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IW 177/81 AUGUSTUS

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THE B-REPRESENTATION OF PIECEWISE POLYNOMIAL PARAMETRIC CURVES AND LOCAL ADAPTION

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Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 65D05, 65D07

ACM-Computing Reviews-category: 5.13

The b-representation of piecewise polynomial parametric curves and local adaption

by

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ABSTRACT

In this paper a description is provided for smooth curves satisfying a number of conditions, based on the theory of b-splines developed by C. de Boor. Local adaption to a new datapoint, cyclic curves and anti-cyclic continuations are the main features. Algorithms are presented as they are derived from the theory developed.

KEY WORDS & PHRASES: *b-splines, parametric curves, computer graphics*

CONTENTS

0. INTRODUCTION

The main object of this paper is to provide a description for smooth curves satisfying a number of conditions, based on the theory of b-splines developed by C. DE BOOR ([1]). A condition may be for instance the occurrence of a given point on the curve, or, whether the curve is cyclic or not. The curves (pp-curves) are parametric curves: $P = P(t) = (p_1(t),\ldots, p_{dim}(t))$; each $p_i(t)$ is a piecewise polynomial function (pp-function).

The basic algorithms for the calculation of pp-functions with the aid of b-splines are described in detail by C.J. RUSMAN ([7]).

In the first chapters a concise description of the theory of pp-functions and b-splines is presented. The numerical aspects (e.g. the conditions of the b-matrices) are descerned, but not further elaborated (see [1], also for references). Special attention is given to a uniform notation per knots, datapoints, breakpoints etc. The theory as given in [1] for instance, lacks this uniformity. This makes it difficult to apply results in a different area. A uniform notation forms a basis for uniform datastructuring in a collection of algorithms. The algorithms as presented in this paper all use the same knot and datapoint organisation.

Chapters 7 and 8 give some reasons why the b-representation of pp-functions (and pp-curves), i.e. the representation by means of b-splines, is preferred to the pp-representation in most cases. If we want to plot a pp-function or pp-curve however, the work is easier done with the pp-representation (Chapter 6). pp-curves are introduced in Chapter 9. These allow for closed curves and so-called anti-cyclic continuations. The cyclic case is dealt with in Chapter 10. Special arrangements are made for anti-cyclic continuations (Chapter 11). For pp-functions methods are suggested to adapt the functions to a new function value (local adaption, Chapter 8). These methods can also be applied to pp-curves. The theory of Béziér-curves is often used to produce curves of a desired shape; the curve is shaped by changing the vertices of the corresponding polygon (see [4]). The Béziér-curve is a special case of pp-curves, the vertices being nothing more than the b-coefficients. So, the methods of local adaption in Chapter 8 give us a wider range of possibilities for producing smooth curves of a desired shape.

Some subjects in this paper are only briefly mentioned or not mentioned at all, but are worthwhile to be worked out or looked at:

- local adaption for pp-curves
- altering the knot-sequence in local adaption
- different methods of anti-cyclic continuation (e.g. with a fixed slope)
- different continuations (e.g. under a given angle).

The theory is described in close connection with the computer programs for the plotting of the pp-functions and pp-curves. To give an impression of what such programs may look like, a complete computer program for the plotting of pp-curves is presented in the appendix. Appendix B contains a newly developed algorithm for dynamically adapting the stepsize to the curvature.

The computer programs were tested and the pictures were drawn at the Mathematisch Centrum, Department of computer science, Amsterdam. I wish to thank the MC for putting its computers and plot-devices at my disposal. I also like to thank Paul ten Hagen, who introduced me to the subject, for his good advices.

1. THEORY OF pp-FUNCTIONS

In this chapter the theory of pp-functions as given in DE BOOR [1] is formulated, using a uniform notation for knots, datapoints and breakpoints. The notation as developed here greatly simplifies formulation of the results as given in Chapters 2 and 3. The algorithms derived from the theory use datastructures which reflect this uniform representation. As a result the various algorithms can be combined without any restructuring of data.

1.1. pp-functions

Let $\Xi = \bigcup_{i=1}^{\ell+1} \xi_i \subset \mathbb{R}$, with $\xi_j < \xi_{j+1}$, $1 \le j \le \ell$. The collection of all functions $P(x)$: $x \to \mathbb{R}$, $\xi_1 \le x < \xi_{\ell+1}$, with the properties:

1) $P_i(x) = \sum_{j=0}^{k-1} \frac{(x-\xi_i)^j}{j!} c(i,j)$, $c(i,j) \in \mathbb{R}$, for $\xi_i \le x < \xi_{i+1}$,
$P = \bigcup_{i=1}^{\ell} P_i$.

2) With every ξ_i, $1 < i \le \ell$, is an integer ν_i, $0 \le \nu_i \le k$, associated;
$N = U\nu_i$;
if $\nu_i > 0$: $P_{i-1}^{(j-1)}(\xi_i) = c(i,j-1)$ for $j = 1,\ldots,\nu_i$

constitutes a linear space $\mathbb{P}(k,\Xi,N)$, with dimension $k\cdot\ell - \sum_{i=2}^{\ell} \nu_i$. k is the *order*, Ξ the set of *breakpoints* and N the set of numbers of *continuation conditions* of $\mathbb{P}(k,\Xi,N)$.
A member of $\mathbb{P}(k,\Xi,N)$ is called a *pp-function* (piecewise polynomial function).
A pp-function with $\nu_i \ge k-1$ for $i = 2,\ldots,\ell$ is a *spline function* or *spline*.
A breakpoint ξ_i with $\nu_i = k$ is a *pseudo-breakpoint*.
The *pp-representation* of a pp-function consists of k, Ξ and a set of *pp-coefficients* $c(i,j)$, $i = 1,\ldots,\ell$; $j = 0,\ldots,k-1$.

1.2. Divided differences

Let $Ut_i \subset \mathbb{R}$, $t_i \le t_{i+1}$. The k-*th divided difference* $[t_i,\ldots,t_{i+k}]g$ of a function g at $t_i,\ldots,t_{i+k}$ is the coefficient of x^k of the polynomial $p_{k+1}(x)$, the *interpolating polynomial*, of order $k+1$ (degree k) with the property:

$$p_{k+1}^{(m_j)}(t_j) = g^{(m_j)}(t_j), \quad m_j = \max(j - ii \mid t_j = t_{ii},\ i \le ii \le i+k) \quad *)$$

Divided differences are most easily computed recursively using the formulas

$$[t_i,\ldots,t_{i+k}]g = \frac{g^{(k)}(t_i)}{k!}, \quad \text{if } t_i = \ldots = t_{i+k}$$

(note that $[t_i]g = g(t_i)$), and

$$[t_i,\ldots,t_{i+k}]g = \frac{[t_{i+1},\ldots,t_{i+k}]g - [t_i,\ldots,t_{i+k-1}]g}{t_{i+k} - t_i}$$

else (see for a derivation [2], p.277-278). The divided differences up to a desired k are commonly arranged in a *divided difference scheme*.

1.3. b-splines

Let $T = Ut_i \subset \mathbb{R}$, $t_i \le t_{i+1}$. The i-th *b-spline* of order k for *knot-sequence* T, $B_{i,k,T}$ (B_i for short), is defined by:

$$B_{i,k,T}(x) = (t_{i+k} - t_i)[t_i,\ldots,t_{i+k}](t-x)_+^{k-1}, \quad x \in \mathbb{R},$$

with $(t-x)_+^{k-1} = (t-x)^{k-1}$ if $x < t$ and 0 else. With the linear space $\mathbb{P}(k,\Xi,N)$ we associate a collection of knot-sequences, $T(k,\Xi,N) = U \bigcup_{i=1}^{n+k} t_i$, with

1) $t_1 \le \ldots \le t_k = \xi_1$, $t_{n+k} \ge \ldots \ge t_{n+1} = \xi_{\ell+1}$.
2) For $\xi_1 < \xi_i < \xi_{\ell+1}$ there is a j so that $t_j = t_{j+1} = \ldots = t_{j+k-\nu_i-1} = \xi_i$; $k-\nu_i$ is the *multiplicity* of $t_j,\ldots,t_{j+k-\nu_i-1}$; we also say $\bigcup_{ii=j}^{j+k-\nu_i-1} t_{ii}$ is a *multiple knot* if $k-\nu_i > 1$.
3) $n = \dim(\mathbb{P}(k,\Xi,N))$.

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*) $(\frac{d}{dt})^r f(t)\big|_{t=t_j}$ is often shortened to $f^{(r)}(t_j)$, although it is actually incorrect.

We now are able to formulate the important theorem, which relates $\bigcup_i B_{i,k,T}$, $T \in T(k,\Xi,N)$, with $\mathbb{P}(k,\Xi,N)$:

THEOREM. $\bigcup_{i=1}^{n} (B_{i,k,T}|_{[\xi_1,\xi_{\ell+1})})$, $T \in T(k,\Xi,N)$, *constitutes a basis for* $\mathbb{P}(k,\Xi,N)$. *(Hence the name "b-spline".)* $n = \dim(\mathbb{P}(k,\Xi,N))$, *so if we can prove that* $B_i|_{[\xi_1,\xi_{\ell+1})} \in \mathbb{P}(k,\Xi,N)$, *we only have to show that* $B_1|_{[\xi_1,\xi_{\ell+1})},\ldots,B_n|_{[\xi_1,\xi_{\ell+1})}$ *are linear independent.*

i) $B_i|_{[\xi_1,\xi_{\ell+1})} \in \mathbb{P}(k,\Xi,N)$.

PROOF. From the divided difference scheme it is clear that there are numbers $d_j \in \mathbb{R}$ so that $[t_i,\ldots,t_{i+k}]g = \sum_{j=1}^{i+k} d_j g^{(m_j)}(t_j)$. So for B_i we get:

$$B_i(x) = (t_{i+k} - t_i) \sum_{j=i}^{i+k} d_j (t_j-x)_+^{k-1-m_j} (k-1)!/(k-1-m_j)!.$$

This is a pp-function of order k with (not necessarily all) breakpoints in Ξ. By the definition of m_j and the construction of T we know that $m_j \le k-\nu_{jj}-1$ if $t_j = \xi_{jj}$. From the fact that $k-1-m_j \ge k-1-k+\nu_{jj}+1 = \nu_{jj}$, it follows that B_i has at least ν_{jj} continuous derivatives at ξ_{jj}, so

$$B_i|_{[\xi_1,\xi_{\ell+1})} \in \mathbb{P}(k,\Xi,N). \qquad \square$$

ii) *Let the linear functional* λ_i *be defined by:*

$$\lambda_i f = \sum_{r=0}^{k-1} (-1)^{k-1-r} \psi^{(k-1-r)}(\tau_i) f^{(r)}(\tau_i),$$

with $\psi(t) = (t_{i+1}-t)\ldots(t_{i+k-1}-t)/(k-1)!$ *and* $t_i < \tau_i < t_{i+k}$.
Then $\lambda_i B_j = \delta_{ij}$ (the Kronecker delta) (DE BOOR & FIX, 1973).

PROOF.

$$\lambda_i (t-x)^{k-1} = \sum_{r=0}^{k-1} (-1)^{k-1-r} \psi^{(k-1-r)}(\tau_i)(k-1)\ldots(k-r)(-1)^r (t-\tau_i)^{k-1-r} \text{*)}$$

$$=$$

*) with the convention $(k-1)\ldots(k) = 1$.

$$= (-1)^{k-1}(k-1)!\sum_{r=0}^{k-1}\{\psi^{(k-1-r)}(\tau_i)/(k-1-r)!\}(t-\tau_i)^{k-1-r}$$

$$= (-1)^{k-1}(k-1)!\psi(t),$$

for ψ is a polynomial of order k. So for $(t-x)_+^{k-1}$ the following equation holds:

$$\lambda_i(t-x)_+^{k-1} = (-1)^{k-1}(k-1)!\psi(t)(t-\tau_i)_+^0.$$

Since

$$(\frac{d}{dt})^{m_r}\lambda_i(t-x)_+^{k-1} = \lambda_i((\frac{d}{dt})^{m_r}(t-x)_+^{k-1}),$$

we have

$$\lambda_i B_j = (t_{j+k}-t_j)\sum_{r=j}^{j+k} d_r\lambda_i\left\{(\frac{d}{dt})^{m_r}(t-x)_+^{k-1}\right\}\Big|_{t=t_r}$$

$$= (t_{j+k}-t_j)\sum_{r=j}^{j+k} dr(-1)^{k-1}(k-1)!(\frac{d}{dt})^{m_r}(\psi(t)(t-\tau_i)_+^0)\Big|_{t=t_r}$$

$$= (t_{j+k}-t_j)(-1)^{k-1}(k-1)![t_j,\ldots,t_{j+k}](\psi(t)(t-\tau_i)_+^0).$$

$[t_j,\ldots,t_{j+k}](\psi(t)(t-\tau_i)_+^0) = 0$ if $j \neq i$; if $j = i$, $\psi(t)(t-\tau_i)_+^0$ agrees with $\psi(x)(x-t_i)/(t_{i+k}-t_i)$ at $t_i.....t_{i+k}$. The coefficient of x^k is

$$\frac{(-1)^{k-1}}{(k-1)!(t_{i+k}-t_i)},$$

so

$$\lambda_i B_i = \frac{(t_{i+k}-t_i)(-1)^{k-1}(k-1)!(-1)^{k-1}}{(k-1)!(t_{i+k}-t_i)} = 1. \qquad \square$$

For $T \in T(k,\Xi,N)$ a pp-function $P \in \mathbb{P}(k,\Xi,N)$ can uniquely be written as $\sum_{i=1}^{n}\alpha_i B_{i,k,T}\big|_{[\xi_1,\xi_{\ell+1})}$, $\alpha_i \in \mathbb{R}$. A *b-representation* of P consists of k, T and n *b-coefficients* α_i, $i = 1,\ldots,n$.

If we know a priori that $P \in \mathbb{P}(k,\Xi,N)$ then the n b-coefficients going with a $T \in T(k,\Xi,N)$ are for instance determined

1) by giving n different function values (i.e., n different abscissae),
2) by giving n_1 different function values and $n-n_1$ additional (continuation) conditions.

1.4. Properties of b-splines

b1) $B_i(x) = 0$ *for* $x < t_i$ *or* $x > t_{i+k}$.

PROOF. For $x < t_i$ we have $(t-x)_+^{k-1} = (t-x)^{k-1}$ on $[t_i,t_{i+k}]$ and $B_i(x) = (t_{i+k}-t_i)[t_i,\ldots,t_{i+k}](t-x)^{k-1} = 0$. (The coefficient of x^k is 0.) For $x > t_{i+k}$ $(t-x)_+^{k-1} \equiv 0$ on $[t_i,t_{i+k}]$ and $B_i(x) = (t_{i+k}-t_i)[t_i,\ldots,t_{i+k}](t-x)_+^{k-1} = 0$ again. Consequently, if $t_j \le x < t_{j+1}$, only $B_{j-k+1},\ldots,B_j$ are possibly non-zero on x. □

b2) $\sum_i B_i(x) = 1$.

PROOF. From b1) follows $\sum_i B_i(x) = \sum_{i=j-k+1}^{j} B_i(x)$ if $t_j \le x < t_{j+1}$.

$$\sum_{i=j-k+1}^{j} B_i(x) = \sum_{i=j-k+1}^{j} ([t_{i+1},\ldots,t_{i+k}](t-x)_+^{k-1} - [t_i,\ldots,t_{i+k-1}](t-x)_+^{k-1})$$

$$= [t_{j+1},\ldots,t_{j+k}](t-x)_+^{k-1} - [t_{j-k+1},\ldots,t_j](t-x)^{k-1}$$

$$= 1 - 0 = 1.$$

□

b3) $$B_{i,k,T}(x) = \frac{t_{i+k}-x}{t_{i+k}-t_{i+1}} B_{i+1,k-1,T}(x) + \frac{x-t_i}{t_{i+k-1}-t_i} B_{i,k-1,T}(x).$$

PROOF. Let us look at the definition of divided difference. The interpolating polynomial p_{k+1} can be written as

$$p_1(x) + \sum_{i=2}^{k+1} (p_i(x) - p_{i-1}(x)) = \sum_{i=1}^{k+1} (x-t_1)\ldots(x-t_{i-1})[t_1,\ldots,t_i]g$$

(the first term is $[t_1]g$). So $\sum_{r=i}^{i+k} (x-t_i)\ldots(x-t_{r-1})[t_i,\ldots,t_r]g \cdot \sum_{s=i}^{i+k} (x-t_{s+1})\ldots(x-t_{i+k})[t_s,\ldots,t_{i+k}]h$ agrees with gh at $t_i,\ldots,t_{i+k}$ and equals 0 for $r > s$ and $x = t_i,\ldots,t_{i+k}$. Therefore $\sum\sum_{r\le s}$ also agrees with gh at $t_i,\ldots,t_{i+k}$ and by the definition of divided difference

$$[t_i,\dots,t_{i+k}]gh = \sum_{r=i}^{i+k} [t_i,\dots,t_r]g[t_r,\dots,t_{i+k}]h.$$

We use this formula for $(t-x)_+^{k-1} = (t-x)(t-x)_+^{k-2}$:

$$\begin{aligned}
B_{i,k,T}(x) &= (t_{i+k}-t_i)[t_i,\dots,t_{i+k}](t-x)_+^{k-1} \\
&= (t_{i+k}-t_i)((t_i-x)[t_i,\dots,t_{i+k}](t-x)_+^{k-2} + [t_{i+1},\dots,t_{i+k}](t-x)_+^{k-2}) \\
&= (t_{i+k}-t_i)((t_i-x)\frac{[t_{i+1},\dots,t_{i+k}](t-x)_+^{k-2} - [t_i,\dots,t_{i+k-1}](t-x)_+^{k-2}}{t_{i+k}-t_i} \\
&\qquad + [t_{i+1},\dots,t_{i+k}](t-x)_+^{k-2}) \\
&= (t_{i+k}-x)[t_{i+1},\dots,t_{i+k}](t-x)_+^{k-2} - (t_i-x)[t_i,\dots,t_{i+k-1}](t-x)_+^{k-2} \\
&= \frac{t_{i+k}-x}{t_{i+k}-t_{i+1}} B_{i+1,k-1,T}(x) + \frac{x-t_i}{t_{i+k-1}-t_i} B_{i,k-1,T}(x). \qquad \square
\end{aligned}$$

It is clear that $B_{i,1,T}(x) = 1$ for $t_i \le x < t_{i+1}$ and 0 else, so, as a consequence, $B_{i,k,T} > 0$ for $t_i < x < t_{i+k}$.

b4) $\frac{d}{dx}(\sum \alpha_i B_{i,k,T}(x)) = \sum (k-1) \frac{\alpha_i - \alpha_{i-1}}{t_{i+k-1}-t_i} B_{i,k-1,T}(x).$

PROOF.

$$\begin{aligned}
\frac{d}{dx} B_{i,k,T}(x) &= \frac{d}{dx}([t_{i+1},\dots,t_{i+k}](t-x)_+^{k-1} - [t_i,\dots,t_{i+k-1}](t-x)_+^{k-1}) \\
&= -(k-1)([t_{i+1},\dots,t_{i+k}](t-x)_+^{k-2} - [t_i,\dots,t_{i+k-1}](t-x)_+^{k-2} \\
&= \frac{k-1}{t_{i+k-1}-t_i} B_{i,k-1,T}(x) - \frac{k-1}{t_{i+k}-t_{i+1}} B_{i+1,k-1,T}(x). \qquad \square
\end{aligned}$$

1.5. Calculation of b-coefficients

Let $P \in \mathbb{P}(k,\Xi,N)$ and $T \in T(k,\Xi,N)$. Furthermore, let *datapoints* τ_i, $i = 1,\ldots,n$, $\xi_1 \le \tau_i < \tau_{i+1} \le \xi_{\ell+1}$, and $P(\tau_i)$, $i = 1,\ldots,n$, be given. ($P(\xi_{\ell+1}) = P_\ell(\xi_{\ell+1})$.)

The questions that rise are:

- How can we calculate the b-coefficients going with T and $P(\tau_i)$, $i = 1,\ldots,n$?
- How must the τ_i's be positioned to make the calculation possible in the first place?

  (It is clear that the problem is not solvable if we take for instance all the τ_i's in the first ξ-interval.)

From property b1) follows

$$P(\tau_i) = \sum_{r=j-k+1}^{j} \alpha_r B_{r,k,T}(\tau_i) \quad \text{if} \quad t_j \le \tau_i < t_{j+1}.$$

For $i = 1,\ldots,n$ we get the n equations needed to determine $\alpha_1,\ldots,\alpha_n$. We write: $(B_{ij})(\alpha_j)^T = (P(\tau_i))^T$, with (B_{ij}) the *b-matrix* of T and τ_i, $i = 1,\ldots,n$. If $\tau_i \le t_i$ or $\tau_i \ge t_{i+k}$, (B_{ij}) is not invertible. Let e.g. $\tau_i \le t_i$. Then $B_{ij} = B_{j,k,T}(\tau_i) = 0$ for $j \ge i$. As τ_i, $i = 1,\ldots,n$, is non-decreasing, we also have $B_{ii_1 j} = 0$ for $1 \le ii < i$ and $j \ge i$, so the last n-i+1 columns only have possible non-zero elements in the last n-i rows and are therefore dependent. Consequently: $t_i < \tau_i < t_{i+k}$. This makes (B_{ij}) banded with band-width 2k-1.

The calculation of B_{ij} is conveniently performed with property b3) (subroutine 'bsplvx'): Suppose $t_j \le \tau_i < t_{j+1}$; starting with $B_{j,1,T}(\tau_i) = 1$ and $B_{ii,1,T}(\tau_i) = 0$ for $ii \ne j$, we can compute $B_{j-k,k+1,T}(\tau_i),\ldots,B_{j,k+1,T}(\tau_i)$ from $B_{j-k+1,k,T}(\tau_i),\ldots,B_{j,k,T}(\tau_i)$, keeping in mind that $B_{j-k,k,T}(\tau_i) = B_{j+1,k,T}(\tau_i) = 0$:

$$B_{j-k+ii-1,k+1,T}(\tau_i) = \frac{t_{j+ii} - \tau_i}{(t_{j+ii}-\tau_i) + (\tau_i - t_{j-k+ii})} B_{j-k+ii,k,T}(\tau_i)$$

$$+ \frac{\tau_i - t_{j-k+ii-1}}{(t_{j+ii-1}-\tau_i) + (\tau_i - t_{j-k+ii-1})} B_{j-k+ii-1,k,T}(\tau_i)$$

$$= \frac{dr_{ii}}{dr_{ii}+d\ell_{ii-k+1}} B_{j-k+ii,k,T}(\tau_i)$$

$$+ \frac{d\ell_{k+2-ii}}{dr_{ii-1}+d\ell_{k+2-ii}} B_{j-k+ii-1,k,T}(\tau_i),$$

with $d\ell_{jj} = \tau_i - t_{j+1-jj}$ and $dr_{jj} = t_{j+jj} - \tau_i$, ii = 1,...,k+1. Because (B_{ij}) is totally positive (no proof), the system can be solved without pivoting.

We store the non-zero elements of (B_{ij}) in an n×(2k-1) matrix. Sometimes the same b-matrix is later on used for another $(P(\tau_i))^T$ and instead of solving the system directly, we first perform an LU-decomposition on the condensed (B_{ij}), which is saved (subroutine 'ludeco'). The theory on matrix-computations is clearly presented in e.g. [3], ch.3. With this LU-decomposition the solution is easily obtained (subroutine 'solsys').

1.6. Conversion b-representation ↔ pp-representation

If we want to plot a pp-function, i.e., to make a picture of it, we evaluate the function values of the pp-function at some points and draw straight lines between them in a certain coordinate-system. It is easier to evaluate the function values starting from the pp-representation, so, if we need many function values, as is usually the case with plotting, it is better to switch over to the pp-representation.

Suppose we have the disposal of a b-representation, k, T and α_i, i = 1,...,n, of a pp-function. The Ξ corresponding with T is easily obtained. Note that $c(i,j) = (\frac{d}{dx})^j P_i(\xi_i)$. So the problem reduces to calculating the function and the derivatives up to k-1 at $\xi_1,\ldots,\xi_\ell$ (subroutine 'ppfppr'). With property b4) we can recursively calculate all the b-coefficients relevant for the function values of the k-1 derivatives at a certain

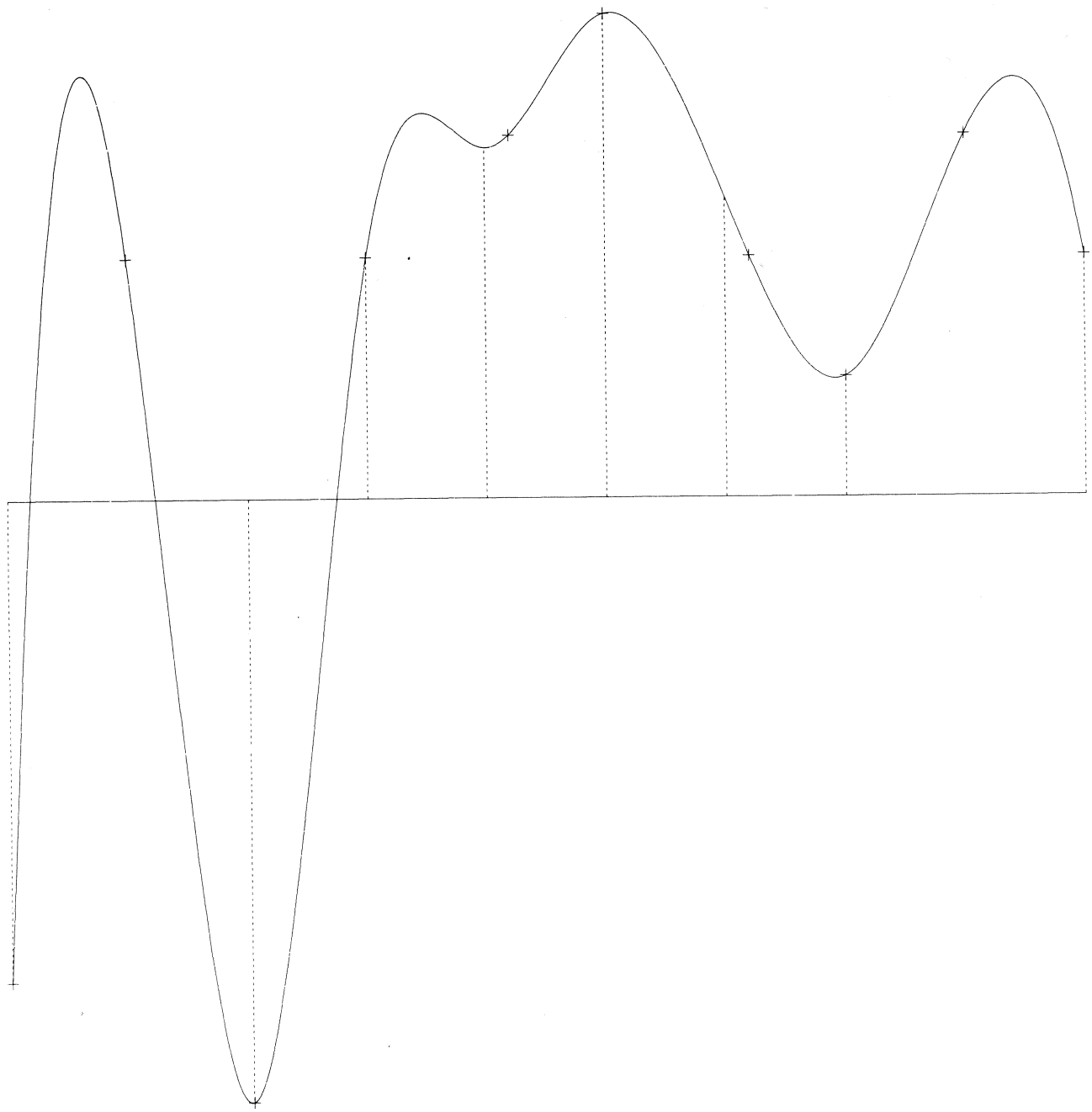


Fig. 1: pp-function of order 4 through 10 datapoints

ξ_{ii}. Let $t_{left} = \xi_{ii} \neq t_{left+1}$. A scratch-matrix s is constructed as follows:

1) $s(i,1) = \alpha_{left-k+i}$, $i = 1,\ldots,k$.

2) $s(i,j+1) = (k-j)\dfrac{s(i+1,j) - s(i,j)}{t_{left+i} - t_{left+i-(k-j)}}$, $j = 1,\ldots,k-1$; $i = 1,\ldots,k-j$;

$t_{left+i} \neq t_{left+i-(k-j)}$.

If $t_{left+i} = t_{left+i-(k-j)}$, $B_{left+i-(k-j),k-j,T}$ was already 0 and $s(i,j+1)$ is simply not calculated. With 'bsplvx' $B_{left-(k-j)+1,k-j,T}(t_{left}),\ldots,$ $B_{left,k-j,T}(t_{left})$, $0 \leq j \leq k-1$, are determined. The pp-coefficients are then:

$$c(ii,j) = \sum_{i=1}^{k-j} s(i,j+1)B_{left+i-(k-j),k-j,T}(t_{left}),\ j = 0,\ldots,k-1.$$

If on the other hand, we have the pp-representation of a pp-function P and want to have a b-representation, the work is done in three steps:

1) a $T \in T(k,\Xi,N)$ is constructed. First we determine N, then we take $k-\nu_i$ subsequent members of T equal to ξ_i.
2) We compute $P(\tau)$ for $\tau = \tau_1,\ldots,\tau_n$, with $\tau_i < \tau_{i+1}$ and $t_i < \tau_i < t_{i+k}$.
3) With T and $P(\tau_1),\ldots,P(\tau_n)$ the b-coefficients are fixed (see Chapter 5).

2. B-REPRESENTATION

2.1. Advantages of the b-representation - cubic splines

We compute the pp-coefficients of the spline function P of order 4 (*cubic spline*), with breakpoints $\xi_1,\ldots,\xi_{\ell+1}$ and function values $P(\xi_i) = p_i$, $i = 1,\ldots,\ell+1$, only using the pp-representation and not a b-spline (or any other) basis. For the $4\times\ell$ unknowns $c(i,j)$, $i = 1,\ldots,\ell$; $j = 0,\ldots,3$, we need as many equations:

1) $c(i,0) = p_i$, $i = 1,\ldots,\ell+1$;

2) $P_{i-1}(\xi_i) = c(i,0)$, $i = 2,\ldots,\ell$;

3) $P^{(1)}_{i-1}(\xi_i) = c(i,1)$, $i = 2,\ldots,\ell$;

4) $P^{(2)}_{i-1}(\xi_i) = c(i,2)$, $i = 2,\ldots,\ell$;

5) $P^{(3)}_1(\xi_2) = c(2,3)$ and $P^{(3)}_{\ell-1}(\xi_\ell) = c(\ell,3)$, making ξ_2 and ξ_ℓ pseudo-breakpoints.

The system is reduced to a set of $\ell+1$ equations with $c(i,1)$, $i = 1,\ldots,\ell+1$ as unknowns ($c(\ell+1,1) = P^{(1)}_\ell(\xi_{\ell+1})$); we can write $P_i(x)$ as follows (see property b3)):

$$P_i(x) = P(\xi_i) + (x-\xi_i)[\xi_i,\xi_i]P(x) + (x-\xi_i)^2[\xi_i,\xi_i,\xi_{i+1}]P(x)$$

$$+ (x-\xi_i)^2(x-\xi_{i+1})[\xi_i,\xi_i,\xi_{i+1},\xi_{i+1}]P(x).$$

With $c(i,1) = [\xi_i,\xi_i]P(x)$ we have:

$$c(i,2) = P^{(2)}(\xi_i)$$

$$= 2[\xi_i,\xi_i,\xi_{i+1}]P(x) - 2(\xi_{i+1}-\xi_i)[\xi_i,\xi_i,\xi_{i+1},\xi_{i+1}]P(x)$$

$$= 2([\xi_i,\xi_{i+1}]P(x) - c(i,1) - 2(c(i,1) + c(i+1,1)$$

$$- 2[\xi_i,\xi_{i+1}]P(x)))/(\xi_{i+1}-\xi_i) \qquad (1)$$

$$c(i,3) = 6(c(i,1) + c(i+1,1) - 2[\xi_i,\xi_{i+1}]P(x))/(\xi_{i+1}-\xi_i)^2 \qquad (2)$$

$$c(i-1,2) + c(i-1,3)(\xi_i-\xi_{i-1}) - c(1,2) = 0,$$

so for the equations of 4) we get

$$c(i-1,1)(\xi_{i+1}-\xi_i) + 2c(i,1)(\xi_{i+1}-\xi_{i-1}) + c(i+1,1)(\xi_i-\xi_{i-1})$$

$$= 3((\xi_{i+1}-\xi_i)[\xi_{i-1},\xi_i]P(x) + (\xi_i-\xi_{i-1})[\xi_i,\xi_{i+1}]P(x)), \quad i = 2,\ldots,\ell.$$

For the two equations of 5) we get:

$$c(1,1)(\xi_3-\xi_2) + c(2,1)(\xi_3-\xi_1)$$

$$= ((\xi_2+2\xi_3-3\xi_1)(\xi_3-\xi_1)[\xi_1,\xi_2]P(x) + (\xi_2-\xi_1)^2[\xi_2,\xi_3]P(x))/(\xi_3-\xi_1)$$

and

$$c(\ell,1)(\xi_{\ell+1}-\xi_{\ell-1}) + c(\ell+1,1)(\xi_\ell-\xi_{\ell-1})$$

$$= (\xi_{\ell+1}-\xi_\ell)^2[\xi_{\ell-1},\xi_\ell]P(x) + (3\xi_{\ell+1}-\xi_\ell-2\xi_{\ell-1})(\xi_\ell-\xi_{\ell-1})\cdot$$

$$\cdot[\xi_\ell,\xi_{\ell+1}]P(x))/(\xi_{\ell+1}-\xi_{\ell-1}).$$

Together we have a system of $\ell+1$ equations with a banded coefficient-matrix, band-width 3, which can be solved without pivoting. c(i,2) and c(i,3), $i = 1,\ldots,\ell$, can then be computed with (1) and (2) (subroutine 'cubspl').

Now for a derivation by means of b-splines. We take a knot-sequence:

$$t_1 \le t_2 \le t_3 \le t_4 = \xi_1,$$
$$t_i = \xi_{i-2},\ i = 5,\ldots,\ell+1,$$
$$\xi_{\ell+1} = t_{\ell+2} \le t_{\ell+3} \le t_{\ell+4} \le t_{\ell+5},$$

and datapoints $\tau_i = \xi_i$, $i = 1,\ldots,\ell+1$. (Note that the first and last two ξ-intervals agree with one t-interval.) After the determination of the b-coefficients, the b-representation can be converted to the pp-representation.

Using pp-representations only, forces us to start with a reduction step, reducing the $4\times\ell$ equations to $\ell+1$ equations. The reduction algorithm changes with different order or continuation conditions and becomes more complicated with increasing order. Using a b-spline basis the alterations of the algorithm are simple and systematic and therefore allow parametrization.

2.2. Advantages of the b-representation - local adaption

Let P be a pp-function and $a \in \mathbb{R}$, $a \neq P(\xi)$, $\xi_1 \leq \xi \leq \xi_{\ell+1}$. Suppose we want to change P into a new pp-function P' so that

a1) $\qquad P'(\xi) = a$

and P' stays as close to P as possible.

We can interprete this in different ways. In case P is a (cubic) spline function, constructed as described in the previous chapter, and ξ is a breakpoint ξ_j, we can compute the spline P' with $P'(\xi_i) = p_i$ for $i \neq j$ and $P'(\xi_j) = a$. Although in some sense the new spline stays as close to P as possible (all but one $P(\xi_i)$, $i = 1,\ldots,\ell+1$, remain the same) the adaption is not local, because the whole spline, except at those ℓ points, changes. Moreover, the spline must be recomputed entirely.

To make the adaption local, function values at ξ_i's adjacent to ξ_j must be 'released'. Now, suppose again we only have the pp-representation of a cubic spline function. If we change the function value at ξ_j, $P'(\xi_j) = a$, it it not possible to keep the spline unchanged for $x < \xi_{j-1}$ and $x \geq \xi_{j+1}$; we get ten equations for the eight pp-coefficients of the two intervals adjacent to ξ_j:

ξ_{j-1}	ξ_j	ξ_{j+1}
$P_{j-2}(\xi_{j-1}) = c(j-1,0)$	$c(j,0) = a$	$P_j(\xi_{j+1}) = c(j+1,0)$
$P^{(1)}_{j-2}(\xi_{j-1}) = c(j-1,1)$	$P_{j-1}(\xi_j) = c(j,0)$	$P^{(1)}_j(\xi_{j+1}) = c(j+1,1)$
$P^{(2)}_{j-2}(\xi_{j-1}) = c(j-1,2)$	$P^{(1)}_{j-1}(\xi_j) = c(j,1)$	$P^{(2)}_j(\xi_{j+1}) = c(j+1,2)$
	$P^{(2)}_{j-1}(\xi_j) = c(j,2)$	

If we release ξ_{j-1} or ξ_{j+1} we get thirteen equations for the twelve pp-coefficients of three intervals adjacent to ξ_i and if we release one more ξ_i, we finally get a solvable system of sixteen equations for sixteen pp-coefficients, for instance:

ξ_{j-3}	ξ_{j-2}	ξ_{j-1}	ξ_j	ξ_{j+1}
$P_{j-4}(\xi_{j-3}) = c(j-3,0)$	$P_{j-3}(\xi_{j-2}) = c(j-2,0)$	$P_{j-2}(\xi_{j-1}) = c(j-1,0)$	$c(j,0) = a$	$P_j(\xi_{j+1}) = c(j+1,0)$
$P^{(1)}_{j-4}(\xi_{j-3}) = c(j-3,1)$	$P^{(1)}_{j-3}(\xi_{j-2}) = c(j-2,1)$	$P^{(1)}_{j-2}(\xi_{j-1}) = c(j-1,1)$	$P_{j-1}(\xi_j) = c(j,0)$	$P^{(1)}_j(\xi_{j+1}) = c(j+1,1)$
$P^{(2)}_{j-4}(\xi_{j-3}) = c(j-3,2)$	$P^{(2)}_{j-3}(\xi_{j-2}) = c(j-2,2)$	$P^{(2)}_{j-2}(\xi_{j-1}) = c(j-1,2)$	$P^{(1)}_{j-1}(\xi_j) = c(j,1)$	$P^{(2)}_j(\xi_{j+1}) = c(j+1,2)$
			$P^{(2)}_{j-1}(\xi_j) = c(j,2)$	

There are three possibilities: the spline will change on (ξ_{j-3},ξ_{j+1}), (ξ_{j-2},ξ_{j+2}) or (ξ_{j-1},ξ_{j+3}). If we raise the order of the spline by one, the above mentioned numbers will be: thirteen equations for ten unknowns, seventeen for fifteen and twentyone for twenty, so we have to release an extra ξ_i. In general we have to change the spline on one of the intervals (ξ_{j-k+i},ξ_{j+i}), $i = 1,\ldots,k-1$, k the order, to keep the adaption local*). We will make no attempt to solve the systems of equations. If we are working with a b-spline basis, everything will become much easier. Let k, T and α_i, $i = 1,\ldots,n$, be a b-representation of the spline (of order k) and let $t_r = \xi_j$. By property b1) we know that $B_i(t_r) \neq 0$ for $i = r-k+1,\ldots,r-1$. Adding $\frac{a-P(t_r)}{B_i(t_r)} B_i(x)$, $r-k+1 \leq i \leq r-1$, to the spline, gives a new spline function, which equals P for $x \leq t_i$ and $x \geq t_{i+k}$ and has function value a at t_r. So, by using a b-representation of the pp-function the same adaption is achieved in a far simpler way.

Although we can thus locally adapt a pp-function to a new function value by changing one b-coefficient, the result may not be what we wanted. Two additional conditions make the adaption more acceptable:

a2) $\max|P'-P| = a.$

a3) $\frac{d}{dx}(P'-P)\Big|_{x=\xi} = 0.$

*) If the function values and derivatives up to k-2 are all zero at the end-points (which is the case for P'-P), we get the old, limited definition of b-splines.

Therefore, the following three options for adaption are suggested (subroutine 'locadp'):

1) ξ is one of the underlying datapoints; the other datapoints are kept unchanged. The pp-function must be recomputed and the adaption will not be local, in general. The same (LU-decomposition of the) b-matrix can be used.

2) Let $i = \min(j \mid \frac{d}{dx} B_{j,k,T}(\xi) \mid > 0)$ $(k > 1)$.

   If

   $$\left|\frac{d}{dx}B_i(\xi)\right| < \left|\frac{d}{dx}B_{i-1}(\xi)\right|,$$

   we add to the pp-function

   $$\frac{a - P(\xi)}{B_i(\xi) - qB_{i-1}(\xi)} B_i(x) - \frac{a - P(\xi)}{B_i(\xi) - qB_{i-1}(\xi)} qB_{i-1}(x),$$

   with

   $$q = \frac{\frac{d}{dx} B_i(\xi)}{\frac{d}{dx} B_{i-1}(\xi)},$$

   else:

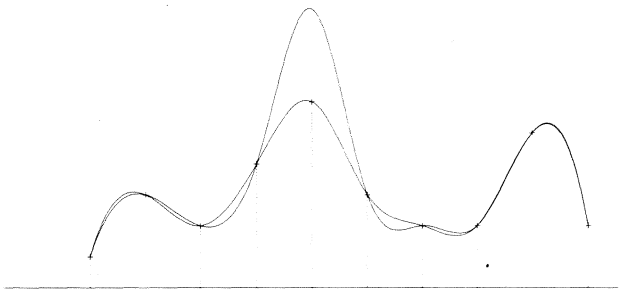
   $$\frac{a - P(\xi)}{B_{i-1}(\xi) - qB_i(\xi)} B_{i-1}(x) - \frac{a - P(\xi)}{B_{i-1}(\xi) - qB_i(\xi)} qB_i(x),$$

   with

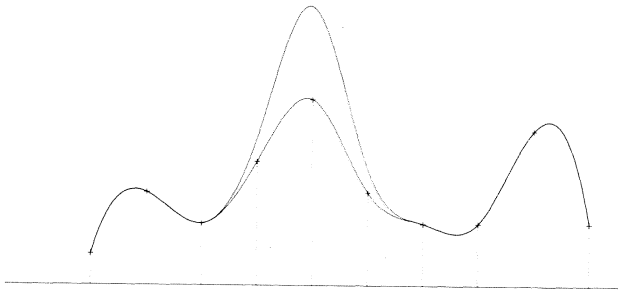
   $$q = \frac{\frac{d}{dx} B_{i-1}(\xi)}{\frac{d}{dx} B_i(\xi)}.$$

   The result is a pp-function satisfying conditions a1) and a3) (and maybe a2)). Note that only one b-coefficient will change, if $\frac{d}{dx} B_{i-1}(\xi) = 0$.

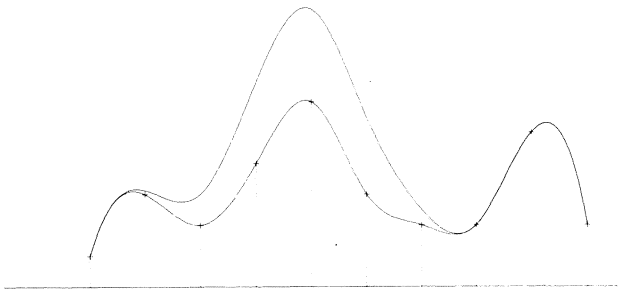
3) Let $t_i \le \xi < t_{i+1}$.
   We make use of property b2): $\sum_{j=i-k+1}^{i} B_j(\xi) = 1$.
   We add: $\sum_{j=i-k+1-s}^{i+t-1} (a-P(\xi))B_j(x)$, $s \ge 0$, $t \ge 0$ ($t = 0$ can be used in case $\xi = t_i$).



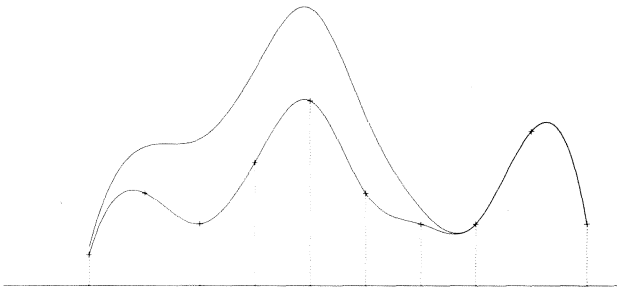
a. option 1.



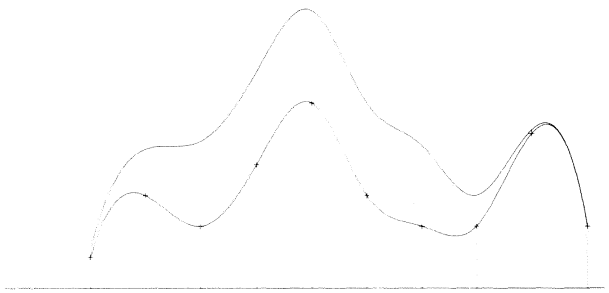
b. option 2.



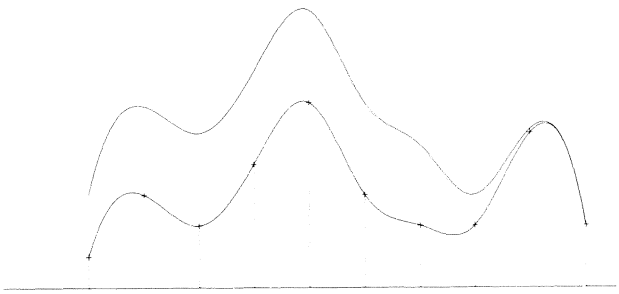
c. option 3: t=0, s=0.



d. option 3: t=0, s=1.



e. option 3: t=1, s=1.



f. option 3: t=1, s=2.

Fig. 2: different methods of local adaption.

In this case a1), a2) and a3) are satisfied ($t > 0$); however, the function changes on a longer Interval as compared with the adapted pp-function in option 2. If s and t are big enough the whole function will shift over a distance a.

For the calculation of $B_{j,k,T}(\xi)$ and $\frac{d}{dx} B_{j,k,T}(\xi)$ the subroutine 'valuex' is used, which returns the function value or the value of a derivative at a given point. The method is roughly that of 'ppfppr'.

3. PARAMETRIC CURVES

3.1. pp-parametric curves

We are going to look at curves in a two dimensional space (d = 2). The discussion is easily generalized to curves in spaces of higher dimension. If p_1 and p_2 both are functions of the same parameter t, $p_1 = p_1(t)$ and $p_2 = p_2(t)$, the curve $P(p_1,p_2) = P(t) = (p_1(t),p_2(t))$ is called a *parametric curve*. An important difference with *implicit curves*, $P(p_1,p_2) = 0$ (P an algebraic expression in p_1 and p_2), is the fact, that parametric curves can have *multiple values*, i.e., $P(t_i) = P(t_j)$, $t_i \neq t_j$, whereas implicit curves can not (see Fig. 5).

Sometimes it is possible to parametrize an implicit cuve. Moreover, for the plotting of an ellipse the parametric form is better suited:

$$P(t) = (a \cos 2\pi t,\ b \sin 2\pi t), \quad 0 \leq t < 1.$$

The points $P(t_i)$ with $t_i = \frac{i}{n}$, i = 0,...,n-i, are nicely distributed over the ellipse. We use the parametric form of ellipses in subroutine 'plotkn' for plotting the knot-markers of pp-parametric curves.

A *pp-parametric curve*, or *pp-curve* for short, P is a parametric curve with both p_1 and p_2 members of the same linear space of pp-functions $\mathbb{P}(k,\Xi,N)$ and $P(t) = (p_1(t),p_2(t))$.

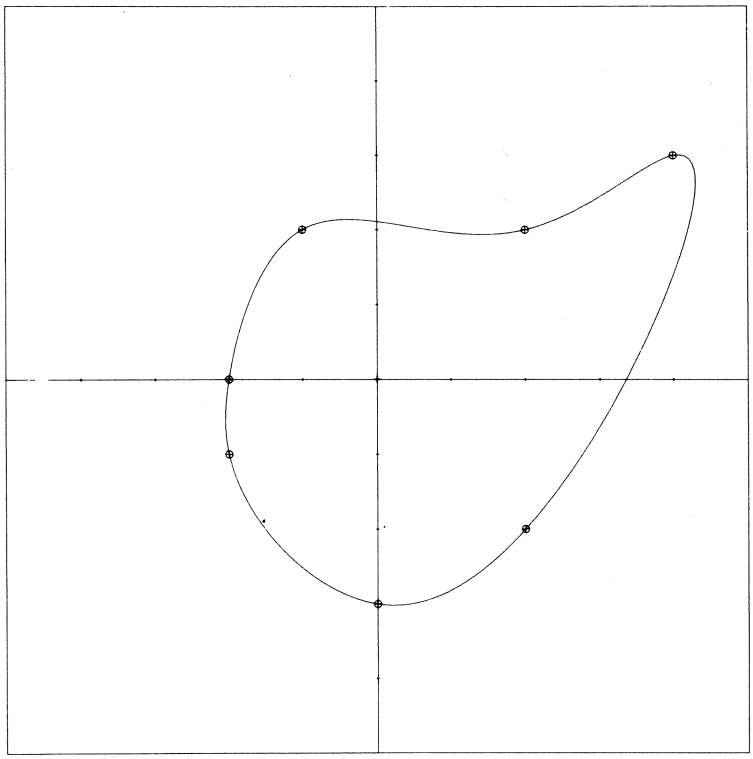
The pp-representation of a pp-curve of dimension d consists of k, Ξ and a set of pp-coefficients c(m,i,j), m = 1,...,d; i = 1,...,ℓ; j = 0,...,k-1. We might add a number ν_1 (or $\nu_{\ell+1}$) to N, $N^+ = N \cup \nu_1$, so that

$$P_\ell^{(j-1)}(\xi_{\ell+1}) = (c(1,1,j-1),c(2,1,j-1))$$

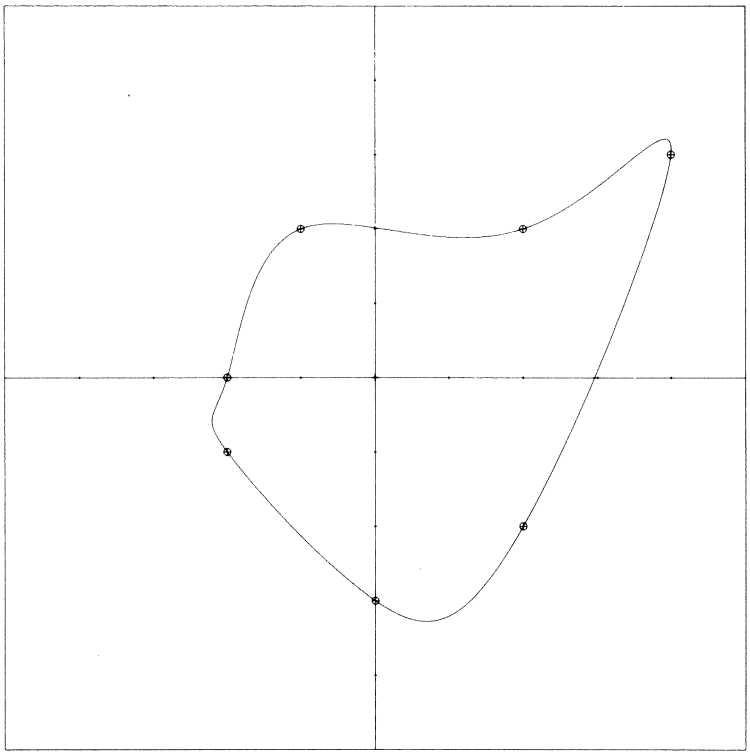
for $j = 1,\ldots,\nu_1$ (if $\nu_1 > 0$), with

$$P_i^{(j-1)}(\xi) = (p_1^{(j-1)}(\xi),p_2^{(j-1)}(\xi)).$$

The collection of pp-curves with dimension d, order k, breakpoints Ξ and numbers of continution conditions N^+ is denoted by $\mathbb{P}(d,k,\Xi,N^+)$. $\mathbb{P}(d,k,\Xi,N) = \mathbb{P}(d,k,\Xi,N^+)\Big|_{\nu_1=0}$. Members of $\mathbb{P}(d,k,\Xi,N^+)$ are said to be *cyclic* if $\nu_i > 0$, i = 1,...,ℓ.



a. datapoints chord-distant (ndist=1)



b. knots equidistant (ndist=2).

Fig. 3: cyclic pp-curve of order 4
(the knots are indicated by o, the datapoints by +).

3.2. The b-representation of cyclic pp-curves

Our aim is to construct a cyclic pp-curve through n-k+1 points $P(\tau_1),P(\tau_2),\ldots,P(\tau_{n-k+1})$, with $\tau_i < \tau_{i+1}$, using the theory of b-splines we have dealt with so far (subroutine 'ppcinc'). The datapoints τ_i, $i = 1,\ldots,n-k+1$ and the knots t_i, $i = 1,\ldots,n+k$, are not fixed for the present.

A datapoint τ_i is placed in the last ξ-interval where $B_{i,k,T^+} \neq 0$ (T^+ is defined below):

$$\xi_j \leq \tau < \xi_{j+1}, \quad B_{i,k,T^+}(\tau_i) \neq 0$$

and

$$B_{i,k,T^+}(\xi_{j+1} + \varepsilon) = 0, \quad \varepsilon > 0.$$

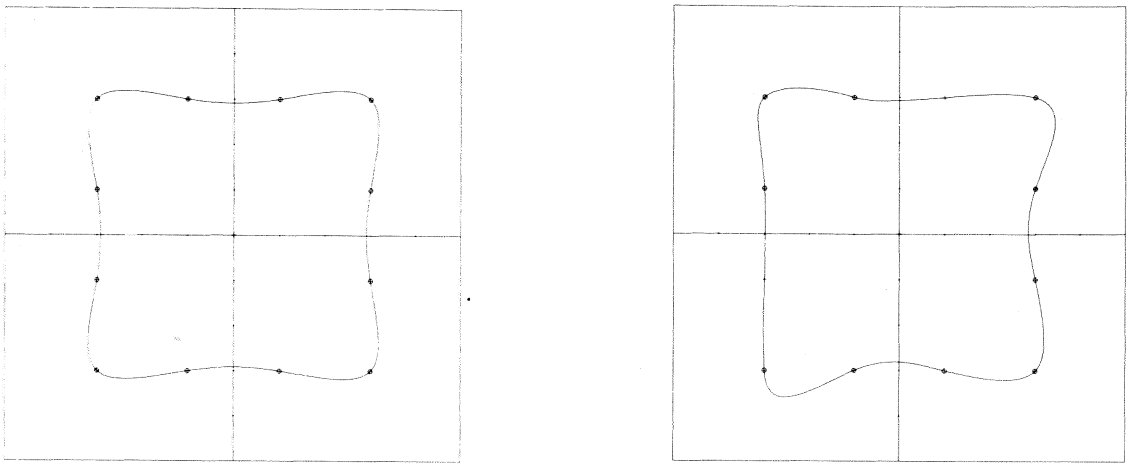
The format of the file, from which the points are read, is kept simple; three kinds of points are distinguished:

code 0 points: points that are datapoints and not knots*)
code 1 points: points that are only knots (*floating knots*)
code 2 points: points that are datapoints and at the same time knots.

An input file may look like:

2	$p_1(\tau_1)$	$p_2(\tau_1)$	knot & datapoint
0	$p_1(\tau_2)$	$p_2(\tau_2)$	datapoint
1			knot
0	$p_1(\tau_3)$	$p_2(\tau_3)$	datapoint
2	$p_1(\tau_4)$	$p_2(\tau_4)$	knot & datapoint
3	dt1	dt2	anti-cyclic continuation (see next chapter)
2	$p_1(\tau_5)$	$p_2(\tau_5)$	knot & datapoint
4			skip interval while plotting
2	$p_1(\tau_6)$	$p_2(\tau_6)$	knot & datapoint
5			end of input.

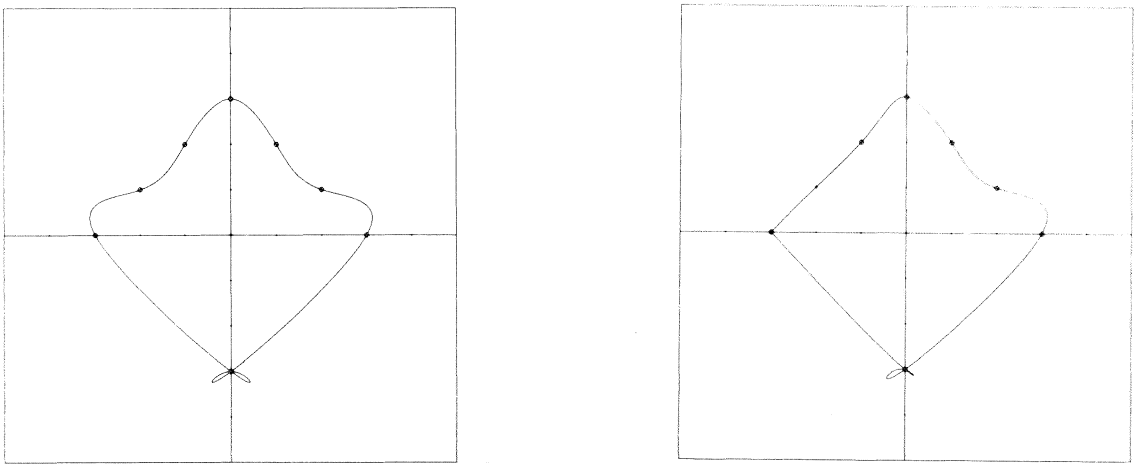
*) We use the word knot (datapoint) for $t_i(\tau_i)$, but also for $P(t_i)(P(\tau_i))$.



a. Without multiple knots

b. With two double knots

Fig. 4: Cyclic pp-curve (order 4, ndist=1) with multiple knots



a. Cyclic

b. Non-cyclic

Fig. 5: pp-curve (order 4, ndist=2) with a triple datapoint.

The following rules must be obeyed:

r1) the first point (item) must be a knot

r2) the last point must be equal to the first

r3) an anti-cyclic continuation must follow a datapoint

r4) a skip must follow a knot

r5) a knot (code 1) must be followed by a datapoint (code 0).

The data are digested according to the following:

1) Two options for the distances between knots and datapoints are considered:
   a) The breakpoints are equidistant: $\xi_{i+1} - \xi_i = c$, $i = 1,\ldots,\ell$.
      Datapoints of code 1 are equally spaced between the breakpoints.
   b) The datapoints are *chord-distant*: $\tau_{i+1}-\tau_i = \|P(\tau_{i+1})-P(\tau_i)\| =$
      $= ((p_1(\tau_{i+1}) - p_1(\tau_i))^2 + (p_2(\tau_{i+1}) - p_2(\tau_i))^2)^{\frac{1}{2}}$.
      Floating knots are placed halfway the two adjacent datapoints.

If the datapoints are chord-distant, an additional restriction is imposed upon the inputfile:

r6) If the chord-distant option holds, two consecutive datapoints must not be equal.

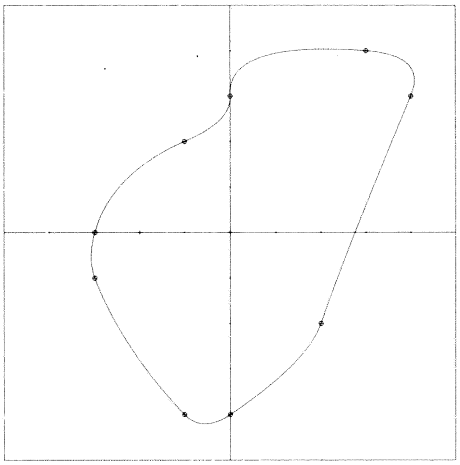
2) The multiplicity of a knot is equal to the number of preceding points of code 0 and 2 counting backwards as far as the first encountered point of code 1 or 2 inclusive.
   The first knot has multiplicity 1.
   So, in the above example we get knots of multiplicity 1,2,1,1 and 1.
   The first knot is t_k ($t_k = 0$), the last t_{n+1}. The first datapoint is τ_1.

3) The extension of the knot-sequence $t_1,\ldots,t_{k-1}$ and $t_{n+2},\ldots,t_{n+k}$, is fixed by:
   a) $t_i = t_k - (t_{n+1} - t_{n+1-k+i})$, $i = 1,\ldots,k-1$.
   b) $t_{n+1+i} = t_{n+1} + (t_{k+i} - t_k)$, $i = 1,\ldots,k-1$.
   The knot-sequence $t_1,\ldots,t_{n+k}$ is denoted by T^+.

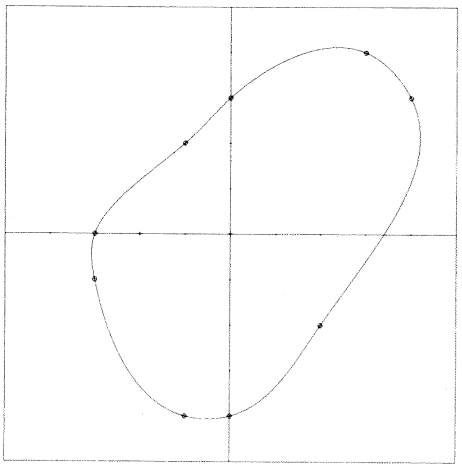
The b-coefficients $\alpha_{m,i}$, $m = 1,\ldots,d$; $i = 1,\ldots,n$, can now be determined:

1. For each co-ordinate we have k-1 *cyclic conditions*:

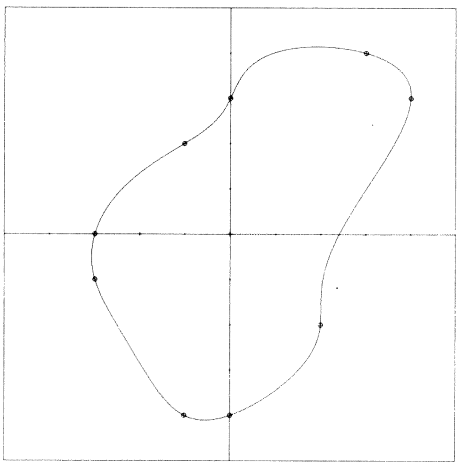
$$\alpha_{m,n-k+1+i} = \alpha_{m,i},\quad 1 \le m \le d,\ i = 1,\ldots,k-1.$$



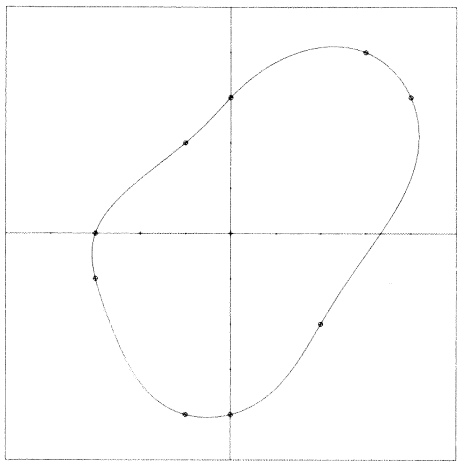
a. order 3



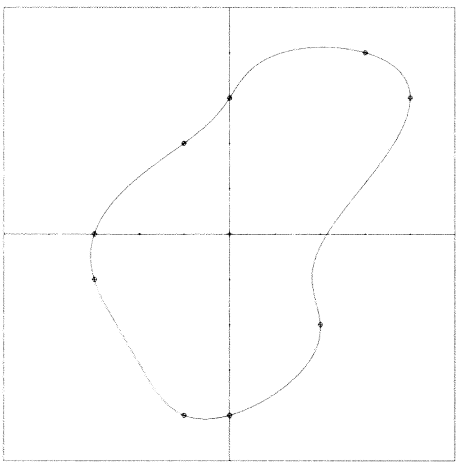
b. order 4



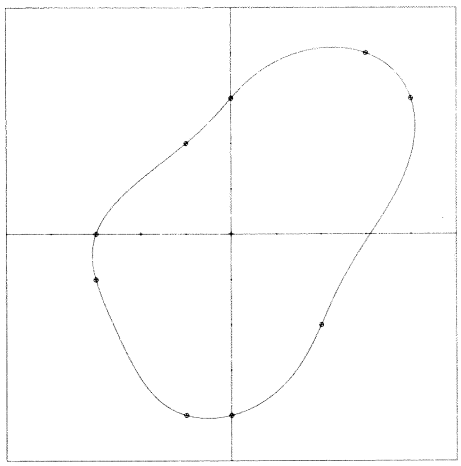
c. order 5



d. order 6



e. order 7



f. order 8

Fig. 6: Cyclic pp-curve (ndist=1)

2. For $\alpha_{m,i}$, m = 1,...,d; i = 1,...,n-k+1, we have d systems of n-k+1 equations:

$$\sum_{r=j-k+1}^{j} \alpha_{m,r} B_{r,k,T^+}(\tau_i) = p_m(\tau_i)$$

(provided that $t_j \leq \tau_i < t_j$), m = 1,...,d; i = 1,...,n-k+1.

As a consequence of the cylic conditions the numbers $B_{r,k,T^+}(\tau_i)$ will not be situated anymore within the band of the b-matrix with band-width 2k-1, if r > n-k+1. We have: $B_{ij} = B_{n-k+1+j,k,T^+}(\tau_i)$, if $1 \leq j \leq k-1$. Only the last k-1 rows may have non-zero elements in the first column.

Two ways of solving the systems are considered:

a. Without pivoting (subroutines "ludecy' and 'solsyc').
   The first n-2k+2 rows are condensed into a (n-2k+2) × (2k-1) matrix, the last k-1 rows are stored in a (k-1) × (n-k+1) matrix.
b. With complete pivoting (subroutines 'ludecp' and 'solscp').
   The permutation which must be performed on the input $(p_m(\tau_1),...,p_m(\tau_{n-k+1}))$ is kept in the 0-th column. The permutation which must be performed on the output $(\alpha_{m,1},...,\alpha_{m,n-k+1})$ is kept in the 0-th row.

In some cases it is possible (not guaranteed) to solve the systems without pivoting, for instance, if all datapoints are knots and the order is not too high.

Good results, i.e., pp-curves that are nicely smooth, are obtained with order 4, simple knots and datapoints chord-distant (see figures).

The same method can be used to produce non-cyclic pp-curves. This is done by reducing the multiplicity of the multiple knot at ξ_2 by [k/2]-1 and that of the multiple knot at $\xi_{\ell+1}$ by k-[k/2]-1. However, there are some limitations:

1. the first and the last read point must be code 2 points
2. the multiplicity of the multiple knot at $\xi_{\ell+1}$ must be k-[k/2]
3. the multiplicity of the multiple knot at ξ_2 must be at least [k/2]
4. the multiplicity of the multiple at $\xi_2,...,\xi_\ell$ must not exceed k-[k/2] (after reducing).

It is not necessary anymore that the first and the last point are equal. The extension of the knot-sequence can be taken arbitrary.

The different methods of local adaption as dealt with in Chapter 8 can also be applied to pp-parametric curves. If a point (p_1,p_2) on the curve is shifted, a procedure must be yielded, which finds the underlying (or one of the underlying) $\xi_\ell, \xi_1 \le \xi < \xi_{\ell+1}$.

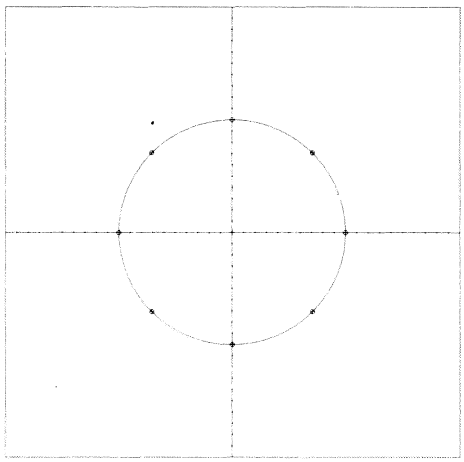


Fig. 7: Cyclic pp-curve (order 4) through eight points equidistant on a circle.

3.3. Anti-cyclic continuation

A parametric curve P has an *anti-cyclic continuation* at ξ, if

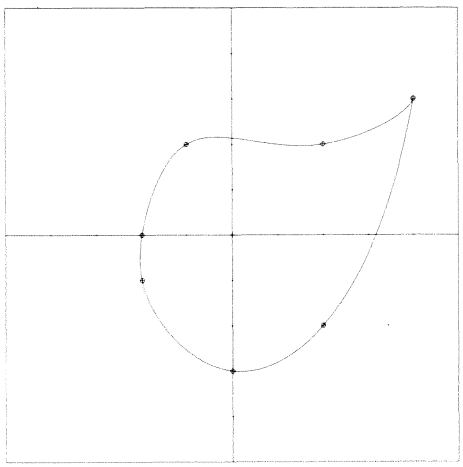
$$\lim_{x \downarrow \xi} P^{(1)}(x) = - \lim_{x \uparrow \xi} P^{(1)}(x),$$

i.e.,

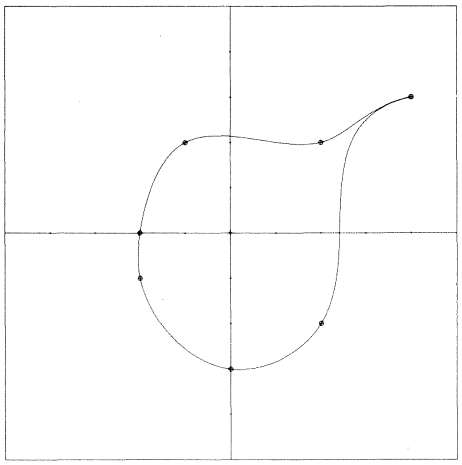
$$\lim_{x \downarrow \xi} p_m^{(1)}(x) = - \lim_{x \uparrow \xi} p_m^{(1)}(x), \quad m = 1,\ldots,d.$$

As to pp-curves there are several ways to impose an anti-cyclic continuation upon the curve. One way is to give anti-cyclic boundary conditions (see [4], p.127). A drawback of this method is, that only one anti-cyclic continuation can be effectuated.

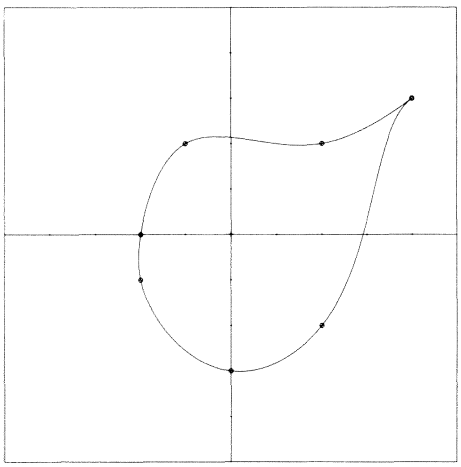
Using the (a) b-representation we can construct an anti-cyclic continuation at a breakpoint ξ_{ii} as follows: Let $\bigcup_{i=j-k+\nu_{ii}+1}^{j} t_i$ be the (multiple) knot going with ξ_{ii}. Between t_j and t_{j+1} two additional knots are inserted, t_a and t_b say, with $t_a - t_j = dt1$, $t_b - t_a = dt2$ and $t_{j+1} - b = t_{j+1} - t_j$ $(dt1, dt2 \geq 0)$. If $dt1 = 0$, then $dt2 \neq 0$. Consequently, two extra b-coefficients (for each co-ordinate) are to be determined.



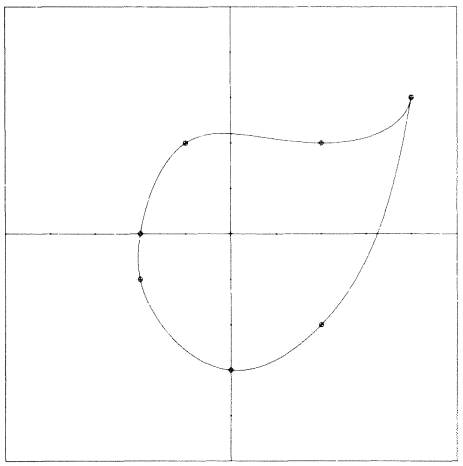
a. (2,0)-anti-cyclic



b. (0,2)-anti-cyclic



c. (1,1)-anti-cyclic



d. (1000,0)-anti-cyclic

Fig. 8: Cyclic pp-curve (order 4, ndist=1) with an anti-cyclic continuation (anti-clockwise)

The anti-cyclic continuation is effectuated by the two conditions:

1) $P(t_b) = P(t_j)$

2) $P^{(1)}(t_b) + P^{(1)}(t_j) = 0.$

After the calculation of the pp-curve the one or two intervals between t_j and t_b are omitted.

We use property b4) to write the second equation as a linear combination of b-coefficients:

$$p_m^{(1)}(t_j) = \frac{d}{dx}\left(\sum_i \alpha_{m,i} B_{i,k}(x)\right)\Big|_{x=t_j} = \sum_i (k-1)\frac{\alpha_{m,i}-\alpha_{m,i-1}}{t_{i+k-1}-t_i} B_{i,k-1}(t_j).$$

Since $B_{i,k-1}(t_j) = 0$ for $i < j-k+2$ or $i > j-1$, we have:

$$p_m^{(1)}(t_j) = (k-1)\sum_{i=j-k+2}^{j-1} \frac{\alpha_{m,i}-\alpha_{m,i-1}}{t_{i+k-1}-t_i} B_{i,k-1}(t_j)$$

$$= (k-1)\left(\frac{\alpha_{m,j-k+2}-\alpha_{m,j-k+1}}{t_{j+1}-t_{j-k+2}} B_{j-k+2,k-1}(t_j) + \ldots + \frac{\alpha_{m,j-1}-\alpha_{m,j-2}}{t_{j+k-2}-t_{j-1}} B_{j-1,k-1}(t_j)\right)$$

$$= (k-1)\,(-w_{j-k+2}\alpha_{m,j-k+1} + w_{j-k+2}\alpha_{m,j-k+2} - w_{j-k+3}\alpha_{m,j-k+2} +$$

$$+ w_{j-k+3}\alpha_{m,j-k+3} - \ldots - w_{j-1}\alpha_{m,k-2} + w_{j-1}\alpha_{m,j-1}),$$

with

$$w_i = \frac{B_{i,k-1}(t_j)}{t_{i+k-1}-t_i}.$$

So, we get

$$p_m^{(1)}(t_j) = (k-1)\,(-w_{j-k+2}\alpha_{m,j-k+1} + \sum_{i=j-k+2}^{j-2} (w_i - w_{i+1})\alpha_{m,i} + w_{j-1}\alpha_{m,j-1}),$$

or

$$p_m^{(1)}(t_j) = \sum_{i=j-k+1}^{j-1} (w_i - w_{i+1})\alpha_{m,i},$$

with

$$w_{j-k+1} = w_j = 0.$$

We can alter the shape of the curve by changing the parameters dt1 and dt2. We could say, that the pp-curve is (dt1,dt2)-anti-cyclic at ξ_{ii} (see Fig. 8). Of course, more than one anti-cyclic continuation is possible (see Fig. 9).

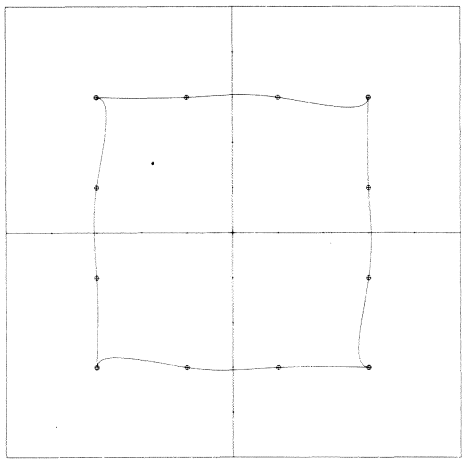


Fig. 9: Cyclic pp-curve (order 4) with four (2,0)-anti-cyclic continuations (anti-clockwise)

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APPENDIX

Remarks

1. The programs are written in C, an Algol/Pascal-like programming language developed by DENNIS RITCHIE at the Bell Laboratories, New Jersey. See for a description [5].
2. Some subroutines form part of the interface CILP with the graphical language ILP: 'pict', 'with', 'mdcontrol', 'draw', 'ward', 'scale', 'endpict', 'newpel' and 'line'. ILP is described in [6].
3. The pictures are drawn on a high-resolution display (HRD).
4. The contents of the appendix is enumerated a) - h) as follows:

a) 'testpr31.c'
   with the subroutines
   'ppcinc' pp-curve interpolation with complete pivoting in the cyclic case.
   'ppfint' pp-function interpolation
   'ludeco' LU-decomposition of banded b-matrix without pivoting
   'solsys' solution of system of equations without pivoting
   'bsplvx' b-spline values at x
   'ludecp' LU-decomposition of b-matrix with complete pivoting
   'solscp' solution of system of equations with complete pivoting
   'ppcppr' conversion of b-representation of pp-curve to pp-representation
   'ppfppr' conversion of b-representation of pp-function to pp-representation
   'plotpc' plotting pp-curve
   'plotdp' plotting datapoint-markers of pp-curve
   'plotkn' plotting knot-markers of pp-curve
   'axes2d' axes 2-dimensional.
b) subroutine 'plotpf' plotting pp-function
c) subroutine 'cubspl' cubic spline interpolation
d) subroutine 'locadp' local adaption of pp-function
e) subroutine 'ludecy' LU-decomposition cyclic without pivoting
f) subroutine 'solsyc' solution of system of equations cyclic without pivoting
g) subroutine 'valuex' value of (derivative of) pp-function at x
h) subroutine 'interv' left endpoint of interval containing x.

APPENDIX A

```
1
2   #include <cilp.h>
3   #include <math.h>
4   #include <stdio.h>
5   #define MAXP 30
6   #define K 4
7   #define KP 5
8   #define KPKM1P 8
9   #define DIMP 3
10  #define DIM2P 5
11  double sqrt();
12  double sin();
13  double cos();
14  FILE *fd, *fopen();
15
16  main()
17  /* calls ppcinc ppcppr plotpc plotkn plotdp axes2d */
18  /* computes and plots pp-curves */
19
20  { double p[MAXP][DIMP], q[MAXP][MAXP], ql[MAXP][KPKM1P];
21    double bcoefp[DIMP][MAXP], t[MAXP], tau[MAXP];
22    double coefp[DIMP][MAXP][KP], breakp[MAXP];
23    double rscale[DIMP], origin[DIMP], pb[MAXP];
24    int dia;
25    int k, l, n, dim, cyc, ndist, plotc, numberstep;
26    fd = fopen("out", "w");
27    scanf(" %f %f", &rscale[1], &rscale[2]);
28    scanf(" %d", &dia);
29    k = K;
30    dim = DIMP - 1;
31    scanf("%d %d %d %d", &cyc, &ndist, &plotc, &numberstep);
32    ppcinc(dim, cyc, ndist, k, &n, p, q, ql, bcoefp, t, tau, pb);
33    ppcppr(dim, bcoefp, t, n, k, coefp, breakp, &l);
34    if (plotc == 1)
35    { pict(2, "pp-curve");
36      with(); scale(rscale[1], rscale[2]);
37      if (dia == 1)
38      { mdcontrol("HRD:diazo");
39        mdcontrol("HRD:feed");
40        mdcontrol("HRD:title:#pp-curve#");
41      }
42      draw();
43    }
44    plotpc(plotc, numberstep, dim, k, 1, coefp, breakp, pb);
45    if (plotc == 1)
46    { plotkn(rscale, breakp, coefp, 1, cyc, pb);
47      plotdp(rscale, p, n, cyc);
48      axes2d(rscale, origin, 20);
```

```
49        ward();
50        endpict();
51      }
52    }
53
54    int ppcinc(dim, cyc, ndist, k, n, p, q, ql, bcoefp, t, tau, pb)
55    /* calls ppfint, ludecp/ludepp and solscp/solspp */
56    /* reads points, continuation conditions and plot-instructions
57     * from an inputfile and calculates the b-coefficients of the
58     * pp-curve through the points, according to the given order,
59     * knot-distance option and cyc option.
60     * see for the format of the inputfile and the restrictions
61     * imposed on it, the following paper: 'the b-representation of
62     * piecewise polynomial parametric curves and local adaption', ch. 3.2.
63     * input : dim : dimension.
64     *         cyc : cyclic (cyc = 1) or not cyclic (cyc = 2).
65     *         ndist : tau is chord-distant (ndist = 1) or t is
66     *             equidistant (ndist = 2).
67     *         k : order.
68     * output : n : number of b-coefficients.
69     *          p[i][0] : pointcodes.
70     *          p[i][1],...,p[i][dim] : datapoints as read from file.
71     *          t : knotsequence.
72     *          tau : abscissae of the datapoints.
73     *          q : b-matrix, (n-k+1) x (n-k+1); used in case cyc = 1.
74     *          ql : condensed b-matrix, n x (2k-1); used in case cyc = 2.
75     *          bcoefp : matrix of b-coefficients, dim x n.
76     */
77
78    int dim, cyc, ndist, *n, k;
79    double p[MAXP][DIMP], q[MAXP][MAXP], ql[MAXP][KPKM1P];
80    double bcoefp[DIMP][MAXP], t[MAXP], tau[MAXP], pb[MAXP];
81    { double sumsq, dist, bcoef[MAXP], gtau[MAXP], tauii, dt;
82      int mult, ii, jj, i, j, m, nmkpl, mm, left, km2div2, tt;
83    /* m : current pointcode;
84     *     m = 0 : datapoint, not a knot.
85     *     m = 1 : floating knot (only if ndist = 2).
86     *     m = 2 : datapoint & knot.
87     *     m = 3 : anti-cyclic continuation.
88     * mm = 0 : last read knot was datapoint
89     * mm = 1 : last read knot was floating
90     */
91      for (i = 1; i < MAXP; ++i)
92      for (j = 1; j < MAXP; ++j)
93      q[i][j] = 0.0;
94      sumsq = 0.0;
95      dist = 0.0;
96      mult = 1;
97      ii = 0;
98      i = k;
99      pb[(tt = 1)] = -1.0;
100     scanf("%d", &m);
101     if (m == 1 || m == 2)
102     { t[k] = 0.0;
103       if (m == 2)
104       { for (j = 1, ++ii ; j <= dim; ++j)
```

```
105         scanf("%f", &p[1][j]);
106         p[1][0] = m;
107         tau[1] = t[k];
108         mm = 0;
109       }
110       else
111       mm = 1;
112       ++i;
113     }
114     else
115     { fprintf(fd, "\nwrong data(1)\n");
116       return(1);
117     }
118     scanf("%d", &m);
119     while (m <= 3)
120     { if ( m == 3)
121       { for (j = 1; j <= 2; ++j)
122         { ++ii;
123           scanf(" %f ", &dt);
124           tau[ii] = t[i] = t[i-1] + dt;
125           if (t[i] > t[i-1]) pb[tt++] = t[i-1];
126           ++i;
127         }
128         p[ii-1][0] = 3.0;
129         p[ii][0] = 2.0;
130         for (jj = 1; jj <= dim; ++jj)
131         { p[ii-1][jj] = 0.0;
132           p[ii][jj] = p[ii-2][jj];
133         }
134         mm = 0;
135         mult = 1;
136       }
137       else if (m != 1)
138       { for ( j = 1, ++ii ; j <= dim; ++j)
139         { scanf("%f", &p[ii][j]);
140           if (ndist == 1)
141           sumsq += (p[ii][j]-p[ii-1][j]) * (p[ii][j]-p[ii-1][j]);
142         }
143         p[ii][0] = m;
144         if (ndist == 1)
145         dist += sqrt(sumsq);
146         sumsq = 0.0;
147       }
148       if (m == 0)
149   /* datapoint, not a knot */
150       { if (ndist == 1)
151         tau[ii] = t[i-1] + dist;
152         ++mult;
153       }
154       else if (m == 1)
155   /* floating knot */
156       { if ((mult == 1 && mm == 1) || ndist == 1)
157     { fprintf(fd, "\nwrong data(2)\n");
158           return(1);
159         }
160         dist = 1.0 / mult;
```

```
161         tau[ii-mult+2] = t[i-1] + dist;
162         t[i] = t[i-1] + 1.0;
163         ++i;
164         for ( j = 3; j <= mult; ++j, ++i)
165         { tau[ii-mult+j] = tau[ii-mult+j-1] + dist;
166           t[i] = t[i-1];
167         }
168         if (mm == 0 && mult > 1)
169         { t[i] = t[i-1];
170           ++i;
171         }
172         mult = 1;
173         mm = 1;
174       }
175       else if (m == 2)
176   /* knot & datapoint */
177       { if (ndist == 1)
178         { tau[ii] = t[i] = t[i-1] + dist;
179           ++i;
180           dist = 0.0;
181         }
182         else
183         { if (mult == 1 && mm == 1)
184     { fprintf(fd, "\nwrong data(3)\n");
185           return(1);
186           }
187           dist = 1.0 / mult;
188           t[i] = t[i-1] + 1.0;
189           tau[ii-mult+1] = t[i-1] + dist;
190           ++i;
191           for (j = 2; j <= mult; ++j)
192           tau[ii-mult+j] = tau[ii-mult+j-1] + dist;
193         }
194         for (j = 3; j <= mult; ++j, ++i)
195         t[i] = t[i-1];
196         if (mm == 0 && mult > 1)
197         { t[i] = t[i-1];
198           ++i;
199         }
200         mult = 1;
201         mm = 0;
202       }
203       scanf("%d", &m);
204     }
205     if (t[i-1] < tau[ii])
206     { fprintf(fd, "\nwrong data(4)\n");
207       return(1);
208     }
209     *n = i - 2;
210   /* printing of the points after digesting */
211   for (j = 1; j <= *n + k; ++ j)
212   { fprintf(fd, "%f   %f   ", t[j], tau[j]);
213     for (m = 0; m <= dim; ++m)
214     fprintf(fd, "%f   ", p[j][m]);
215     fprintf(fd, "\n");
216   }
```

```
217   fprintf(fd, "\n");
218     if (cyc == 2)
219     { km2div2 = (k - 2) / 2;
220       if (t[i-1] > tau[ii]) --*n;
221       if (t[k+1] < t[k+1+km2div2] || t[*n+1-km2div2] < t[*n+1])
222       { fprintf(fd, "\nwrong data(5)\n"); return(1); }
223       for (i = 1; i <= *n - k - k + 3; ++i)
224       t[i+k] = t[i+k+km2div2];
225       *n -= k-2;
226     }
227     if (*n < 2 * k)
228     { fprintf(fd, "\nwrong data(6)\n");
229       return(1);
230     }
231   /* extension t */
232     for (i = 1; i < k; ++i)
233     { t[i] = t[k] - (t[*n+1] - t[*n+1-k+i]);
234       t[*n+1+i] = t[*n+1] + (t[k+i] - t[k]);
235     }
236   /* printing of the points after extension t */
237   for (i = 1; i <= *n + k; ++ i)
238   { fprintf(fd, "%f   %f   ", t[i], tau[i]);
239     for (j = 0; j <= dim; ++j)
240     fprintf(fd, "%f   ", p[i][j]);
241     fprintf(fd, "\n");
242   }
243   fprintf(fd, "\n");
244     if (cyc == 2)
245     for (j = 1; j <= dim; ++j)
246     { for ( i = 1; i <= *n+1; ++i)
247       gtau[i] = p[i][j];
248       if (ppfint(tau, gtau, t, *n, k, q1, bcoefp[j]) == 2)
249       return(2);
250     }
251     else
252     { nmkp1 = *n - k + 1;
253   /* fill matrix */
254       left = k;
255       for (ii = 1; ii <= nmkp1; ++ii)
256       { if (p[ii][0] != 3.0)
257         { tauii = tau[ii];
258           for (; tauii >= t[left+1]; ++left);
259           bsp1vx(t, k, 1, tauii, left, bcoef);
260           for (i = left - k + 1, j = 1; j <= k; ++i, ++j)
261           { if (i > nmkp1) i -= nmkp1;
262             q[ii][i] += bcoef[j];
263           }
264         }
265         else
266         for (j = 1; j <= 2; ++j)
267         { bcoef[0] = bcoef[k-1] = 0.0;
268           bsp1vx(t, k-1, 1, tauii, left, bcoef);
269           for (i = left - k + 2, jj = 1; jj <= k - 1; ++i, ++jj)
270           { bcoef[jj] /= (t[i+k-1] - t[i]);
271             if (i > nmkp1 + 1) i -= nmkp1;
272             q[ii][i-1] -= (bcoef[jj-1] - bcoef[jj]) * (k - 1);
```

```
273           }
274           if (j == 1)
275           { if (t[left] < tauii) left += 3;
276             else if (t[left] == t[left-1]) ++left;
277             else left += 2;
278             tauii = t[left];
279           }
280         }
281       }
282   /* printing of the b-matrix before lu-decomposition */
283   for (j = 0; j <= nmkpl;)
284   { for (i = 0; i <= nmkpl; ++i)
285     { for (ii = j; ii <= j + 5 && ii <= nmkpl; ++ii)
286       fprintf(fd, "  %f", q[i][ii]);
287       fprintf(fd, "\n");
288     }
289     fprintf(fd, "\n");
290     j += 6;
291   }
292       m = ludecp(q, nmkpl);
293   /* printing of the b-matrix after lu-decomposition */
294   for (j = 0; j <= nmkpl;)
295   { for (i = 0; i <= nmkpl; ++i)
296     { for (ii = j; ii <= j + 5 && ii <= nmkpl; ++ii)
297       fprintf(fd, "  %f", q[i][ii]);
298       fprintf(fd, "\n");
299     }
300     fprintf(fd, "\n");
301     j += 6;
302   }
303       if (m == 2)
304       { fprintf(fd, "\nb-matrix in ppcinc not invertible\n");
305         return(2);
306       }
307       for (j = 1; j <= dim; ++j)
308       { for ( i = 1; i <= nmkpl; ++i)
309         bcoef[i] = p[i][j];
310         solscp(q, nmkpl, bcoef);
311         for (i = 1; i <= nmkpl; ++i)
312         { bcoefp[j][i] = bcoef[i];
313           if (i < k)
314           bcoefp[j][nmkpl+i] = bcoef[i];
315         }
316       }
317     }
318     return(0);
319   }
320
321   int ppfint(tau, gtau, t, n, k, q, bcoef)
322   /* calls bsplvx solsys ludeco */
323   /* calculates the n b-coefficients (bcoef) of a pp-function with:
324    *     knotsequence t,
325    *     datapoints (tau[i], gtau[i]), i=1,...,n.
326    *     order k.
327    * q is the condensed b-matrix (n x (2k-1)).
328    */
```

```
329   double tau[MAXP], gtau[MAXP];
330   double t[MAXP], q[MAXP][KPKM1P], bcoef[MAXP];
331   int n, k;
332   { int i, npl,  ilplmx, j, jj, kml, left;
333     double taui;
334     npl = n+1;
335     kml = k-1;
336     left = k;
337     for (i = 1; i <= n; ++ i)
338     for (j = 1; j <= k+kml; ++j)
339     q[i][j] = 0.0;
340     for (i = 1; i <= n; ++i)
341     { taui = tau[i];
342       ilplmx = ((i + k) < npl) ? (i + k) : npl;
343       left = (left > i) ? left : i;
344       if (taui < t[left])
345       { fprintf(fd, "b-matrix in ppfint not invertible (1)\n");
346         return(2);
347       }
348       do
349       { if (taui < t[left + 1]) break;
350         else ++left;
351       }
352       while (left < ilplmx);
353       if (left >= ilplmx)
354       { --left;
355         if (taui > t[left+1])
356         { fprintf(fd, "b-matrix in ppfint not invertible (2)\n");
357           return(2);
358         }
359       }
360       bsplvx(t, k, 1, taui, left, bcoef);
361       for (j = left-i+1, jj = 1; jj <= k; ++j, ++jj)
362       q[i][j] = bcoef[jj];
363     }
364     if (ludeco(q, k, n) == 2)
365     { fprintf(fd, "b-matrix in ppfint not invertible (3)\n");
366       return(2);
367     }
368     for (i = 1; i <= n; ++i)
369     bcoef[i] = gtau [i];
370     solsys(q, k, n, bcoef);
371     return(1);
372   }
373
374   int ludeco(q, k, n)
375   /* lu-decomposition of banded n x n matrix, bandwidth 2k-1,
376    * without pivoting; the band is stored in q (n x (2k-1)).
377    */
378   double q[MAXP][KPKM1P];
379   int k, n;
380   { int l, nr, i, j, jj, kml, ipl;
381     double qik;
382     kml = k-1;
383     for (i = 1; i <= n; ++i)
384     { l = k;
```

```
385       ipl = i+1;
386       nr = ((nr = i+kml) <= n) ? nr : n;
387       if ((qik = q[i][k]) == 0) return(2);
388       for (j = ipl; j <= nr; ++j)
389       { l -= 1;
390         if (q[j][l] != 0.0)
391         { q[j][l] /= qik;
392           for (jj = 1; jj <= kml; ++jj)
393           q[j][l+jj] -= q[j][l] * q[i][k+jj];
394         }
395       }
396     }
397     return(1);
398   }
399
400   solsys(q, k, n, bcoef)
401   /* solves a system of equations;
402    * to be used with ludeco;
403    * the right-values are expected in bcoef, the solution
404    * is put back in bcoef.
405    */
406   double q[MAXP][KPKM1P], bcoef[MAXP];
407   int k, n;
408   { int l, i, j, kml, kpl, kpkml;
409     kml = k-1;
410     kpl = k+1;
411     kpkml = k+kml;
412   /* forward step */
413     for (i = 2; i <= n; ++i)
414     { l = (k > i)? (k-i+1) : 1;
415       for (j = 1; j <= kml; ++j)
416       bcoef[i] -= q[i][j] * bcoef[j-k+i];
417     }
418   /* backward step */
419     for (i = n; i >= 1; --i)
420     { l = (n-i < k)? (n-i+k) : kpkml;
421       for (j = 1; j >= kpl; --j)
422       if (q[i][j] != 0.0)
423       bcoef[i] -= q[i][j]  * bcoef[i+j-k];
424       bcoef [i] /= q[i][k];
425     }
426   }
427
428   bsplvx(t, jhigh, indexx, x, left, biatx)
429   /* calculates the values of all possibly non-zero b-splines
430    * at x of order
431    *        jhigh, if index = 1,
432    *        max(jhigh, j+1), if index = 2.
433    * further input: t, the knot sequence; left, an integer,
434    * such that t[left] <= x < t[left+1].
435    * if index = 1, the calculation starts from the beginning
436    * (i.e. with order = 1);
437    * if index = 2, the calculation continues where it left of.
438    * The value of j and the auxiliary arrays deltal and
439    * deltar are therefore saved.
440    * output: biatx[i], 1 <= i <= order, with
```

```
441     * biatx[i] = b[left-order+i][order](x).
442     */
443   double t[MAXP], x, biatx[KP];
444   int jhigh, indexx, left;
445   { static double deltal[20], deltar[20];
446     static int j;
447     int i, jpl;
448     double saved, term;
449     if (indexx == 1)
450     { j = 1;
451       biatx[1] = 1;
452     }
453     if ((indexx == 1 && j < jhigh) || indexx == 2)
454     do
455     { jpl = j + 1;
456       deltar[j] = t[left+j] - x;
457       deltal[j] = x - t[left+1-j];
458       saved = 0.0;
459       for (i = 1; i <= j; ++i)
460       { term = biatx[i] / (deltar[i] + deltal[jpl-i]);
461         biatx[i] = saved + deltar[i] * term;
462         saved = deltal[jpl-i] * term;
463       }
464       biatx[jpl] = saved;
465       j = jpl;
466     }
467     while (j < jhigh);
468   }
469
470   int ludecp(q, n)
471   /* lu-decomposition with complete pivoting of an n x n matrix q;
472    * the permutation to be performed on the input (output)
473    * is kept in the 0-th column (row).
474    */
475   double q[MAXP][MAXP];
476   int n;
477   { int i, j, pc, pr, ii, ipl;
478     double pivot, hulp, abspiv, absq;
479     q[1][0] = q[0][1] = 1.0;
480     for (i = 2; i <= n; ++i)
481     q[i][0] = q[0][i] = q[0][i-1] + 1.0;
482     for (i = 1; i <= n; ++i)
483     { ipl = i + 1;
484       pivot = q[i][i];
485       abspiv = (pivot > 0.0)? pivot : -pivot;
486       pr = pc = i;
487       for (ii = i; ii <= n; ++ii)
488       for (j = i; j <= n; ++j)
489       { absq = (q[ii][j] > 0.0)? q[ii][j] : -q[ii][j];
490         if (absq > abspiv)
491         { pc = j;
492           pr = ii;
493           pivot = q[ii][j];
494           abspiv = (pivot > 0.0)? pivot : -pivot;
495         }
496       }
```

```
497       if (pivot == 0.0) return(2);
498       if (pc != i)
499       for (j = 0; j <= n; ++j)
500       { hulp = q[j][i];
501         q[j][i] = q[j][pc];
502         q[j][pc] = hulp;
503       }
504       if (pr != i)
505       for (j = 0; j <= n; ++j)
506       { hulp = q[i][j];
507         q[i][j] = q[pr][j];
508         q[pr][j] = hulp;
509       }
510       for (j = ipl; j <= n; ++j)
511       if (q[j][i] != 0.0)
512       { q[j][i] /= pivot;
513         for (ii = ipl; ii <= n; ++ii)
514         q[j][ii] -= q[j][i] * q[i][ii];
515       }
516     }
517     return(1);
518   }
519
520   solscp(q, n, bcoef)
521   /* solves a system of equations;
522    * to be used with ludecp;
523    * the right-values are expected in bcoef, the solution
524    * is put in bcoef again.
525    */
526   double q[MAXP][MAXP], bcoef[MAXP];
527   int n;
528   { int i, j;
529     double hulp[MAXP];
530     for (i = 1; i <= n; ++i)
531     hulp[i] = bcoef[(int) q[i][0]];
532     for (i = 1; i <= n; ++i)
533     bcoef[i] = hulp[i];
534     for (i = 2; i <= n; ++i)
535     for (j = 1; j < i; ++j)
536     if (q[i][j] != 0.0)
537     bcoef[i] -= q[i][j] * bcoef[j];
538     for (i = n; i >= 1; --i)
539     { for (j = n; j > i; --j)
540       if (q[i][j] != 0.0)
541       bcoef[i] -= q[i][j] * bcoef[j];
542       bcoef[i] /= q[i][i];
543     }
544     for (i = 1; i <= n; ++i)
545     hulp[(int) q[0][i]] = bcoef[i];
546     for (i = 1; i <= n; ++i)
547     bcoef[i] = hulp[i];
548   }
549
550   ppcppr(dim, bcoefp, t, n, k, coefp, breakp, 1)
551   /* calls ppfppr */
552   /* computes the pp-representation of a pp-curve starting from a
```

```
553     * b-representation.
554     * input : dim : dimension.
555     *          bcoefp : b-coefficients
556     *          t : knotsequence.
557     *          n : number of b-coefficients.
558     *          k : the order.
559     * output : coefp : pp-coefficients (dim x l x k)
560     *           breakp : sequence of breakpoints.
561     *           l : number of intervals.
562     */
563    int n, k, *l, dim;
564    double t[MAXP], coefp[DIMP][MAXP][KP], breakp[MAXP];
565    double bcoefp[DIMP][MAXP];
566    { int i;
567      for (i = 1; i <= dim; ++i)
568      ppfppr(t, bcoefp[i], n, k, breakp, coefp[i], l);
569    }
570
571    ppfppr(t, bcoef, n, k, breakp, coef, l)
572    /* calls bsplvx */
573    /* computes the pp-representation of a pp-function starting
574     * from a b-representation;
575     * input : t : knotsequence.
576     *          bcoef : b-coefficients.
577     *          k : the order.
578     *          n : number of b-coefficients.
579     * output : breakp : sequence of breakpoints.
580     *           coef : the pp-coefficients.
581     *           l : number of intervals.
582     */
583    int n, k, *l;
584    double t[MAXP], bcoef[MAXP], breakp[MAXP], coef[MAXP][KP];
585    { double scrtch[KP][KP], diff, sum, biatx[KP];
586      int left, lsofar, i, j, kmj, jpl;
587      lsofar = 0;
588      breakp[l] = t[k];
589      for (left = k; left <= n; ++left)
590      if (t[left+1] > t[left])
591      { lsofar += 1;
592        breakp[lsofar+1] = t[left+1];
593        if (k == 1)
594        coef[lsofar][1] = bcoef[left];
595        else
596        { for (i = 1; i <= k; ++i)
597          scrtch[i][1] = bcoef[left-k+i];
598          for (jpl = 2; jpl <= k; ++jpl)
599          { j = jpl -1;
600            kmj = k - j;
601            for (i = 1; i <= kmj; ++i)
602            { diff = t[left+i] - t[left+i-kmj];
603              if (diff > 0.0)
604              scrtch[i][jpl]=((scrtch[i+1][j]-scrtch[i][j])/diff)*kmj;
605            }
606          }
607          bsplvx(t, 1, 1, t[left], left, biatx);
608          coef[lsofar][k] = scrtch[1][k];
```

```
609         for (jpl = 2; jpl <= k; ++jpl)
610         { bsplvx(t, jpl, 2, t[left], left, biatx);
611           kmj = k + 1 - jpl;
612           sum = 0.0;
613           for (i = 1; i <= jpl; ++i)
614           sum += biatx[i] * scrtch[i][kmj];
615           coef[lsofar][kmj] = sum;
616         }
617       }
618     }
619     *l = lsofar;
620   }
621
622   plotpc(plotc, numberstep, dim, k, l, coefp, breakp, pb)
623   /* plots a pp-curve;
624    * plotc = 1 : the calculated (points on the) curve is (are)
625    *             given to a display.
626    * plotc = 2 : the points are put on a file.
627    * numberstep : steps per interval.
628    * pb : values of the left endpoints of intervals to be skipped.
629    */
630   double coefp[DIMP][MAXP][KP], breakp[MAXP], pb[MAXP];
631   int plotc, numberstep, dim, k, l;
632   { double parpp[DIMP], h, dh;
633     int i, j, m, jj, tt;
634     if (plotc == 1 && dim != 2)
635     { fprintf(fd, "\nplotting only if dim = 2\n");
636       return;
637     }
638     if (plotc == 2)
639     { fprintf(fd, "\n          t");
640       for (i = 1; i <= dim; ++i)
641       fprintf(fd, "       p%1d(t)", i);
642       fprintf(fd, "\n\n");
643     }
644     tt = 1;
645     for (i = 1; i <= l; ++i)
646     if (breakp[i] == pb[tt]) ++tt;
647     else
648     { dh = (breakp[i+1] - breakp[i]) / numberstep;
649       h = 0.0;
650       for (jj = 0; jj <= numberstep; ++jj)
651       { for (j = 1; j <= dim; ++j)
652         { parpp[j] = 0.0;
653           for (m = k; m >= 1; --m)
654           parpp[j] = (parpp[j] / m) * h + coefp[j][i][m];
655         }
656         if (plotc == 1)
657         line(parpp[1], parpp[2]) ;
658         else if (plotc == 2)
659         { fprintf(fd, " %10.6f", breakp[i] + h);
660           for (j = 1; j <= dim; ++j)
661           fprintf(fd, "  %10.6f", parpp[j]);
662           fprintf(fd, "\n");
663         }
664         h += dh;
```

```
665       }
666     }
667     for (i = 1; i < DIMP; ++i)
668     coefp[i][l+1][1] = parpp[i];
669   }
670
671   plotdp(rscale, p, n, cyc)
672   /* plots the datapoint-markers (little crosses) of a pp-curve;
673    * rscale : scale factors.
674    */
675   double p[MAXP][DIMP], rscale[DIMP];
676   int n, cyc;
677   { int i;
678     double dpx, dpy, p1, p2;
679     dpx = 2.0 / (rscale[1] * 200);
680     dpy = 2.0 / (rscale[2] * 200);
681     if (cyc == 2) ++n;
682     for (i = 1; i <= n; ++i)
683     if (p[i][0] == 3) ++i;
684     else
685     { p1 = p[i][1];
686       p2 = p[i][2];
687       newpel();
688       line(p1, p2 - dpy);
689       line(p1, p2 + dpy);
690       newpel();
691       line(p1 - dpx, p2);
692       line(p1 + dpx, p2);
693     }
694   }
695
696   plotkn(rscale, breakp, coefp, l, cyc, pb)
697   /* plots the knot-markers (little circles) of a pp-curve;
698    * rscale : scale factors.
699    */
700   double coefp[DIMP][MAXP][KP], breakp[MAXP], rscale[DIMP];
701   double pb[MAXP];
702   int l, cyc;
703   { int i, j, tt;
704     double dsx, dsy, px, py, dth;
705     tt = 1;
706     dth = 3.1416 / 8;
707     dsx = 2.0 / rscale[1];
708     dsy = 2.0 / rscale[2];
709     for (i = 1; i <= l + 1; ++i)
710     if (pb[tt] == breakp[i]) ++tt;
711     else
712     { if ((i == l+1) && (cyc == 1)) return;
713       px = coefp[1][i][1];
714       py = coefp[2][i][1];
715       newpel();
716       for (j = 0; j <= 16; ++j)
717       line(px + dsx*sin(j*dth)/200, py + dsy*cos(j*dth)/200);
718     }
719   }
720
```

```
721  axes2d(rscale, or, nstr)
722  /* plots axes with origin (or[1], or[2]) and scale factors
723   * rscale[1] and rscale[2];
724   * approximately nstr points p[i] are indicated on the
725   * axes, such that p[i] = 1[i] * 10^k, 1[i] and k integers.
726   */
727  int nstr;
728  double or[DIMP], rscale[DIMP];
729  { int i, j;
730    double w[DIMP], fac;
731    for (i = 1; i < DIMP; ++i)
732    w[i] = or[i];
733    for (i = 1; i < DIMP; ++i)
734    { w[i] = 1.0 / (rscale[i] * nstr);
735      j = 0;
736      fac = 1.0;
737      while (fac > w[i])
738      { --j; fac /= 10; }
739      while (fac < w[i])
740      { ++j; fac *= 10; }
741      newpel();
742      w[i] = -1.0 / rscale[i];
743      line(w[1], w[2]);
744      w[0] = w[i] = -w[i];
745      line(w[1], w[2]);
746      while (w[i] > 0.0) w[i] -= fac;
747      w[i % 2 + 1] += 1.0 / (rscale[i % 2 + 1] * 200);
748      for (w[i] = -w[0] + fac + w[i]; w[i] < w[0]; w[i] += fac)
749      { newpel();
750        line(w[1], w[2]);
751        w[i % 2 + 1] -= 1.0 / (rscale[i % 2 + 1] * 100);
752        line(w[1], w[2]);
753        w[i % 2 + 1] += 1.0 / (rscale[i % 2 + 1] * 100);
754      }
755      w[i] = or[i];
756    }
757    newpel();
758    for (i = 1; i < DIMP; ++i)
759    w[i] = 1.0 / rscale[i];
760    line(-w[1], -w[2]);
761    line(-w[1], w[2]);
762    line(w[1], w[2]);
763    line(w[1], -w[2]);
764    line(-w[1], -w[2]);
765  }
```

APPENDIX B

Subroutine 'plotpf'

This subroutine plots a pp-function. There are three options:

1) For each interval a fixed number (numberstep +1) of points is evaluated:

$$\left(\xi_j + \frac{\xi_{j+1}-\xi_j}{\text{numberstep}} \cdot i,\ P_j\left(\xi_j + \frac{\xi_{j+1}-\xi_j}{\text{numberstep}} \cdot i\right)\right),\ i = 0,\ldots,\text{numberstep}.$$

The points are put on a file (plotcode 3).

2) Idem.
The points are given to a display, which draws straight lines between them (plotcode 2).

3) The distribution of the points is adapted to the curvature of the function (plotcode 1).
If the function is plotted with (not too many) mpoints equidistant on each interval, then, on places where the function has a strong curvature, it will show, that the plotting is done by drawing little straight lines.

Two parameters are given to the procedure

maxdev: maximum deviation in slope (in rad):
$|\text{atan}\,(P_j^{(1)}(x_{i+1}))-\text{atan}\,(P_j^{(1)}(x_i))| \leq \text{maxdev}$,
x_i, x_{i+1} two consecutive abscissae.
Consequently, the angle between two consecutive lines lies between -2 maxdev and +2 maxdev, if $P_j^{(2)}(x) \neq 0$ on the intervals corresponding with the lines.

maxdh: $|x_{i+1}-x_i| \leq \text{maxdh}$.

Suppose we have reached a point $(x_i, P_j(x_i))$ in the plotting process. $dh := x_i - x_{i-1}$ (if $x_i = \xi_j$ we take $dh := \text{maxdh}$). If $|\text{atan}(P_j^{(1)}(x_i+dh)) - \text{atan}(P_j^{(1)}(x_i))| > \text{maxdev}$, the value of dh is altered: $dh := \frac{1}{2}dh$, else $dh := 2dh$.

Again we look at the angle. If the >-sign changes into ≤, a line is drawn between $(x_i, P_j(x_i))$ and $(x_{i+1}, P_j(x_{i+1}))$, $x_{i+1} = x_i + dh$.
If the ≤-sign changes into >, a line is drawn between $(x_i, P_j(x_i))$ and

$(x_{i+1}, P_j(x_{i+1}))$, $x_{i+1} = x_i + \frac{1}{2}dh$.
If there is no change of sign, the halving or doubling of dh continues until the sign changes.

Of course, provisions are made in case dh > maxdh, or $x_i+dh > \xi_{j+1}$.
Figure 1 is drawn by this procedure. The numbers of steps per ξ-interval turned out to be: 139, 37, 75, 51, 46, 51 and 96 (maxdh = 0.1, maxdev = 0.05).

```
 1
 2  plotpf(plotcode, numberstep, maxdh, maxdev, k, l, coef, breakp)
 3  /* plots a pp-function.
 4   * input : k, l, coef and breakp : the pp-representation
 5   *              of the function.
 6   *          numberstep : steps per interval (only used if
 7   *              plotcode = 2 or plotcode = 3).
 8   *          maxdh : maximum step-width (only used if plotcode = 1).
 9   *          maxdev : maximum value of the angle (in rad) between
10   *              the slopes of the curve at two consecutive points
11   *              (only used if plotcode = 1).
12   *          plotcode = 3 : (numberstep + 1) equidistant points per
13   *              interval are evaluated and put on a file.
14   *          plotcode = 2 : idem, the points are given to a display
15   *              (which draws straight lines between them).
16   *          plotcode = 1 : the distances between the points depend
17   *              on the curvature of the function. The points are
18   *              given to a display.
19   */
20  int k, l, plotcode, numberstep;
21  double maxdh, maxdev, coef[MAXP][KP], breakp[MAXP];
22  { int i, j, pc, a, nl, m;
23    double h, dh, ppx, ppf, ppfa, darc, darcn, darct;
24    if (plotcode == 1)
25    for (i = 1; i <= l; ++i)
26    { h = 0.0; dh = maxdh; pc = 2; a = 0;
27      nl = 0;
28      ppf = coef[i][1];
29      darc = atan(coef[i][2]);
30      newpel();
31      line(breakp[i], ppf);
32      while (breakp[i] + h < breakp[i+1])
33      { if (a++ > 1000)
34        { fprintf(fd, "\n while in loop\n"); return; }
35        ppfa = 0.0;
36        for (j = k; j >= 2; --j)
37        ppfa = (ppfa / (j-1)) * (h + dh) + coef[i][j];
38        darcn = atan(ppfa);
39        if (((darcn>darc)? (darcn-darc) : (darc-darcn)) > maxdev)
40        { dh /= 2;
41          if (pc == 0)
42          { h += dh;
43            ppf = 0.0;
44            for (j = k; j >= 1; --j)
45            ppf = (ppf / j) * h + coef[i][j];
46            darc = darct;
```

```
47              pc = 2;
48              line(breakp[i] + h, ppf);
49              ++nl;
50            }
51            else pc = 1;
52          }
53          else
54          { if (pc==1 || dh>=maxdh || breakp[i]+h+dh>=breakp[i+1])
55            { if (breakp[i] + h + dh >= breakp[i+1])
56              h = breakp[i+1] - breakp[i];
57              else
58              h += dh;
59              ppf = 0.0;
60              for (j = k; j >= 1; --j)
61              ppf = (ppf / j) * h + coef[i][j];
62              darc = darcn;
63              line(breakp[i] + h, ppf);
64              ++nl;
65              pc = 2;
66            }
67            else
68            { pc = 0;
69              if (dh < maxdh) dh *= 2;
70            }
71          }
72          darct = darcn;
73        }
74        fprintf(fd, "     %4d\n", nl);
75      }
76      else
77      for (i = 1; i <= 1; ++i)
78      { dh = (breakp[i+1] - breakp[i]) / numberstep;
79        h = 0.0;
80        if (plotcode == 2) newpel();
81        for (m = 0; m <= numberstep; ++m)
82        { ppf = 0.0;
83          for (j = k; j >= 1; --j)
84          ppf = (ppf / j) * h + coef[i][j];
85          ppx = breakp[i] + h;
86          if (plotcode == 2)
87          line(ppx, ppf);
88          else if (plotcode == 3)
89          fprintf(fd, "\n %10.6f   %10.6f", ppx, ppf);
90          h += dh;
91        }
92      }
93      coef[1+1][1] = ppf;
94    }
```

APPENDIX C

```
1
2   cubspl(tau, c, n)
3   /* cubic spline interpolation.
4    * input: tau[i], 1 <= i <= n, the knots,
5    *        (tau[i], c[i][1]), the given points,
6    *        n, number of points (n >= 3).
7    * output: c[i][2], c[i][3] and c[i][4], 1 <= i <= n-1,
8    *        the values of the first, second and third
9    *        derivatives at the left end points.
10   * for tau[i] <= x <= tau[i+1] we have
11   *        f(x) (= p[i](x)) = c[i][1] + (x - tau[i]) * c[i][2] +
12   *          + 1/2 * (x - tau[i])^2 * c[i][3] +
13   *          + 1/6 * (x - tau[i])^3 * c[i][4].
14   * the not-a-knot boundary condition is being used.
15   */
16  int n;
17  double tau[MAXP], c[MAXP][KP];
18  { int l, m;
19    double g, dtau, divdf3, divdf1;
20    l = n-1;
21    for (m = 2; m <= n; ++m)
22    { c[m][3] = tau[m] - tau[m-1];
23      c[m][4] = (c[m][1] - c[m-1][1]) / c[m][3];
24    }
25  /* calculation of the diagonal- (c[m][4]), next-to-diagonal-
26   * (c[m][3]) and right-elements (c[m][2]) together with the
27   * forward step of the Gauss-elimination.
28   */
29    c[1][4] = c[3][3];
30    c[1][3] = c[2][3] + c[3][3];
31    c[1][2] = (c[2][3] + 2 * c[1][3]) * c[2][4] * c[3][3];
32    c[1][2] += c[2][3] * c[2][3] * c[3][4];
33    c[1][2] /= c[1][3];
34    for (m = 2; m <= l; ++m)
35    { g = (-c[m+1][3]) / c[m-1][4];
36      c[m][2] = g * c[m-1][2];
37      c[m][2] += 3*(c[m][3] * c[m+1][4] + c[m+1][3] * c[m][4]);
38      c[m][4] = g * c[m-1][3] + 2 * (c[m][3] + c[m+1][3]);
39    }
40    g = c[n-1][3] + c[n][3];
41    c[n][2] = (c[n][3] + 2 * g) * c[n][4] * c[n-1][3];
42    c[n][2] += c[n][3] * c[n][3] * (c[n-1][1] - c[n-2][1]);
43    c[n][2] /= c[n-1][3] * g;
44          /* c[n-1][4] was already overwritten */
45    c[n][4] = c[n-1][3];
46    g = (-g) / c[n-1][4];
47    c[n][4] = g * c[n-1][3] + c[n][4];
48    c[n][2] = (g * c[n-1][2] + c[n][2]) / c[n][4];
```

```
49    /* completion Gauss-elimination */
50      for (m = 1; m >= 1; --m)
51      c[m][2] = (c[m][2] - c[m][3] * c[m+1][2]) / c[m][4];
52    /* calculation of the functionvalues of the second and third
53     * derivatives in the left endpoints
54     */
55      for (m = 2; m <= n; ++m)
56      { dtau = c[m][3];
57        divdf1 = (c[m][1] - c[m-1][1]) / dtau;
58        divdf3 = c[m-1][2] + c[m][2] - 2 * divdf1;
59        c[m-1][3] = 2 * (divdf1 - c[m-1][2] - divdf3) / dtau;
60        c[m-1][4] = (divdf3 / dtau) * 6.0 / dtau;
61      }
62    }
```

APPENDIX D

```
1
2   locadp(x, fx, adapcode, t, n, k, bcoef, q, tau, gtau)
3   /* calls interv solsys valuex */
4   /* adapts (a b-representation of) a pp-function to a new
5    * function value.
6    * input : (x, fx) : new point.
7    *         t, n, k and bcoef : b-representation of the
8    *             pp-function.
9    *         q : the condensed b-matrix.
10   *         (tau[i], gtau[i]) : the underlying datapoints.
11   *         adapcode : adaption-code.
12   *             adapcode = 0 : x = tau[j] and the other
13   *               datapoints are kept unchanged (if adapcode = 0
14   *               and x != tau[j], 'locadp' will protest).
15   *             adapcode = -1 : two consecutive b-coefficients
16   *               change.
17   *             adapcode >= 1 : (k-2+adapcode) consecutive
18   *               b-coefficients change.
19   * (see for a description: 'the b-representation of piecewise
20   * polynomial parametric curves and local adaption', ch. 2.2)
21   */
22  double x, fx, t[MAXP], q[MAXP][KPKMlP], bcoef[MAXP];
23  double tau[MAXP], gtau[MAXP];
24  int n, k, adapcode;
25  { int i, left, taui, right;
26    double bsplc[MAXP], der, derpr, diff, bval, bvalpr;
27    double valuex();
28    interv(t, n, x, &left);
29    if (x < t[k] || x > t[n+1])
30    { printf("\n(%f, %f) not in interval", x, fx);
31      printf(" [%f, %f]\n", t[k], t[n+1]);
32      return;
33    }
34    if (adapcode == 0)
35    { interv(tau, n, x, &taui);
36      if (x != tau[taui])
37      { printf("\nadapcode = 0 en x != tau[i]\n");
38        return;
39      }
40      gtau[taui] = fx;
41      for (i = 1; i <= n; ++i)
42      bcoef[i] = gtau[i];
43      solsys(q, k, n, bcoef);
44    }
45    else if (adapcode == -1)
46    { for ( i = 1; i <= n; ++i)
47      bsplc[i] = 0.0;
48      for (i = left-k+1; i <= left; ++i)
```

```
49        { bsplc[i] = 1.0;
50          der = valuex(t, bsplc, n, k, x, 1);
51          if (der > 0.0) break;
52          derpr = der;
53          bsplc[i] = 0.0;
54        }
55        diff = fx - valuex(t, bcoef, n, k, x, 0);
56        bval = valuex(t, bsplc, n, k, x, 0);
57        bsplc[i] = 0.0;
58        bsplc[i-1] = 1.0;
59        bvalpr = valuex(t, bsplc, n, k, x, 0);
60      printf("\n   %f   %f   %f   %f \n", bval, bvalpr, der, derpr);
61        if (der < derpr)
62        { diff /= (bval - (der / derpr) * bvalpr);
63          bcoef[i-1] -= diff * (der / derpr);
64          bcoef[i] += diff;
65        }
66        else
67        { diff /= (bvalpr - (derpr / der) * bval);
68          bcoef[i-1] += diff;
69          bcoef[i] -= diff * (derpr /  der);
70        }
71        for (left = 1; tau[left] <= t[i-1]; ++left);
72        for (right = n; tau[right] >= t[i+k]; --right);
73        for (i = left; i <= right; ++i)
74        gtau[i] = valuex(t, bcoef, n, k, tau[i], 0);
75      }
76      else
77      { diff = fx - valuex(t, bcoef, n, k, x, 0);
78        if (x == t[left] && adapcode > 1) --left;
79        i = left;
80        adapcode -= 2;
81        left = left - k + 1 - (adapcode / 2);
82        left = (left < 1) ? 1 : left;
83        right = i + adapcode - (adapcode / 2);
84        right = (right > n) ? n : right;
85        for (i = left; i <= right; ++i)
86        bcoef[i] += diff;
87        for (i = 1; tau[i] <= t[left]; ++i);
88        left = i;
89        for (i = n; tau[i] >= t[right+k]; --i);
90        right = i;
91        for (i = left; i <= right; ++i)
92        gtau[i] = valuex(t, bcoef, n, k, tau[i], 0);
93      }
94    }
```

APPENDIX E

```
1
2   int ludecy(q1, q2, k, n)
3   /* lu-decomposition without pivoting of an n x n b-matrix, which
4    * is banded (bandwidth 2k-1) in the first n-k+1 rows (condensed
5    * in q1); the last k-1 rows are stored in q2.
6    */
7   double q1[MAXP][KPKM1P], q2[K][MAXP];
8   int k, n;
9   { int kp1, km1, nmkp1, i, ipkm1, ip1, m, j, jj, l, nr, ipnmkp1;
10    double pivot;
11    kp1 = k + 1;
12    km1 = k - 1;
13    nmkp1 = n - k + 1;
14    for (m = 1; m <= km1 && q2[m][1] == 0.0; ++m);
15    /* m kan k zijn */
16    for (i = 1; i <= nmkp1; ++i)
17    { l = k;
18      ipkm1 = i + k - 1;
19      ip1 = i + 1;
20      nr = (ipkm1 <= nmkp1)? ipkm1 : nmkp1;
21      if ((pivot = q1[i][k]) == 0.0) return(2);
22      for (j = ip1; j <= nr; ++j)
23      { l -= 1;
24        if (q1[j][l] != 0.0)
25        { q1[j][l] /= pivot;
26          for (jj = 1; jj <= km1; ++jj)
27          q1[j][l+jj] -= q1[j][l] * q1[i][k+jj];
28        }
29      }
30      for (j = m; j <= km1; ++j)
31      { q2[j][i] /= pivot;
32        for (jj = ip1, l = kp1; jj <= ipkm1; ++jj, ++l)
33        q2[j][jj] -= q2[j][i] * q1[i][l];
34      }
35    }
36    for (i = 1; i <= km1; ++i)
37    { ipnmkp1 = i + nmkp1;
38      if ((pivot = q2[i][ipnmkp1]) == 0.0) return(2);
39      ip1 = i + 1;
40      for (j = ip1; j <= km1; ++j)
41      if (q2[j][ipnmkp1] != 0.0)
42      { q2[j][ipnmkp1] /= pivot;
43        for (jj = nmkp1 + ip1; jj <= n; ++jj)
44        q2[j][jj] -= q2[j][ipnmkp1] * q2[i][jj];
45      }
46    }
47    return(1);
48  }
```

APPENDIX F

```
1
2   solsyc(ql, q2, k, n, bcoef)
3   /* solves a system of equations;
4    * to be used with ludecy;
5    * the right-values are expected in bcoef, the solution is put
6    * in bcoef again.
7    */
8   double ql[MAXP][KPKM1P], q2[K][MAXP], bcoef[MAXP];
9   int k, n;
10  { int kml, nmkpl, l, i, j, ipnmkpl, jmkpipnmkpl, kpl, kpkml;
11    kml = k-1;
12    nmkpl = n - kml;
13    kpkml = k + kml;
14    kpl = k + 1;
15    for (i = 2; i <= nmkpl; ++i)
16    { l = (k > i) ? (kpl - i) : l;
17      for (j = 1; j <= kml; ++j)
18      if (ql[i][j] != 0.0)
19      bcoef[i] -= ql[i][j] * bcoef[j-k+i];
20    }
21    for (i = 1; i <= kml; ++i)
22    { ipnmkpl = i + nmkpl;
23      if (q2[i][1] == 0.0)
24      for (j = 1; j <= kml; ++j)
25      { jmkpipnmkpl = ipnmkpl - k + j;
26        if (q2[i][jmkpipnmkpl] != 0.0)
27        bcoef[ipnmkpl] -= q2[i][jmkpipnmkpl] * bcoef[jmkpipnmkpl];
28      }
29      else
30      for (j = 1; j < ipnmkpl; ++j)
31      bcoef[ipnmkpl] -= q2[i][j] * bcoef[j];
32    }
33    for (i = kml; i >= 1; --i)
34    { ipnmkpl = i + nmkpl;
35      for (j = n; j > ipnmkpl; --j)
36      if (q2[i][j] != 0.0) bcoef[ipnmkpl] -= q2[i][j] * bcoef[j];
37      bcoef[ipnmkpl] /= q2[i][ipnmkpl];
38    }
39    for (i = nmkpl; i >= 1; --i)
40    { for (j = kpkml; j >= kpl; --j)
41      if (ql[i][j] != 0.0) bcoef[i] -= ql[i][j] * bcoef[i+j-k];
42      bcoef[i] /= ql[i][k];
43    }
44  }
```

APPENDIX G

```
 1
 2  double valuex(t, bcoef, n, k, x, jderiv)
 3  /* calls interv */
 4  /* returns the value at x of the jderiv-th derivative of the
 5   * the pp-function with
 6   *           order k,
 7   *           knot sequence t[i], 1 <= i <= n+k,
 8   *           b-coefficients bcoef[j], 1 <= j <= n.
 9   */
10  double t[MAXP], bcoef[MAXP], x;
11  int n, k, jderiv;
12  { double dl[KP], dr[KP], aj[KP];
13    int kml, jcmin, imk, i, j, jcmax, nmi, jc, kmj, ilo, jj;
14    if (jderiv >= k) return(0.0);
15    if (interv(t, n+k, x, &i) != 0) return(0.0);
16    kml = k-1;
17    if (kml == 0) return(bcoef[i]);
18    jcmin = 1;
19    imk = i-k;
20    if (imk >= 0)
21    for (j = 1; j <= kml; ++j)
22    dl[j] = x - t[i+1-j];
23    else
24    { jcmin = 1 -imk;
25      for (j = 1; j <= i; ++j)
26      dl[j] = x - t[i+1-j];
27      for (j = i; j <= kml; ++j)
28      { aj[k-j] = 0.0;
29        dl[j] = dl[i];
30      }
31    }
32    jcmax = k;
33    nmi = n - i;
34    if (nmi >= 0)
35    for ( j = 1; j <= kml; ++ j)
36    dr[j] = t[i+j] - x;
37    else
38    { jcmax = k + nmi;
39      for (j = 1; j <= jcmax; ++j)
40      dr[j] = t[i+j] - x;
41      for (j = jcmax; j <= kml; ++j)
42      { aj[j+1] = 0.0;
43        dr[j] = dr[jcmax];
44      }
45    }
46    for (jc = jcmin; jc <= jcmax; ++jc)
47    aj[jc] = bcoef[imk+jc];
48    for (j = 1; j <= jderiv; ++j)
```

```
49      { kmj = ilo = k-j;
50        for (jj = 1; jj <= kmj; ++jj)
51        { aj[jj] = ((aj[jj+1] - aj[jj]) / (dl[ilo] + dr[jj])) * kmj;
52          ilo -= 1;
53        }
54      }
55      for (j = jderiv + 1; j <= kml; ++j)
56      { kmj = ilo = k - j;
57        for (jj = 1; jj <= kmj; ++jj)
58        { aj[jj] = aj[jj+1] * dl[ilo] + aj[jj] * dr[jj];
59          aj[jj] /= (dl[ilo] + dr[jj]);
60          ilo -= 1;
61        }
62      }
63      return(aj[1]);
64    }
```

APPENDIX H

```
 1
 2   int interv(xt, lxt, x, left)
 3   /* input: xt[i], xt[i] <= xt[i+1], 1 <= i <= lxt.
 4    * returns -1, if x < xt[1],
 5    *          0, if xt[1] <= x <= xt[lxt] and
 6    *          1, if x >= xt[lxt].
 7    * computes in an economic way *left, such that
 8    *   xt[*left] <= x < xt[*left+1], if
 9    *   xt[1] <= x < xt[lxt].
10    */
11   int *left, lxt;
12   double xt[MAXP], x;
13   { static int ilo;
14     int ihi, istep, middle;
15     if (ilo == 0) ilo = 1;
16     ihi = ilo + 1;
17     if (ihi >= lxt)
18     { if (x >= xt[lxt]) { *left = lxt; return(1); }
19       if (lxt <= 1) { *left = 1; return(-1); }
20       ilo = lxt-1;
21       ihi = lxt;
22     }
23     if (x < xt[ihi] && x >= xt[ilo])
24     { *left = ilo; return(0); }
25     else if (x < xt[ilo])
26     { istep = 1;
27       do
28       { ihi = ilo;
29         ilo = ihi - istep;
30         if (ilo <= 1)
31         { ilo = 1;
32           if (x < xt[1]) { *left =1; return(-1); }
33         }
34         istep += istep;
35       }
36       while (x < xt[ilo]);
37     }
38     else
39     { istep = 1;
40       do
41       { ilo = ihi;
42         ihi = ilo + istep;
43         if (ihi >= lxt)
44         { ihi = lxt;
45           if (x >= xt[lxt]) { *left = lxt; return(1); }
46         }
47         istep += istep;
48       }
```

```
49        while (x >= xt[ihi]);
50      }
51      for (;;)
52      { middle = (ilo + ihi) / 2;
53        if (middle == ilo) { *left = ilo; return(0); }
54        if (x < xt[middle])
55        ihi = middle;
56        else ilo = middle;
57      }
58    }
```