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A note on non-generators of full AFL's *)

by

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ABSTRACT

We compare two definitions of non-generator for full AFL's, leading to two sets of non-generators for each full AFL K. The main result gives a necessary and sufficient condition on K such that these sets coincide.

KEY WORDS & PHRASES: *full AFL (full Abstract Family of Languages), non-generator, splitting full AFL, full principal AFL*

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A main tool in studying full Abstract Families of Languages (or full AFL's) which are included in a given full AFL K, is the notion of *non-generator*; cf.[5] and Section 6.6 of [3]. In these two references - to which we also refer for all unexplained notation and terminology in this note - the set $\hat{N}_g(K)$ of non-generators of K is defined by

$$\hat{N}_g(K) = \{L \in K \mid \hat{F}(L) \neq K\}.$$

As usual, for each set X of languages, $\hat{F}(X)$ is the smallest full AFL which includes X. In case X equals {L} we write $\hat{F}(L)$ instead of $\hat{F}(\{L\})$. Note that, if $L \in \hat{F}(X)$ then there exists a finite subset X_f of X such that $L \in \hat{F}(X_f)$.

It is easy to see that $\hat{N}_g(K) = \cup \{K_1 \mid K_1 \subset K, K_1 \text{ is a full AFL}\}$, and that K is full principal if and only if $\hat{N}_g(K) \subset K$ (We use "$\subset$" to denote proper inclusion).

The concept of non-generator originally occurs in algebra where it is defined in a different way; cf. e.g. [6,1]. This definition of the set $\hat{N}(K)$ of non-generators of K reads in AFL-notation as

$$\hat{N}(K) = \{L \in K \mid \text{for each subset X of K: if } \hat{F}(X \cup \{L\}) = K, \text{ then } \hat{F}(X) = K\}.$$

The aim of this note is to investigate the relation between $\hat{N}_g(K)$ and $\hat{N}(K)$ for a given full AFL K.

Notice that these sets differ for the smallest full AFL REG (i.e., the family of regular languages): $\hat{N}_g(REG) = \emptyset$, whereas $\hat{N}(REG) = REG$, since each subset of REG generates REG.

We first consider some elementary properties of $\hat{N}(K)$ and $\hat{N}_g(K)$ for a nonregular full AFL K, i.e., a full AFL K satisfying $REG \subset K$.

PROPOSITION 1. *Let* K *be a nonregular full AFL. Then*

(1) $\hat{N}(K) \subseteq \hat{N}_g(K)$,

(2) $\hat{N}(K)$ *is a full AFL,*

(3) $\hat{N}_g(K)$ *is a full trio closed under Kleene* $*$.

PROOF. (1) Let L be in $\hat{N}(K)$. Then for $X = \emptyset$ we have that $\hat{F}(L) = K$ implies $\hat{F}(\emptyset) = K$. But $\hat{F}(\emptyset) = REG$ which contradicts the assumption that $REG \subset K$. Hence $\hat{F}(L) \neq K$, i.e., $L \in \hat{N}_g(K)$.
(2) The statement is a direct application of Corollary 3.4.2 from [6] to full AFL's. For the sake of completeness we repeat the proof (translated into AFL-terminology).

Let f be any of the full AFL-operations. And let $L_1,\ldots,L_n$ be in $\hat{N}(K)$, where n is the arity of f, i.e., either $n = 1$ (Kleene $*$, homomorphism, inverse homomorphism, intersection with regular set) or $n = 2$ (union, concatenation). If X is a subset of K such that $X \cup \{f(L_1,\ldots,L_n)\}$ generates K, then $X \cup \{L_1,\ldots,L_n\}$ also generates $K (n \leq 2)$. Since $L_2 \in \hat{N}(K)$, we have $K = \hat{F}(X \cup \{L_1\})$. And similarly, $K = \hat{F}(X)$ as $L_1 \in \hat{N}(K)$. Hence $f(L_1,\ldots,L_n)$ is in $\hat{N}(K)$.
(3) The fact that K is nonregular implies that $REG \subseteq \hat{N}_g(K)$. So it remains to show that $\hat{N}_g(K)$ is closed under the unary operations homomorphism, inverse homomorphism, intersection with regular sets, and Kleene $*$. Let f be any of these operations, and let L be in $\hat{N}_g(K)$. Suppose $\hat{F}(f(L)) = K$. Then we have $\hat{F}(L) = K$ which contradicts the fact that $L \in \hat{N}_g(K)$. Hence $\hat{F}(f(L)) \neq K$, i.e. $f(L) \in \hat{N}_g(K)$. □

The cases in which $\hat{N}_g(K)$ is a full AFL for a given nonregular full principal AFL K, have been characterized by GREIBACH in [5] (cf. also Corollary 4 below) from which we also quote the following definition.

A full AFL K *splits* if there exist incomparable full AFL's K_1 and K_2 such that $K = \hat{F}(K_1 \cup K_2)$.

We call such a pair (K_1,K_2) a *split* of K. We say that a split (K_1,K_2) of K is *principal* if either K_1 or K_2 is a full principal AFL. Thus a split (K_1,K_2) is *nonprincipal* if K_1 and K_2 are both not full principal.

We are now ready for the main result of this note.

THEOREM 2. *Let* K *be a nonregular full AFL. Then* $\hat{N}_g(K) = \hat{N}(K)$ *if and only if each split of* K *is nonprincipal.*

PROOF. Assume that $\hat{N}_g(K) = \hat{N}(K)$. Let (K_1,K_2) be a split of K, i.e. $K = \hat{F}(K_1 \cup K_2)$ with $K_1 \subset K$, $K_2 \subset K$ where K_1 and K_2 are incomparable full AFL's. Suppose K_1 is principal: $K_1 = \hat{F}(L_1)$ for some L_1 in K_1. Then

$L_1 \in \hat{N}_g(K)$, as $K_1 \subset K$. Due to the assumption, $L_1 \in \hat{N}(K)$, i.e., for each subset X of K, $\hat{F}(X \cup \{L_1\}) = K$ implies $\hat{F}(X) = K$. Take X equal to K_2. Then $\hat{F}(K_2 \cup \{L_1\}) = \hat{F}(K_2 \cup \hat{F}(L_1)) = K$ implies $\hat{F}(K_2) = K_2 = K$. But $K_2 \subset K$. Therefore K_1 (and symmetrically, K_2) is not principal. Hence (K_1, K_2) is a nonprincipal split.

Conversely, assume that each split of K is nonprincipal. By Proposition 1(1) we have $\hat{N}(K) \subseteq \hat{N}_g(K)$. In order to show the reverse inclusion, let L be in $\hat{N}_g(K)$: $\hat{F}(L) \subset K$.

If X is a subset of K such that $\hat{F}(X \cup \{L\}) = K$, then either $\hat{F}(X) \subset K$ or $\hat{F}(X) = K$. Suppose $\hat{F}(X) \subset K$, then $\hat{F}(X)$ and $\hat{F}(L)$ must be comparable (Otherwise K splits into $\hat{F}(X)$ and $\hat{F}(L)$, viz. $K = \hat{F}(X \cup \{L\}) = \hat{F}(\hat{F}(X) \cup \hat{F}(L))$, and hence K possesses a principal split). I.e., either $\hat{F}(X) \subseteq \hat{F}(L) \subset K$, or $\hat{F}(L) \subseteq \hat{F}(X) \subset K$. However, both alternatives contradict the fact that $\hat{F}(X \cup \{L\}) = K$. Therefore $\hat{F}(X) = K$, i.e., $L \in \hat{N}(K)$, and $\hat{N}_g(K) \subseteq \hat{N}(K)$. □

We now consider the case in which K is a full principal AFL. The next lemma has already been proved in Theorem 3.1 of [5]. However, for completeness' sake we give a direct proof.

<u>LEMMA 3</u>. *Let* K *be a nonregular full principal AFL. Then each split of* K *is nonprincipal if and only if* K *does not split.*

<u>PROOF</u>. The "if" part is obvious.

To prove the "only if" part, assume that each split of K is nonprincipal. Suppose K splits into K_1 and K_2. Since K is full principal there is a language L in K such that $\hat{F}(L) = K$. Then there exist finite sets $X_i \subset K_i$ $(i = 1,2)$ such that $L \in \hat{F}(X_1 \cup X_2)$ and so $K = \hat{F}(X_1 \cup X_2)$. Since $\hat{F}(X_i) \subseteq K_i$, K splits into $\hat{F}(X_1)$ and $\hat{F}(X_2)$ which are (even) both principal.

This contradicts the assumption that each split of K is nonprincipal. Therefore K does not split. □

The following corollary extends Theorem 3.1 of [5].

<u>COROLLARY 4</u>. *Let* K *be a nonregular full principal AFL. Then the following propositions are equivalent.*

(1) $\hat{N}_g(K) = \hat{N}(K)$.

(2) $\hat{N}_g(K)$ *is a full AFL.*
(3) $\hat{N}_g(K)$ *is closed under union.*
(4) K *does not split.*
(5) $\hat{N}_g(K)$ *is the largest full AFL which is properly included in* K.

PROOF. By Proposition 1, (1) implies (2), and (2) and (3) are equivalent. It is easy to see that (2) implies (5). Obviously, (4) follows from (5). Finally, by Theorem 2 and Lemma 3, (4) implies (1). □

EXAMPLES. We show that there exist full AFL's K satisfying
(1) $\hat{N}(K) \subset \hat{N}_g(K) \subset K$,
(2) $\hat{N}(K) \subset \hat{N}_g(K) = K$,
(3) $\hat{N}(K) = \hat{N}_g(K) \subset K$,
(4) $\hat{N}(K) = \hat{N}_g(K) = K$ and K does not split, and
(5) $\hat{N}(K) = \hat{N}_g(K) = K$ and K splits.

The latter example shows that in Theorem 2 the condition that each split of K is nonprincipal cannot be replaced by the condition that K does not split (as can be done for full principal AFL's; cf. Corollary 4).

We first note (see [2] VII. 4,5 and VIII. 7) that there exist two incomparable full principal AFL's Ocl and $\hat{F}$(Lin), where Ocl is the family of one-counter languages and Lin the family of linear languages, such that their substitution-closures Fcl and Qrt are incomparable nonprincipal full substitution-closed AFL's (And even $\hat{F}$(Lin) and Fcl are incomparable, and so are Qrt and Ocl).

We also recall the fact that every full substitution-closed AFL K is *fully prime* (Theorem 2.3 of [4]), i.e., if $K \subseteq \hat{F}(K_1 \cup K_2)$ then $K \subseteq K_1$ or $K \subseteq K_2$ (where K_1 and K_2 are arbitrary full AFL's). Clearly, if K is fully prime then K does not split.

(1). Take two incomparable full principal AFL's $\hat{F}(L_1)$ and $\hat{F}(L_2)$, and consider $K = \hat{F}(\{L_1, L_2\}) = \hat{F}(\hat{F}(L_1) \cup \hat{F}(L_2))$. Clearly, K splits and is full principal. Hence, $\hat{N}(K) \subset \hat{N}_g(K) \subset K$ by Corollary 4.

(2). Take two incomparable full AFL's K_1 and K_2, such that K_1 is nonprincipal and substitution-closed, and K_2 is full principal. Consider $K = \hat{F}(K_1 \cup K_2)$. Since K has a principal split, $\hat{N}(K) \subset \hat{N}_g(K)$ by Theorem 2. To show that $\hat{N}_g(K) = K$, assume that K is principal, i.e., $\hat{F}(K_1 \cup K_2) = \hat{F}(L)$.

Then there exist finite sets $X_i \subseteq K_i$ such that $\hat{F}(K_1 \cup K_2) = \hat{F}(X_1 \cup X_2)$. Since K_1 is fully prime, $K_1 \subseteq \hat{F}(X_1)$ or $K_1 \subseteq \hat{F}(X_2)$. But this implies that K_1 is principal or $K_1 \subseteq K_2$, which is a contradiction.

(3) and (4). Each full substitution-closed AFL K is fully prime and hence does not split. So, by Theorem 2, $\hat{N}(K) = \hat{N}_g(K)$. Consider, e.g., the family of context-free languages and Qrt, respectively.

(5). Take two incomparable nonprincipal full substitution-closed AFL's K_1 and K_2, and consider $K = \hat{F}(K_1 \cup K_2)$. Thus K has a (nonprincipal) split. To show that $\hat{N}(K) = K$, take an arbitrary $L \in K$ and $X \subseteq K$ such that $\hat{F}(X \cup \{L\}) = K$. Since both K_1 and K_2 are fully prime, it follows that $K_i \subseteq \hat{F}(X)$ or $K_i \subseteq \hat{F}(L)$. It now suffices to show that K_i is not included in $\hat{F}(L)$: then $K_1 \cup K_2 \subseteq \hat{F}(X)$ and hence $K = \hat{F}(X)$, so $L \in \hat{N}(K)$. Suppose that $K_1 \subseteq \hat{F}(L)$ (The proof for K_2 is similar). Since $L \in \hat{F}(K_1 \cup K_2)$ there exist finite sets $X_i \subseteq K_i$ such that $L \in \hat{F}(X_1 \cup X_2)$. Hence $K_1 \subseteq \hat{F}(X_1 \cup X_2)$. As K_1 is fully prime, $K_1 \subseteq \hat{F}(X_1)$ or $K_1 \subseteq \hat{F}(X_2)$. Therefore K_1 is principal or $K_1 \subseteq K_2$, which is a contradiction. □

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