$E(G) \subseteq E(G').$$

Now suppose that $v \overset{u}{\underset{G'}{\Rightarrow}} z$ and $v, z \in W^*$ and no intermediate word in the derivation belongs to W^*. According to the productions the last table applied must have been P_2 and the word v' it was applied to belongs to $(\bar{\bar{\bar{W}}}_\lambda \times \bar{\bar{\bar{W}}} \times \{0, 1, 2, \ldots, k\})^*$ since otherwise F would occur in z. But the only way to derive such a v' by application of tables P_1 and P_2 under the given assumptions yields a v' such that if $v' \overset{2}{\underset{G'}{\Rightarrow}} z$ then $v \underset{G}{\Rightarrow} z$ as careful scrutiny of the production rules shows. [In fact if $u = u'2$ then under the assumptions

$$u' \in 2\ 1^*\ (2^+(11)^+)^{lg(v)-1}\ 2^+1(12^*1)^* \quad]$$

Hence $E(G') \subseteq E(G)$ which together with the previous implication shows $E(G') = E(G)$. □

The inclusion relations between RE, CS, DLBA, CF, REG, ED1L, EPD1L and H_λEPD1L can be found in VITÁNYI, 1975. The connected parts of the diagram from ETOL downwards follow by various combinations of Lemma 3.2 and Theorems 6.4-6.7 from NIELSEN et. al., 1975. ETOL has deterministic tape complexity O(n) and therefore ETOL $\subseteq$ DLBA; since moreover ETOL is a full AFL and DLBA is not the inclusion is strict, VAN LEEUWEN, 1975. The only thing remaining to be shown is:

LEMMA 8. X *and* Y *are incomparable for all* X *and* Y *such that* X $\in$ {ED1L, EPD1L, H_λEPD1L} *and* Y $\in$ {ETOL, EDTOL}.

PROOF. REG $\not\subseteq$ X (VITÁNYI, 1975), but according to the established part of the diagram REG $\subsetneq$ Y. By definition EPDOL $\subseteq$ X $\cap$ Y (EPDOL is not displayed in the diagram). Since the homomorphic closure of X is equal to RE (by the fact that H_sPD1L = RE) and the homomorphic closure of Y is contained in ETOL (by definition and the fact that ETOL is a full AFL) there are languages in X which are not in Y. Hence X and Y have a nonempty intersection and neither contains the other. □

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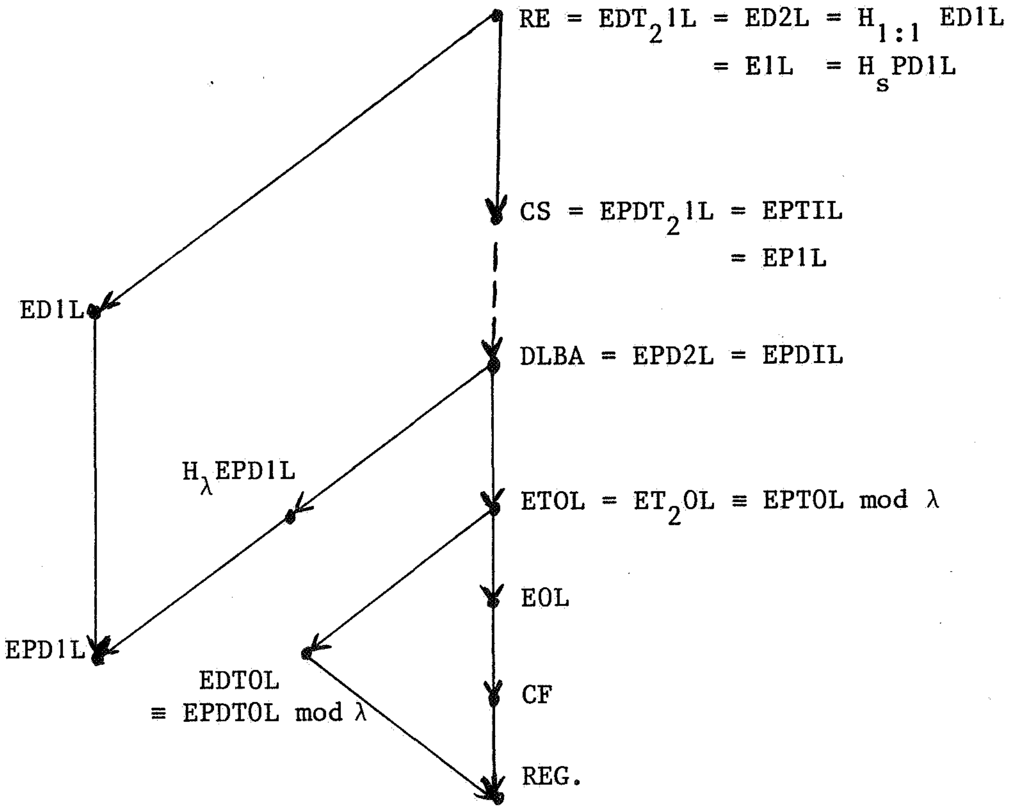
RE = EDT2 1L = ED2L = H1:1 ED1L
= E1L = Hs PD1L
CS = EPDT2 1L = EPT1L
= EP1L
ED1L
DLBA = EPD2L = EPD1L
Hλ EPD1L
ETOL = ET2OL ≡ EPTOL mod λ
EOL
EPD1L
EDTOL
≡ EPDTOL mod λ
CF
REG.

fig. 1