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DECIDING EQUIVALENCE OF FUNCTIONAL SCHEMES FOR PARALLEL PROGRAMS

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DECIDING EQUIVALENCE OF FUNCTIONAL SCHEMES FOR PARALLEL PROGRAMS*)

by

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ABSTRACT

This paper presents two formal systems for proving equivalence of parallel programs. Most of the axioms and proof rules of the first system are taken from [1,2,3]. The second system, which is an extension of the first system, is developed on the base of the proof system constructed in [3]. We obtain a completeness result for a certain subset of expressions of the second system. In particular, this subset includes all expressions of the first system. The method we use for proving equivalence of parallel programs exhibits a formal resemblance to the method used by SALOMAA [4] for proving equivalence of expressions in the algebra of regular events.

KEY WORDS & PHRASES: *parallel program schemes, program equivalence, function interpretation, equational characterization, formal proof system, completeness*

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*) This report will be submitted for publication elsewhere.

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1. INTRODUCTION

A functional approach to programming of parallel algorithms is developed in this paper. The language of functional schemes (FS) defined in section 2 includes a parallel operation (denoted there by *). A sound system of axioms and proof rules is given, which allows to make transformations of expressions into equivalent ones. Expressions of FS are interpreted as partial functions.

Due to the operation *, FS-expressions have not an 'automaton' representation, similar to that for regular events [4]. In other words, they can not be characterized by finite sets of equations of a certain kind.

A more powerful system (FS1) is developed in the subsequent sections. FS1 is an extension of FS, that is, every FS-expression is also an FS1-expression. FS1 is introduced to be able to characterize the FS1-expressions (or at least the subset of FS-expressions) by a finite set of equations analogous to that for regular events, and furthermore to be able to use the property of equational characterization for proving equivalence of parallel program schemes.

In section 3 the definition of expressions of FS1 and the set of axioms and proofs rules of FS1 are given. With respect to the interpretation of FS1 the soundness of this system is proved.

The notion *equational characterization* for FS1-expressions is introduced in section 4. It is settled that each FS-expression is equationally characterized in FS1.

The main result of this paper is obtained in section 5. It is the following completeness result: if two FS-expressions A and B are semantically equivalent then the formula (equation) A = B is derivable within FS1.

The paper ends with an example on which the proof method for equivalence of FS-expressions is demonstrated.

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2. THE FORMAL SYSTEM FS

In this section we shall define the syntax of FS, give a formal proof system for formulas of FS, and define a function interpretation of FS-expressions.

2.1. The language FS

The alphabet of FS consists of:

(i) a set of function variables, VAR, with typical elements a,b,c,...;

(ii) two function constants: e and $\emptyset$;

(iii) connective signs: $\cdot, +, *, \{\ \}$;

(iv) parentheses.

The set of expressions EXP with typical elements A,B,C,..., is defined by

$$A ::= a \mid e \mid \emptyset \mid A_1 \cdot A_2) \mid (A_1 * A_2) \mid (A_1 + A_2) \mid \{A\}.$$

We shall omit the sign "$\cdot$" in the sequel. Also we shall omit parentheses when no confusion can arise, assuming the following order of priority of the operations: $\cdot, *, +$.

Looking forward we shall give an informal explanation of the interpretation of these expressions:

- a function variable a is interpreted as a *partial function*;
- the constant e is interpreted as the *identity function*;
- the constant $\emptyset$ is interpreted as the *nowhere defined function*;
- (AB) means the *composition* of two expressions (interpreted as partial functions) and the value of (AB)(α) for some argument α, is equal to the value B(A(α)) (not A(B(α))!);

- $(A*B)$ is called the *concatenation* of two expressions and the value of $(A*B)(\alpha)$ is equal to the set of ordered pairs $\langle\beta_1,\beta_2\rangle$, where $\beta_1 \in A(\alpha)$ and $\beta_2 \in B(\alpha)$;
- $(A+B)$ is the *union* of two expressions and the value of $(A+B)(\alpha)$ is equal to $A(\alpha) \cup B(\alpha)$;
- $\{A\}$ is called the *iteration* of A and denotes $\sum_{i=0}^{\infty} A^i$, where $A^0 \equiv e$, $A^{i+1} = A^i A$.

The set of *formulas* FORM with typical elements f,... is defined by

$$f ::= A = B.$$

2.2. Formal proof system for formulas of FS

The set of *axioms* Ax is given by

A1. $A + A = A$

A2. $A + B = B + A$

A3. $(A + B) + C = A + (B + C)$

A4. $A + \emptyset = A$

A5. $A\emptyset = \emptyset A = \emptyset$

A6. $Ae = eA = A$

A7. $(AB)C = A(BC)$

A8. $A * \emptyset = \emptyset * A = \emptyset$

A9. $(A + B)C = (AC) + (BC)$

A10. $A(B + C) = (AB) + (AC)$

A11. $(A + B) * C = (A * C) + (B * C)$

A12. $A * (B + C) = (A * B) + (A * C)$

A13. $A(B * C) = (AB) * (AC)$ if A does not contain + or { }

A14. $\{A\} = e + A\{A\}$

A15. $\{A + e\} = \{A\}$

A16. $\{e\} = e$.

2.2.1. DEFINITION. A *possesses the constant* e if A satisfies one of the following conditions:

(i) $A \equiv e$

(ii) $A \equiv A_1 + A_2$ and at least one of A_i $(i = 1,2)$ possesses e

(iii) $A \equiv A_1A_2$ and both A_1 and A_2 possess e

(iv) $A \equiv \{B\}$ where B is an arbitrary expression.

The set of *proof rules Pr* consists of:

R1. (*Replacement rule*). Let A occur in B and A = C. If D is the expression obtained from B by replacing A by C, then B = D.

R2. (*Solution of an equation*). From A = B A + C, where B does not posses e, one may infer A = {B} C. (This rule is also known as *Arden's rule*).

2.3. Interpretations of FS

Let M be a nonempty finite set. The set M^∞ with typical elements $\alpha,\beta,\gamma,\ldots$ is defined by

$$\alpha ::= m \mid \alpha m \mid \langle \alpha_1, \alpha_2 \rangle$$

where $m \in M$.

The interpretation of the constants e, $\emptyset$ and function variables $a \in VAR$ is given by a function ρ, of type

$$\rho : X \rightarrow (M^\infty \rightarrow M^\infty)$$

where $X = VAR \cup \{e\} \cup \{\emptyset\}$.

Then

(i) $\rho(a)$ is a partial function, of type

$$\rho(a) : M^\infty \rightarrow M^\infty$$

(ii) $\rho(e)$ is the identity function, of type

$$\rho(e) : M^\infty \rightarrow M^\infty$$

(iii) $\rho(\emptyset)$ is the nowhere defined function.

The interpretation of expressions of FS is defined by a function $\phi_{M,\rho}$, of type

$$\phi_{M,\rho} : EXP \rightarrow (P(M^\infty) \rightarrow P(M^\infty))$$

where $P(M^\infty) = \{\tilde{\alpha} \mid \tilde{\alpha} \subseteq M^\infty\}$, i.e. $P(M^\infty)$ is the power set of M^∞ with typical elements $\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\ldots$.

Then

(i) $\phi_{M,\rho}(x)(\tilde{\alpha}) = \{\rho(x)(\alpha) \mid \alpha \in \tilde{\alpha}\}$, where $x \in X$

(ii) $\phi_{M,\rho}(AB)(\tilde{\alpha}) = \phi_{M,\rho}(B)(\phi_{M,\rho}(A)(\tilde{\alpha}))$

(iii) $\phi_{M,\rho}(A * B)(\tilde{\alpha}) = \{<\beta_1,\beta_2> \mid \beta_1 \in \phi_{M,\rho}(A)(\{\gamma\})$ and $\beta_2 \in \phi_{M,\rho}(B)(\{\gamma\})$, for some $\gamma \in \tilde{\alpha}\}$

(iv) $\phi_{M,\rho}(A + B)(\tilde{\alpha}) = \phi_{M,\rho}(A)(\tilde{\alpha}) \cup \phi_{M,\rho}(B)(\tilde{\alpha})$

(v) $\phi_{M,\rho}(\{A\})(\tilde{\alpha}) = \bigcup_{i=1}^{\infty} \phi_{M,\rho}(A^i)(\tilde{\alpha})$, where $A^0 = e$, $A^{i+1} = A^i A$

The interpretation of formulas of FS is given by a function $F_{M,\rho}$ of type

$$F_{M,\rho}: \text{FORM} \to (P(M^{\infty}) \to T)$$

where the set T = {true,false}.

Then

$$F_{M,\rho}(A = B)(\tilde{\alpha}) = (\phi_{M,\rho}(A)(\{\gamma\}) = \phi_{M,\rho}(B)(\{\gamma\})) \text{ for all } \gamma \in \tilde{\alpha}.$$

2.3.1. <u>NOTATION</u>. The assertion that a formula f is valid for all M and ρ we denote by writing

$$\models f.$$

2.3.2. <u>NOTATION</u>. The assertion that a formula f is formally derivable within the system FS we denote by writing

$$\vdash_{FS} f.$$

The usual notion of soundness of the formal system with the set of axioms *Ax* and proof rules *Pr* is equivalent to the assertion

$$\vdash_{FS} f \Rightarrow \models f.$$

To prove this assertion it is enough to prove it for all axioms *Ax* and to show that all proof rules *Pr* preserve the validity of this assertion.
We omit here the proof of the soundness of FS. We shall give below the similar proof of the soundness of FS1.

The usual aim is to have a complete formal system, i.e. a system for which the following holds:

$$\models f \Rightarrow \vdash_{FS} f.$$

In the sequel we shall define a system FS1 which extends the system FS and show the following:

$$\models A = B \Leftrightarrow \vdash_{FS1} A = B$$

where $A \in EXP$ and $B \in EXP$.

3. THE FORMAL SYSTEM FS1

In this section we shall define the syntax of the system FS1, give a formal proof system for formulas of FS1 and an interpretation of expressions of FS1.

3.1. The language FS1

The alphabet of FS1 consists of:

(i) the set of variables VAR (as for FS);

(ii) function constants: e and $\emptyset$;

(iii) connective signs:

- $\cdot$ (*composition*),
- $+$ (*union*)
- $\vee$ (*fork*),
- $|$ (*separate union*),
- $\wedge$ (*join*),
- $*$ (*concatenation*),
- $\{\ \}$ (*iteration*);

(iv) parentheses.

Each expression of FS1 will be characterized by its *type* as follows.

The set of input (output) types TYPE with typical elements t,... is defined by

$$t ::= \tau \mid (t_1, t_2).$$

Here τ is an atomic type. Then, by $t_1 \Rightarrow t_2$ we denote the set of all expressions of FS1, which have $t_1 \in$ TYPE and $t_2 \in$ TYPE as input and output type, respectively. The set of *expressions* EXP1 with typical elements A,B,C,..., is defined as $\bigcup_{t_i,t_j \in \text{TYPE}} (t_i \Rightarrow t_j)$ as follows:

(i) $a \in \tau \Rightarrow \tau$ for all $a \in$ VAR;

$e \in \tau \Rightarrow \tau$;

(ii) $\emptyset \in t_1 \Rightarrow t_2$ for all $t_1, t_2 \in$ TYPE;

(iii) if $A \in t_1 \Rightarrow t_2$ and $B \in t_2 \Rightarrow t_3$ then $(A \cdot B) \in t_1 \Rightarrow t_3$;

(iv) if $A \in t_1 \Rightarrow t_2$ and $B \in t_1 \Rightarrow t_2$ then $(A + B) \in t_1 \Rightarrow t_2$;

(v) if $A \in t_1 \Rightarrow t_2$ and $B \in t_1 \Rightarrow t_3$ then $(A \vee B) \in t_1 \Rightarrow (t_2, t_3)$;

(vi) if $A \in t_1 \Rightarrow t_2$ and $B \in t_3 \Rightarrow t_4$ then $(A \mid B) \in t_1, t_3) \Rightarrow (t_2, t_4)$;

(vii) if $A \in t_1 \Rightarrow \tau$ and $B \in t_2 \Rightarrow \tau$ then $(A \wedge B) \in (t_1, t_2) \Rightarrow \tau$;

(viii) if $A \in t_1 \Rightarrow \tau$ and $B \in t_1 \Rightarrow \tau$ then $(A * B) \in t_1 \Rightarrow \tau$;

(ix) if $A \in t_1 \Rightarrow t_1$ then $\{A\} \in t_1 \Rightarrow t_1$.

Again we shall omit the sign "$\cdot$" in the sequel, and likewise we shall omit parentheses when no confusion can arise.
To aid the intuitive understanding, we give in Figure 3.1.1 the diagrams corresponding to FS1 expressions.
The set of *identity expressions*, IDEN, with typical elements ε,..., is described by

$$\varepsilon ::= e \mid (\varepsilon_1 \mid \varepsilon_2) \mid \varepsilon_1 \varepsilon_2.$$

The set of *formulas* FORM1 with typical elements f,..., is defined by

$$f ::= A = B \quad \text{where } A, B \in \text{EXP1}.$$

3.2. Formal proof system for formulas of FS1

The set of *axioms* *Ax1* is given by:

Ax1. $A + A = A$

Ax2. $A + B = B + A$

Ax3. $(A + B) + C = A + (B + C)$

Ax4. $A + \emptyset = A$

Ax5. $A\emptyset = \emptyset A = \emptyset$ (continued on p. 9)

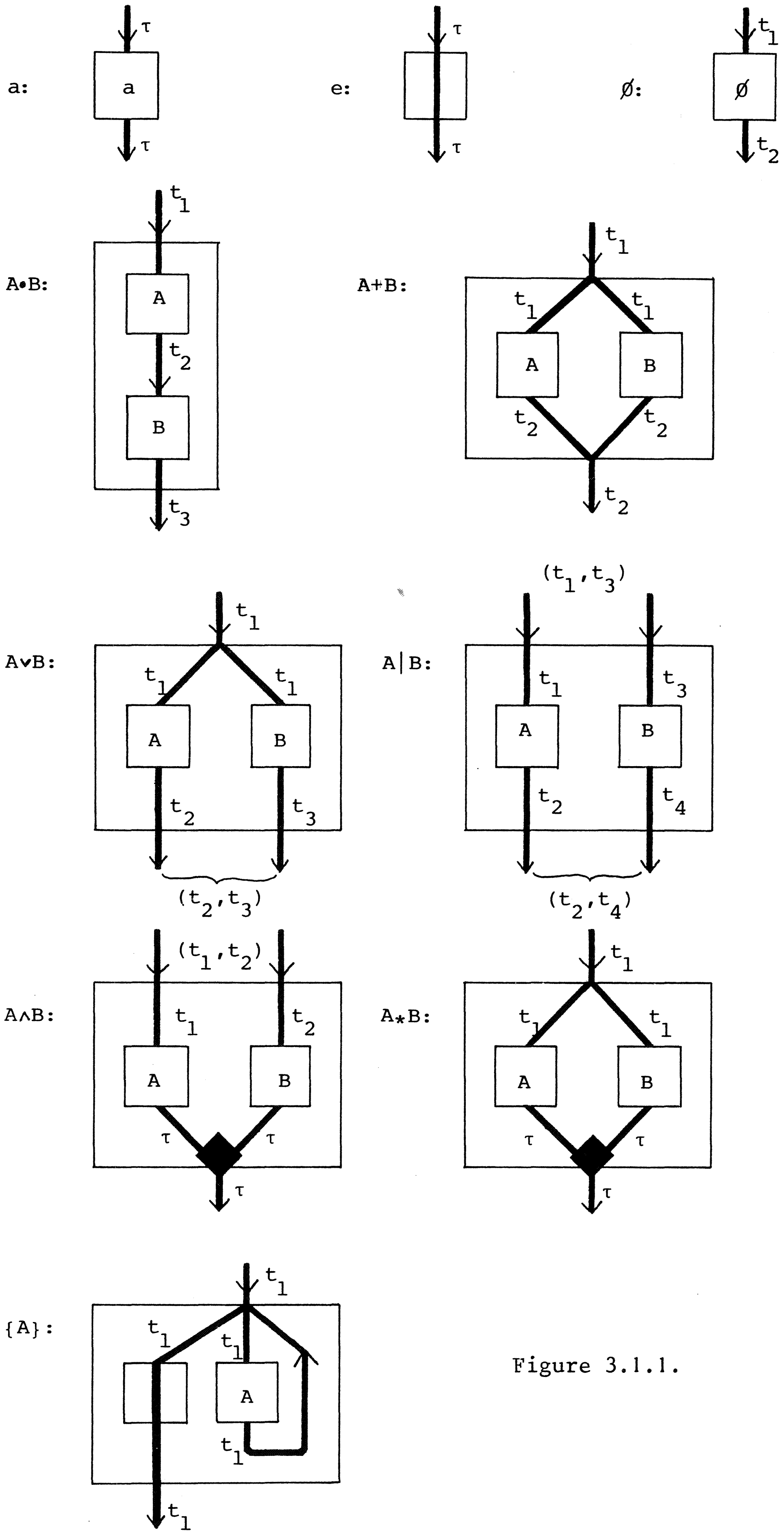

Figure 3.1.1.

Ax6. $A\varepsilon = \varepsilon A = A$

Ax7. $(A B) C = A (B C)$

Ax8. $\emptyset \mid A = A \mid \emptyset = \emptyset$

Ax9. $(A + B) C = (A C) + (B C)$

Ax10. $A(B + C) = (A B) + (A C)$

Ax11. $(A + B) \mid C = (A \mid C) + (B \mid C)$

Ax12. $A \mid (B + C) = (A \mid B) + (A \mid C)$

Ax13. $(A \mid B)(C \mid D) = (A C) \mid (B D)$

Ax14. $(A \vee B)(C \mid D) = (A C) \vee (B D)$

Ax15. $(A \mid B)(C \wedge D) = (A C) \wedge (B D)$

Ax16. $A(B \vee C) = (A B) \vee (A C)$ if Λ does not contain + and {}

Ax17. $A C * B D = (A \vee B)(C \wedge D)$

Ax18. $\{A\} = \varepsilon + A\{A\}$

Ax19. $\{A + \varepsilon\} = \{A\}$

Ax20. $\{\varepsilon\} = \varepsilon$.

3.2.1. DEFINITION. A *possesses* ε if A satisfies one of the following conditions:

(i) $A \equiv \varepsilon$

(ii) $A \equiv A_1 + A_2$ and at least one of $A_i (i = 1,2)$ possesses ε

(III) $A \equiv A_1 A_2$ and both A_1 and A_2 possess ε

(iv) $A \equiv \{B\}$ where B is an arbitrary expression.

The set of *proof rules* *Pr1* consists of:

R1. (*Replacement rule*). Let A occur in B and A = C. If D is the expression obtained from B by replacing A by C, then B = D.

R2. (*Solution of an equation*). From $A = B A + C$, where B does not possess ε, one may infer $A = \{B\} C$.

3.3. Interpretations of FS1

Let M be a nonempty finite set. The set M' with typical elements Q,..., is dfined by

$$Q ::= m \mid Qm \mid \langle Q_1, Q_2 \rangle$$

where $m \in M$.

The set M^{∞} with typical elements $\alpha,\beta,\gamma,\ldots$ is defined by

$$\alpha ::= Q \mid (\alpha_1;\alpha_2).$$

(We assume that no confusion will arise from using here the same notation M^{∞}, as in section 2.3 for another set. This set plays a rile analogous to that in section 2.3.)

The interpretation of the constants e, $\emptyset$ and function variables is given by a function ρ, of type

$$\rho: X \rightarrow (M^{\infty} \rightarrow M^{\infty})$$

where $X = VAR \cup \{e\} \cup \{\emptyset\}$

Then

(i) $\rho(a)$ is a partial function, of type

$$\rho(a) : M' \rightarrow M'$$

(ii) $\rho(e)$ is the identity function of type

$$\rho(e) : M' \rightarrow M'$$

(iii) $\rho(\emptyset)$ is the nowhere defined function.

The interpretation of FS1-expressions is given by a function $\phi 1_{M,\rho}$ of type

$$\phi 1_{M,\rho} : EXP1 \rightarrow (P(M^{\infty}) \rightarrow P(M^{\infty}))$$

where $P(M^{\infty})$ is the power set of M^{∞} with typical elements $\tilde{Q}$ and $\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\ldots$.

Now we define:

(i) $\phi 1_{M,\rho}(a)(\tilde{Q}) = \{\rho(x)(Q) \mid Q \in \tilde{Q}\}$, where $a \in VAR$

(ii) $\phi 1_{M,\rho}(e)(\tilde{Q}) = \{\rho(e)(Q) \mid Q \in \tilde{Q}\}$

(iii) $\phi 1_{M,\rho}(\emptyset)(\tilde{Q})$ is empty

(iv) $\phi 1_{M,\rho}(A\,B)\,(\tilde{\alpha}) = \phi 1_{M,\rho}(B)(\phi_{M,\rho}(A)(\tilde{\alpha}))$

(v) $\phi 1_{M,\rho}(A \vee B)(\tilde{\alpha}) = \{(\beta_1;\beta_2) \mid \beta_1 \in \phi 1_{M,\rho}(A)(\{\gamma\})$ and $\beta_2 \in \phi 1_{M,\rho}(B)(\{\gamma\}), \gamma \in \tilde{\alpha}\}$

(vi) $\phi 1_{M,\rho}(A \mid B)(\tilde{\alpha}_1;\tilde{\alpha}_2) = \{(\beta_1;\beta_2 \mid \beta_1 \in \phi 1_{M,\rho}(A)(\tilde{\alpha}_1)$ and $\beta_2 \in \phi 1_{M,\rho}(B)(\tilde{\alpha}_2)\}$

(vii) $\phi 1_{M,\rho}(A \wedge B)(\tilde{\alpha}_1;\tilde{\alpha}_2) = \{<\beta_1,\beta_2> \mid \beta_1 \in \phi 1_{M,\rho}(A)(\tilde{\alpha}_1)$ and $\beta_2 \in \phi 1_{M,\rho}(B)(\tilde{\alpha}_2)\}$

(viii) $\phi 1_{M,\rho}(A * B)(\tilde{\alpha}) = \{<\beta_1,\beta_2> \mid \beta_1 \in \phi 1_{M,\rho}(A)(\{\gamma\})$ and $\beta_2 \in \phi 1_{M,\rho}(B)(\{\gamma\}), \gamma \in \tilde{\alpha}\}$

(ix) $\phi 1_{M,\rho}(A+B)(\tilde{\alpha}) = \phi 1_{M,\rho}(A)(\tilde{\alpha}) \cup \phi 1_{M,\rho}(B)(\tilde{\alpha})$.

The interpretation of formulas of FS1 is given by a function $F1_{M,\rho}$ of type

$$F1_{M,\rho} : \text{FORM1} \to (P(M^{\infty}) \to T).$$

Then

$$F1_{M,\rho}(A=B)(\tilde{\alpha}) = (\phi 1_{M,\rho}(A)(\{\gamma\}) = \phi 1_{M,\rho}(B)(\{\gamma\}) \text{ for all } \gamma \in \tilde{\alpha}.$$

As usual we write $\models f$, when the formula f is valid for all M and ρ.

3.4. Soundness of FS1

To prove soundness of FS1 it is necessary to prove that for any possible M and ρ and $\tilde{\alpha} \in P(M^{\infty})$ all axioms are valid and all proof rules preserve validity.

Validity of all axioms can be checked immediately. Validity of formulas, which can be derived using the first proof rule, is apparent. It remains to prove that the second proof rule preserves validity also.

3.4.1. NOTATION. (i) The assertion that $\phi 1_{M,\phi}(A)(\tilde{\alpha}) \subseteq \phi 1_{M,\rho}(B)(\tilde{\alpha})$ for any M,ρ and any singleton set $\{\gamma\},\gamma \in \tilde{\alpha}$, is denoted by writing

$$\models A \subseteq B.$$

(ii) We shall write in the sequel $\phi 1_{M,\rho}(A)(\alpha), \alpha \in M^{\infty}$ instead of $\phi 1_{M,\rho}(A)(\{\alpha\})$, to denote the meaning of the function $\phi 1_{M,\rho}(A)$ on the singleton set $\{\alpha\} \in P(M^{\infty})$.

3.4.2. <u>LEMMA</u>. *If* $\models A = BA + C$ *then* $\models \{B\}C \subseteq A$.

<u>PROOF</u>. Let $\beta \in \phi 1_{M,\rho}(\{B\}C)(\alpha)$ for some M,ρ and $\alpha \in M^{\infty}$. This means that $\beta \in \phi 1_{M,\rho}(B^n C)(\alpha)$ for some n. Then, making n replacements of A by (BA + C) in the right hand side of the equation A = BA + C, we obtain

$$A = B^{n+1}A + B^n C + \dots + BC + C.$$

Thus, $\beta \in \phi l_{M,\rho}(B^nC)(\alpha) \subseteq \phi l_{M,\rho}(A)(\alpha)$. Since this inclusion holds for any M,ρ and $\alpha \in M^\infty$ we conclude that

$$\models \{B\}C \subseteq A. \qquad \square$$

3.4.3. DEFINITION. The *length* of any element $\alpha \in M^\infty$ equals the number of elements from M occurring in α.

3.4.4. LEMMA. *If* $\models A = BA + C$ *and* B *does not possess* ε *then* $\models A \subseteq \{B\}C$.

PROOF. Let $\beta \in \phi l_{M,\rho}(A)(\alpha)$ for some M,ρ and $\alpha \in M^\infty$. To prove the required inclusion we shall introduce a new interpretation $\phi l_{M',\rho'}$, where $M' = M \cup \{m\}$ ($m \notin M$) and ρ' is defined below.

First we introduce some notations: m^n denotes a string of n consecutive m's; furthermore, we use the (ambiguous) notation $\alpha^n \in M'^\infty$ if α^n contains exactly n occurrences of the new symbol m and yields $\alpha \in M^\infty$ after erasing these occurrences.

Now ρ' is defined as follows:

$$\text{for all } n \geq 0,\ a \in VAR \text{ and } \gamma \in M^\infty$$
$$\rho'(a)(\gamma^n) = \{\beta m^{n+1} \mid \beta \in \rho(a)(\gamma)\}.$$

The point of the interpretation $\phi l_{M',\rho'}$ is that it increases the number of occurrences of the symbol m in any resulting element after passing each variable $a \in VAR$.

Thus, if $\beta \in \phi l_{M,\rho}(A)(\alpha)$ then for some n, $\beta^n \in \phi l_{M',\rho'}(A)(\alpha)$. Let the length of the element β^n be k. Replace A in the right hand side of the equation $A = BA + C$, k times. Then we obtain:

$$A = B^{k+1}A + B^kC + B^{k-1}C + \ldots + BC + C.$$

Since B does not possess ε each summand of B contains either a variable or operations $\vee, \wedge, *$, where $\vee$ and $\wedge$ are paired. In both cases the length of any resulting element which belongs to $\phi l_{M',\rho'}(B^{k+1}A)$ is more than k, as, in the first case, it contains at least k+1 occurrences of the symbol m, and, in the second case, each operation $\vee$ (or $*$) doubles the length of an input element after each execution of B.

Thus, we have:

$$\beta^n \notin \phi 1_{M',\phi'}(B^{k+1}A)(\alpha).$$

Hence

$$\beta^n \in \phi 1_{M',\phi'} (B^kC + B^{k-1}C + \ldots + BC + C)(\alpha)$$
$$\subseteq \phi 1_{M',\rho'} (\{B\}C)(\alpha).$$

As this inclusion holds for any $\alpha \in M'^{\infty}$, we conclude that:

$$\phi 1_{M',\rho'}(A)(\alpha) \subseteq \phi 1_{M',\rho'}(\{B\}C)(\alpha)$$

and also

$$\phi 1_{M,\rho}(A)(\alpha) \subseteq \phi 1_{M,\phi}(\{B\}C)(\alpha).$$

As this inclusion holds for all M and ρ, we have

$$\models A \subseteq \{B\}C. \qquad \square$$

The soundness of the second proof rule (section 3.2, Rule 2) immediately follows from Lemma 3.4.2 and Lemma 3.4.4, and we also have the following theorem.

3.4.5. <u>THEOREM</u>. *The formal system* FS1 *is sound, i.e.*

$$\vdash_{FS1} f \Rightarrow \models f.$$

3.4. <u>Correspondence between FS and FS1</u>

3.5.1. <u>LEMMA</u>. FS *is a subsystem of* FS1.

<u>PROOF</u>. It suffices to prove:
(a) EXP $\subseteq$ EXP1; (b) all axioms *Ax* and proof rules *Pr* are derivable in FS1.

(a) is apparent from the definitions of EXP and EXP1.

(b) Axioms A1 ÷ A7, A9, A10, A14 ÷ A16 and proof rules *Pr* follow straight from the corresponding axioms Ax1 ÷ Ax7, Ax9, Ax10, Ax18 ÷ Ax20 and proof rules Pr1. Axioms A8, A11 ÷ A13 follow immediately from Ax17 and Ax6, Ax8, Ax11 ÷ Ax16. □

3.5.2. PROPOSITION. *For any* M,ρ *and* $\alpha \in M^{\infty}$:

(a) $\phi_{M,\rho}(A)(\alpha) = \phi 1_{M,\rho}(A)(\alpha)$ for any $A \in EXP$;

(b) $F_{M,\rho}(A = B)(\alpha) = F1_{M,\rho}(A = B)(\alpha)$ for any $A \in EXP$, $B \in EXP$.

PROOF.
(a) Immediate from the definitions of $\phi_{M,\rho}$ and $\phi 1_{M,\rho}$.
(b) Immediate from (a). □

This reduces the problem of equivalence for the FS-expressions to the same problem for the corresponding FS1-expressions.

The remainder of our paper is devoted to a proof of the following completeness result:

$$\models A = B \;\Rightarrow\; \vdash_{FS1} A = B$$

where $a \in EXP$ and $B \in EXP$.

To this end we shall introduce the notion of equational characterization and prove that each $A \in EXP$ can be equationally characterized. We shall do it in a way analogous to that for regular events in SALOMAA [4].

4. EQUATIONAL CHARACTERIZATIONS

First we shall give some auxiliary definitions and propositions.

4.1. DEFINITION (i) An expression $A \in EXP1$, which does not contain + and iterations of type $t \Rightarrow t$, where $t \not\equiv \tau$, is *open* if it satisfies one of the following conditions:

1) $A \equiv \{B\}$ for some $B \in EXP$;
2) A has one of the forms: $A_1 \vee A_2$, $A_1 \mid A_2$, $A_1 \wedge A_2$, $A_1 * A_2$, and at least one of A_i $(i = 1,2)$ is open;

3) $A \equiv eA_1$ and A_1 is open:

4) $A \equiv A_1A_2$, where $A_1 \in t \Rightarrow \tau$, $A_1 \notin$ IDEN and A_1 is open.

5) $A \equiv A_1A_2$, where A_1 has one of the forms: $A_1' \vee A_1''$, $A_1' \mid A_1''$, and, correspondingly, A_2 has one of the forms: $A_2' \mid {}_2''$, $A_2' \wedge A_2''$, and, at least, one of the expressions $A_1'A_2'$ or $A_1''A_2''$ is open.
(So this means that there is an iteration at the beginning of some parallel branch, possibly preceded by an identity expression).

(ii) An expression $A \in$ EXP1, which does not contain + and iterations of type $t \Rightarrow t$, where $t \not\equiv \tau$ is *closed*, if it is not open.

NOTATION. In view of Ax 3 , we can write sums of several expressions associatively. In the sequel we use the notation $\sum_{i=1}^{n} A_i$ to denote $A_1 + A_2 + \ldots + A_n$.

4.2. PROPOSITION. *Let* $A \in$ EXP1 *where the only iterations contained by* A *are of type* $\tau \Rightarrow \tau$. *Then for some closed* $A_1, A_2, \ldots, A_n$ $(n \geq 1)$ *not containing* + :

$$\vdash_{FS1} A = \sum_{i=1}^{n} A_i. \qquad (*)$$

PROOF. Induction on the structure of A.
Basis: for $a \in$ VAR, an identity expression $\varepsilon \in$ IDEN and $\emptyset$, the assertion (*) holds trivially.
Induction step: Let A_1 and A_2 satisfy (*). Then we have to prove that:
1) for $A_1 \mid A_2$, $A_1 \vee A_2$, $A_1 \wedge A_2$, $A_1 * A_2$, $A_1 + A_2$, A_1A_2 the assertion (*) holds; (2) if $A_1 \in \tau \Rightarrow \tau$, then for $\{A_1\}$ the assertion (*) also holds.
1) Let $A \equiv A_1 \mid A_2$. By the induction hypothesis, there are some closed $A_{11}, A_{12}, \ldots, A_{1n}$ and $A_{21}, A_{22}, \ldots, A_{2m}$, such that:

$$\vdash_{FS1} A = (\sum_{i=1}^{n} A_{1i}) \mid (\sum_{j=1}^{m} A_{2j}).$$

Using distributive laws (Ax 11,12) and Ax 2,3, we obtain:

$$\vdash_{FS1} A = \sum_{i=1}^{n} \sum_{j=1}^{m} (A_{1i} \mid A_{2j})$$

where for each i and j, $A_{1i} \mid A_{2j}$ is closed (see Definition 4.1).

For the next three cases the proof is analogous. We have only to use instead of axioms Ax 11,12 the corresponding distributive laws, which are easily derivable in FS1.

In the case $A \equiv A_1 + A_2$ the assertion (*) evidently holds.

Let $A \equiv A_1A_2$. Then, by the induction hypothesis:

$$\vdash_{FS1} A = (\sum_{i=1}^{n} A_{1i})(\sum_{j=1}^{m} A_{2j})$$

where all A_{1i} $(i = 1,2,\dots,n)$ and A_{2j} $(j = 1,2,\dots,m)$, are closed. Using axioms Ax 9,10, we obtain:

$$\vdash_{FS1} A = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{1i}A_{2j}$$

where, by Definition 4.1, $A_{1i}A_{2j}$ is closed for each i and j.

2) Let $A \equiv \{A_1\}$. By the induction hypothesis:

$$\vdash_{FS1} A = \{\sum_{i=1}^{n} A_{1i}\}$$

where all A_{1i} $(i = 1,2,\dots,n)$ are closed and do not contain +. Using axiom Ax 19, we obtain either,

$$\vdash_{FS1} A = \{\varepsilon\}$$

or:

$$\vdash_{FS1} A = \{\sum_{j=1}^{m} A_{1j}\}$$

where $m \leq n$ and all $A_{1j} \notin$ IDEN and $A_{1j} \in \tau \Rightarrow \tau$.

In the first case, by Ax 20, we obtain:

$$\vdash_{FS1} A = \varepsilon$$

and, therefore, $\{A_1\}$ satisfies the assertion (*).

Using Ax 18 and Ax 9 in the second case, we derive:

$$\vdash_{FS1} A = \sum_{j=1}^{m} A_{1j}\{\sum_{j=1}^{m} A_{1j}\} + \varepsilon$$

where, since for each j ($j = 1,2,\ldots,m$), $A_{1j} \notin$ IDEN, $A_{1j} \in \tau \Rightarrow \tau$ and is closed, we have, by Definition 4.1, that $A_{1j}\{\sum_{j=1}^{m} A_{1j}\}$ is closed. □

Now we shall introduce the notions of *prefix* and *suffix* of a closed expression $A \in$ EXP1. We write $\overline{A}$ and $\underline{A}$ to denote the prefix and suffix of A, respectively.

4.3. DEFINITION. The prefix and suffix of a closed expression which does not contain + and iterations of type $t \Rightarrow t$, where $t \not\equiv \tau$, are defined by:

(i) a) $\overline{a} = a$, where $a \in$ VAR

$\underline{a} = e$

b) $\overline{\varepsilon} = \varepsilon$

$\underline{\varepsilon} = \varepsilon$

c) $\overline{\emptyset} = \emptyset$

$\underline{\emptyset} = \varepsilon$

(ii) a) $\left.\begin{array}{l} \overline{(A \wedge B)} = (\overline{A} \mid \overline{B}) \\ \underline{(A \wedge B)} = (\underline{A} \wedge \underline{B}) \end{array}\right\}$ if $A \notin$ IDEN or $B \notin$ IDEN

b) $\overline{(\varepsilon \wedge \varepsilon)} = (\varepsilon \wedge \varepsilon)$

$\underline{(\varepsilon \wedge \varepsilon)} = \varepsilon$

(iii) $\left.\begin{array}{l} \overline{A \mid B} = \overline{A} \mid \overline{B} \\ \underline{A \mid B} = \underline{A} \mid \underline{B} \end{array}\right\}$ if $A \notin$ IDEN or $B \notin$ IDEN

(iv) a) $\left.\begin{array}{l} \overline{(A * B)} = \overline{A} \\ \underline{(A * B)} = (\underline{A} * \underline{B}) \end{array}\right\}$ if $\overline{A} \equiv \overline{B} \notin$ IDEN

b) $\left.\begin{array}{l} \overline{(A * B)} \quad (\overline{A} \vee \overline{B}) \\ \underline{(A * B)} = (\underline{A} \wedge \underline{B}) \end{array}\right\}$ if $\overline{A} \not\equiv \overline{B}$ and: $A \notin$ IDEN or $B \notin$ IDEN

c) $\overline{(\varepsilon * \varepsilon)} = (\varepsilon \vee \varepsilon)$

$\underline{(\varepsilon * \varepsilon)} = (\varepsilon \wedge \varepsilon)$

(v) a) $\left.\begin{array}{l} \overline{(A \vee B)} = \overline{A} \\ \underline{(A \vee B)} = (\underline{A} \vee \underline{B}) \end{array}\right\}$ if $\overline{A} \equiv \overline{B} \notin$ IDEN

b) $\left.\begin{array}{l} \overline{(A \vee B)} \quad (\overline{A} \vee \overline{B}) \\ \underline{(A \vee B)} = (\underline{A} \mid \underline{B}) \end{array}\right\}$ if $\overline{A} \not\equiv \overline{B}$ and: $A \notin$ IDEN or $B \notin$ IDEN

c) $\overline{(\varepsilon \vee \varepsilon)} = (\varepsilon \wedge \varepsilon)$

$\underline{(\varepsilon \vee \varepsilon)} = \varepsilon_1$, where if $\varepsilon \in t_1 \Rightarrow t_1$ then $\varepsilon_1 \in (t_1,t_1) \Rightarrow (t_1,t_1)$

(vi) a) $\left.\begin{array}{l}\overline{AB} = \overline{A}B \\ \underline{AB} = A\underline{B}\end{array}\right\}$ if $A \in t \Rightarrow \tau$ and $A \notin$ IDEN

b) $\left.\begin{array}{l}\overline{AB} = \overline{B} \\ \underline{AB} = \underline{B}\end{array}\right\}$ if $A \in$ IDEN

c) $\overline{(A_1 \vee A_2)(B_1 \mid B_2)} = \overline{(A_1B_1 \vee A_2B_2)}$
$\underline{(A_1 \vee A_2)(B_1 \mid B_2)} = \underline{(A_1B_1 \vee A_2B_2)}$

d) $\overline{(A_1 \vee A_2)(B_1 \wedge B_2)} = \overline{(A_1B_1 * A_2B_2)}$
$\underline{(A_1 \vee A_2)(B_1 \wedge B_2)} = \underline{(A_1B_1 * A_2B_2)}$

e) $\overline{(A_1 \mid A_2)(B_1 \mid B_2)} = \overline{(A_1B_1 \mid A_2B_2)}$
$\underline{(A_1 \mid A_2)(B_1 \mid B_2)} = \underline{(A_1B_1 \mid A_2B_2)}$

f) $\overline{(A_1 \mid A_2)(B_1 \wedge B_2)} = \overline{(A_1B_1 \wedge A_2B_2)}$
$\underline{(A_1 \mid A_2)(B_1 \wedge B_2)} = \underline{(A_1B_1 \wedge A_2B_2)}$

4.4. REMARK. By means of this definition we take as a prefix one variable, if it is possible, from each parallel branch of the expression, supplementing the other parallel branches by ε. In cases when $A \equiv A_1 * A_2$ (clause (iv)) or $A \equiv A_1 \vee A_2$ (clause (v)) and $\overline{A}_1 \equiv \overline{A}_2 \notin$ IDEN, $\overline{A}_1$ is taken as a prefix of the whole expression A.

4.5. EXAMPLE. Let $A \equiv ab * ac$. Then $\overline{A} = a$ and $\underline{A} = b * c$.

Now we shall expand the notions of prefix and suffix, given in Definition 4.3, to cover also some expressions which contain +. We do not need, nor want, to define $\underline{A}$, $\overline{A}$ for all expressions; e.g. not in case $A \equiv aB + cD$. However, for a sum of expressions, which have the same prefix, we shall define prefix and suffix as in the next definition.

4.6. DEFINITION. (i) $\overline{A}$ and $\underline{A}$ are a prefix and a suffix of A, respectively, if they are obtained by Definition 4.3; (ii) if $\overline{A},\overline{B}$ and $\underline{A},\underline{B}$ are prefixes and suffices of A and B, respectively, and $\overline{A} \equiv \overline{B}$, then $\overline{A + B} = \overline{A}$ and $\underline{A + B} = \underline{A} + \underline{B}$.

4.7. PROPOSITION. *For all* A *such that* $\overline{A}, \underline{A}$ *are defined*:

$$\vdash_{FS1} A = \overline{A}\ \underline{A}\ .$$

PROOF. Straightforward from Definition 4.6. □

4.8. EXAMPLE. Let $A \equiv (a_1 \vee e)(b_1b_2 \wedge a_2b_3)c + A_1c * a_2b_1$.
Let $A_1 \equiv (a_1 \vee e)(b_1b_2 \wedge a_2b_3)c$ and $A_2 \equiv a_1c * a_2b_1$. So A_1,A_2 are closed and hence we can take their prefixes and suffixes. By Definition 4.3, we have

$\overline{A_1}$ = (by vi,a)

$\overline{(a_1 \vee e)(b_1b_2 \wedge a_2b_3)}$ = (by vi,d)

$\overline{a_1b_1b_2 * ea_2b_3}$ = (by vi,b)

$\overline{a_1b_1b_2} \vee \overline{ea_2b_3}$ = (by vi,a)

$\overline{a_1} \vee \overline{ea_2}$ = (by i,a and vi,b)

$a_1 \vee a_2$

$\underline{A_1}$ = (by vi,a)

$\underline{(a_1 \vee e)(b_1b_2 \wedge a_2b_3)c}$ = (by vi,d)

$\underline{(a_1b_1b_2 * ea_2b_3)c}$ = (by iv,b)

$\underline{a_1b_1b_2} \wedge \underline{ea_2b_3)c}$ = (by vi,a)

$(\underline{a_1}b_1b_2 \wedge \underline{ea_2}b_3)c$ = (by i,a and vi,b)

$(eb_1b_2 \wedge eb_3)c$.

Thus, $\vdash_{FS1} A_1 = \overline{A_1}\,\underline{A_1} = (a_1 \vee a_2)(eb_1b_2 \wedge eb_3)c$ by Proposition 4.7.
Furthermore,

$\overline{A_2}$ = (by iv,b)

$\overline{a_1c} \vee \overline{a_2b_1}$ = (by vi,a)

$\overline{a_1} \vee \overline{a_2}$ = (by i,a)

$a_1 \vee a_2$.

$\underline{A_2}$ = (by iv,b)

$\underline{a_1c} \wedge \underline{a_2b_1}$ = (by vi,a)

$\underline{a_1}c \wedge \underline{a_2}b_1$ = (by i,a)

$ec \wedge eb_1$.

Thus, $\vdash_{FS1} A_2 = \overline{A_2}\,\underline{A_2} = (a_1 \vee a_2)(ec \wedge eb_1)$ by Proposition 4.7.

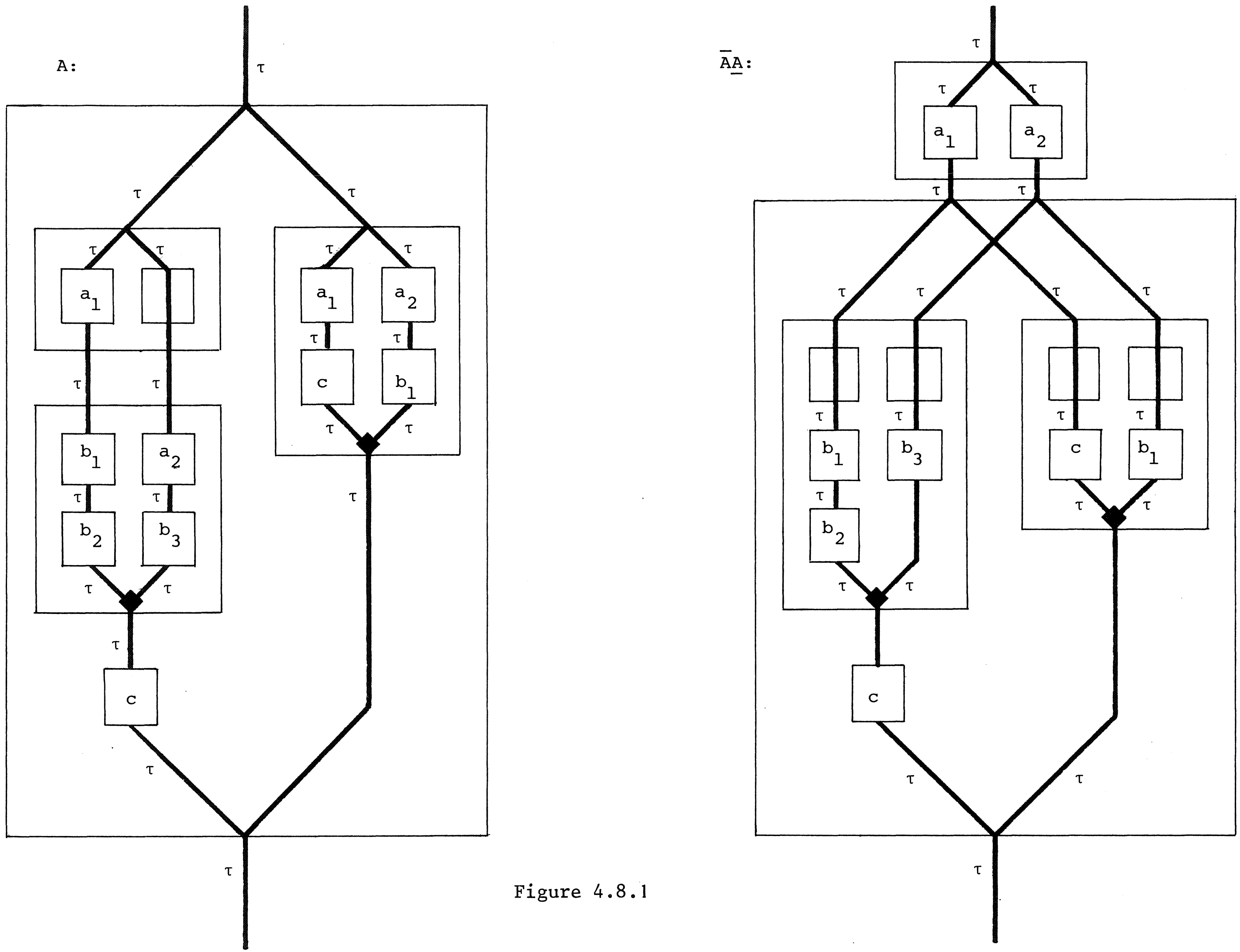

Figure 4.8.1

Since $\overline{A_1} \equiv \overline{A_2}$, we have, by Definition 4.6:

$$\overline{A} = \overline{A_1} = (a_1 \vee a_2)$$

$$\underline{A} = \underline{A_1} + \underline{A_2} = (eb_1b_2 \wedge eb_3)c + (ec \wedge eb_1).$$

Therefore, $\vdash_{FS1} A = \overline{A}\underline{A} = (a_1 \vee a_2)((eb_1b_2 \wedge eb_3)c + (ec \wedge eb_1))$ by Proposition 4.7.

The diagrams in Figure 4.8.1 demonstrate graphically what has happened in the preceding example.

4.9. DEFINITION. An expression $A \in$ EXP1 is *equationally characterized* if there exists a finite set of expressions $A_1, A_2, \ldots, A_n$, such that $A \equiv A_1$ and

$$\vdash_{FS1} A_i = \sum_{j \in S(N)} \overline{A_{ij}}\, \underline{A_{ij}} + \delta(A_i) \qquad (i = 1,2,\ldots,n)$$

where:

- $\delta(A_i) \in$ IDEN or $\delta(A_i) \equiv \emptyset$;
- $N = \{1,2,\ldots\}$ and $S(N)$ is a finite subset of N;
- $\forall i,j\ \exists k$, $1 \le k \le n$, such that $\underline{A_{ij}} \equiv A_k$;
- $\forall_i$ the $\overline{A_{ij}}$ are pairwise syntactically different.

4.10. THEOREM. *Every* $A \in$ EXP *is equationally characterized.*

PROOF. Induction on the structure of A.
Basis: for $\alpha \in$ VAR, an identity expression $\varepsilon \in$ IDEN and $\emptyset$, the equational characterization holds trivially.
Induction step: let $A \in$ EXP and $B \in$ EXP be equationally characterized. Then we have to prove that $A * B$, $A + B$, AB and $\{A\}$ are equationally characterized.

By the induction hypothesis, there are finite sets $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_m$, such that $A \equiv A_1$, $B \equiv B_1$ and

$$\vdash_{FS1} A_i = \sum_{j \in S(N)} \overline{A_{ij}}\, \underline{A_{ij}} + \delta(A_i) \qquad (i = 1,2,\ldots,n) \qquad (**)$$

$$\vdash_{FS1} B_i = \sum_{k \in S(N)} \overline{B_{ik}}\, \underline{B_{ik}} + \delta(B_i) \qquad (i = 1,2,\ldots,m) \qquad (***)$$

<u>Case 1</u>. $A * B$. We denote

$$\eta(u,v) \equiv A_u * B_v \tag{1}$$

$$\xi(u,v) \equiv A_u \wedge B_v \tag{1'}$$

$$(u = 0,1,2,\ldots,n;\ v = 0,1,2,\ldots,m)$$

We write A_0 and B_0 instead of ε. The number of expressions (1) and (1') is finite.

For $\eta(u,v)$ we have the following:

$$\vdash_{FS1} \eta(u,v) = \Big(\sum_{j\in S(N)} \overline{A_{uj}}\,\underline{A_{uj}} + \delta(A_u)\Big) * \Big(\sum_{k\in S(N)} \overline{B_{vk}}\,\underline{B_{vk}} + \delta(B_v)\Big).$$

Using the distributive laws (axioms A12,13 of FS, which are derivable in FS1) and Ax 2,3, we obtain:

$$\vdash_{FS1} \eta(u,v) = \sum_{j\in S(N)}\sum_{k\in S(N)} (\overline{A_{uj}}\,\underline{A_{uj}} * \overline{B_{vk}}\,\underline{B_{vk}}) + \sum_{j\in S(N)} (\overline{A_{uj}}\,\underline{A_{uj}} * \delta(B_v))$$
$$+ \sum_{k\in S(N)} (\delta(A_u) * \overline{B_{vk}}\,\underline{B_{vk}}) + (\delta(A_u) * \delta(B_v)).$$

Then, by Definition 4.6 of prefix and suffix:

$$\vdash_{FS1} \eta(u,v) = \sum_{j\in S(N)\&k\in S(N)} \overline{A_{uj}}\,(\underline{A_{uj}} * \underline{B_{vk}})$$

$$+ \sum_{j\in S(N)}\sum_{k\in S(N)} (\overline{A_{uj}} \vee \overline{B_{vk}})(\underline{A_{uj}} \wedge \underline{B_{vk}}) + \sum_{j\in S(N)} (\overline{A_{uj}} \vee \delta(B_v))(\underline{A_{uj}} \wedge \varepsilon)$$
$$+ \sum_{k\in S(N)} (\delta(A_u) \vee \overline{B_{vk}})(\varepsilon \wedge \underline{B_{vk}}) + (\delta(A_u) \vee \delta(B_v))(\varepsilon \wedge \varepsilon)$$

where all expressions $(\underline{A_{uj}} * B_{vk})$ are as in (1), and $(\underline{A_{uj}} \wedge \underline{B_{vk}})$, $(A_{uj} \wedge \varepsilon)$, $(\varepsilon \wedge B_{vk})$, $(\varepsilon \wedge \varepsilon)$ are as in (1').

For $\xi(u,v)$, by assumption, we have:

$$\vdash_{FS1} \xi(u,v) = \Big(\sum_{j\in S(N)} \overline{A_{uj}}\,\underline{A_{uj}} + \delta(A_u)\Big) \wedge \Big(\sum_{k\in S(N)} \overline{B_{vk}}\,\underline{B_{vk}} + \delta(B_v)\Big).$$

Using the distributive laws $((A+B) \wedge C = (A \wedge C) + (B \wedge C)$ and $A \wedge (B+C) = (A \wedge B) + (A \wedge C)$, axioms Ax 2,3 and Definition 4.6 of prefix and suffix, we obtain:

$$\vdash_{FS1} \xi(u,v) = \sum_{j\in S(N)} \sum_{k\in S(N)} (\overline{A_{uj}} \mid \overline{B_{vk}})(\underline{A_{uj}} \wedge \underline{B_{vk}}) + \sum_{j\in S(N)} (\overline{A_{uj}} \mid \delta(B_v))(\underline{A_{uj}} \wedge \varepsilon)$$

$$+ \sum_{k\in S(N)} (\delta(A_u) \mid \overline{B_{vk}})(\varepsilon \wedge \underline{B_{vk}}) + (\delta(A_u) \wedge \delta(B_v))$$

where all expressions $(\underline{A_{uj}} \wedge \underline{B_{vk}})$, $(\underline{A_{uj}} \wedge \varepsilon)$, $(\varepsilon \wedge \underline{B_{vk}})$ and $(\delta(A_u) \wedge \delta(B_v))$ are as in (1'). Since $\eta(1,1) \equiv A * B$, we conclude that $A * B$ is equationally characterized.

<u>Case 2</u>. $A + B$. We denote

$$\xi(u,v) \equiv A_u + B_v \tag{2}$$
$$(u = 0,1,2,\dots,n;\ v = 0,1,2,\dots,m).$$

We write $(A_0 + B_v)$ instead of B_v and $(A_u + B_0)$ instead of A_u. The number of expressions (2) is finite. By the induction hypothesis, we have:

$$\vdash_{FS1} \xi(u,v) = (\sum_{j\in S(N)} \overline{A_{uj}}\, \underline{A_{uj}} + \delta(A_u)) + (\sum_{k\in S(N)} \overline{B_{vk}}\, \underline{B_{vk}} + \delta(B_v)).$$

Using Ax 1,2,3,10 and Definition 4.6 of prefix and suffix, we obtain:

$$\vdash_{FS1} \xi(u,v) = \sum_{j\in S(N)} \overline{A_{uj}}\, \underline{A'_{uj}} + \sum_{j\in S(N)} \overline{A_{uj}}\, \underline{A_{uj}} + \sum_{k\in S(N)} \overline{B_{vk}}\, \underline{B_{vk}} + \delta(u,v)$$

where all expressions $\underline{A'_{uj}} \equiv \underline{A_{uj}} + \underline{B_{vk}}$ for some j and k, are as in (2), $\underline{A_{uj}}$ and $\underline{B_{vk}}$ are also as in (2). Since $\xi(1,1) \equiv A + B$, we conclude that $A + B$ is equationally characterized.

<u>Case 3</u>. AB. We denote

$$\eta(u,v_1,\dots,v_r) \equiv A_u B + B_{v_1} + \dots + B_{v_r} \tag{3}$$
$$(u = 0,1,\dots,n,\ r \geq 0,\ 1 \leq v_i \leq m,\ i = 1,2,\dots,r).$$

We write $A_0B + B_{v_1} + \dots + B_{v_r}$ instead of $B_{v_1} + \dots + B_{v_r}$. The number of expressions (3) is finite. By the induction hypothesis, we have:

$$\vdash_{FS1} \eta(u,v_1,\dots,v_r) = \sum_{j\in S(N)} \overline{A_{uj}}\,\underline{A_{uj}} + \delta(A_u))B$$

$$+ \sum_{k\in S(N)} \overline{B_{v_1k}}\,\underline{B_{v_1k}} + \delta(B_{v_1}) + \dots + \sum_{k\in S(N)} \overline{B_{v_rk}}\,\underline{B_{v_rk}} + \delta(B_{v_r}).$$

Using Ax 9 and Ax 2, the following is derivable:

$$\vdash_{FS1} \eta(u,v_1,\dots,v_r) = \sum_{j\in S(N)} \overline{A_{uj}}\,\underline{A_{uj}}\,B + \sum_{k\in S(N)} \overline{B_{v_1k}}\,\underline{B_{v_1k}} + \dots +$$

$$\sum_{k\in S(N)} \overline{B_{v_rk}}\,\underline{B_{v_rk}} + \delta(A_u)B + \delta(B_{v_1}) + \dots + \delta(B_{v_r})$$

where all $\overline{A_{uj}}\,\underline{A_{uj}} \in t \Rightarrow \tau$ and $\overline{A_{uj}}\,\underline{A_{uj}} \notin$ IDEN, and for all $p = 1,2,\dots,r$, $\overline{B_{v_pk}}\,\underline{B_{v_pk}} \in t \Rightarrow \tau$ and $\overline{B_{v_pk}}\,\underline{B_{v_pk}} \notin$ IDEN. If $\delta(A_u) \in$ IDEN, then replace the last occurrence of B by its representation (***) for B_1. Then, using axioms Ax 1,2,3,7,10 and Definition 4.6 of prefix and suffix, we obtain:

$$\vdash_{FS1} \eta(u,v_1,\dots,v_r) = \sum_{j\in S(N)} \overline{A_{uj}}\,\underline{A'_{uj}} + \sum_{q\in S(N)} \sum_{j\in S(N)} \overline{B_{qj}}\,\underline{B'_{qj}} +$$

$$\delta(u,v,\dots,v_r)$$

where all expressions $\underline{A'_{uj}}$ and $\underline{B'_{qj}}$ are as in (3). Since $\eta(1,1) \equiv AB$, we conclude that AB is equationally characterized.

<u>Case 4</u>. {A}. We denote $\xi(0) \equiv \{A\}$,

$$\xi(u_1,\dots,u_r) \equiv (A_{u_1} + \dots + A_{u_r})\{A\} \tag{4}$$

$$(r \geq 0,\ 1 \leq u_i \leq n,\ i = 1,2,\dots,r).$$

The number of expressions (4) is finite. By the induction hypothesis, we have:

$$\vdash_{FS1} \xi(0) = \{\sum_{j\in S(N)} \overline{A_{1j}}\,\underline{A_{1j}} + \delta(A_1)\}.$$

Using Ax 19, we obtain either:

$$\vdash_{FS1} \xi(0) = \{\varepsilon\}$$

or:

$$\vdash_{FS1} \xi(0) = \{\sum_{j\in S(N)} \overline{A_{1j}}\ \underline{A_{1j}}\}$$

where all expressions $\overline{A_{1j}}\ \underline{A_{1j}} \in \tau \Rightarrow \tau$ and $\overline{A_{1j}}\ \underline{A_{1j}} \notin$ IDEN.

In the first case, by Ax 10, we obtain:

$$\vdash_{FS1} \xi(0) = \varepsilon$$

and, therefore, {A} is equationally characterized.

Using in the second case Ax 18, then Ax 19 (to restore the original form of A) and Ax 9, we derive:

$$\vdash_{FS1} \xi(0) = \sum_{j\in S(N)} \overline{A_{1j}}\ \underline{A_{1j}}\ \{A\} + \varepsilon.$$

Then, using Ax 2,3,7,10 and Definition 4.6 of prefix and suffix, we obtain:

$$\vdash_{FS1} \xi(0) = \sum_{j\in S(N)} \overline{A_{1j}}\ \underline{A'_{1j}} + \delta(0) \qquad (5)$$

where all expression $\underline{A'_{1j}}$ are as in (4).

For $\xi(u_1,\ldots,u_r)$, by the induction hypothesis, we have:

$$\vdash_{FS1} \xi(u_1,\ldots,u_r) = (\sum_{j\in S(N)} \overline{A_{u_1j}}\ \underline{A_{u_1j}} + \delta(A_{u_1}) + \ldots + \sum_{j\in S(N)} \overline{A_{u_rj}}\ \underline{A_{u_rj}} + \delta(A_{u_r}))\{A\}.$$

Using Ax 2,3,7,9, we obtain:

$$\vdash_{FS1} \xi(u_1,\ldots,u_r) = \sum_{j\in S(N)} \overline{A_{u_1j}}\ \underline{A_{u_1j}}\ \{A\} + \ldots + \sum_{j\in S(N)} \overline{A_{u_rj}}\ \underline{A_{u_rj}}\ \{A\} + (\delta(A_{u_1}) + \ldots + \delta(A_{u_r}))\{A\}$$

where for all $j \in S(N)$ and $p = 1,2,\ldots,r$, $\overline{A_{u_p j}}\ \underline{A_{u_p j}} \in t \Rightarrow \tau$ and $\overline{A_{u_p j}}\ \underline{A_{u_p j}} \notin$ IDEN.

Now consider the last summand $(\delta(A_{u_1}) \ldots + \delta(A_{u_r}))\{A\}$. If all $\delta(A_{u_p})$, $p = 1,2,\ldots,r$, are $\emptyset$, this summand vanishes. If one of $\delta(A_{u_p}) \equiv \varepsilon$, we replace $\{A\}$ in this summand by its representation (5). Then, using Ax 1,10 and Definition 4.6 of prefix and suffix, we obtain:

$$\vdash_{FS1} \xi(u_1,\ldots,u_r) = \sum_{i \in S(N)} \sum_{j \in S(N)} \overline{A_{ij}}\ \underline{A'_{ij}} + \delta(u_1,\ldots,u_r)$$

where all expressions $\underline{A'_{ij}}$ are as in (4). Therefore, $\{A\}$ is equationally characterized. □

4.11. <u>EXAMPLE</u>. Let $A \equiv \{a_1b_1b_2 * a_2b_3\}c$.

Suppose $A_1 \equiv A$. Then:

$A_1 = \{a_1b_1b_2 * a_2b_3\}c =$ (by Proposition 4.2)
$\quad c + (a_1b_1b_2 * a_2b_3)\{a_1b_1b_2 * a_2b_3\}c =$ (by Definition 4.3)
$\quad cA_2 + (a_1 \vee a_2)A_3$

$A_2 = e$

$A_3 = (eb_1b_2 \wedge eb_3)\{a_1b_1b_2 * a_2b_3\}c =$ (by Definition 4.3)
$\quad (b_1 \mid b_3)A_4$

$A_4 = (eb_2 \wedge e)\{a_1b_1b_2 * a_2b_3\}c =$ (by Definition 4.3)
$\quad (b_2 \mid e)A_5$

$A_5 = (e \wedge e)\{a_1b_1b_2 * a_2b_3\}c =$ (by Definition 4.3)
$\quad (e \wedge e)A_6$

$A_6 = e\{a_1b_1b_2 * a_2b_3\}c =$ (by Proposition 4.2)
$\quad c + (a_1b_1b_2 * a_2b_3)\{a_1b_1b_2 * a_2b_3\}c =$ (by Definition 4.3)
$\quad cA_2 + (a_1 \vee a_2)A_3$

5. EQUIVALENCE OF FS EXPRESSIONS

In this section we shall show that if $A \in$ EXP, $B \in$ EXP and $\models A = B$, then there are sets of expressions $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots, B_n$, such that $A \equiv A_1$ and $B \equiv B_1$ and

$$\vdash_{FS1} A_i = \sum_{j \in S(N)} \overline{A_{ij}}\ \underline{A_{ij}} + \delta(A_i) \qquad (i = 1,2,\ldots,n) \qquad (*)$$

$$\vdash_{FS1} B_i = \sum_{j \in S(N)} \overline{B_{ij}}\ \underline{B_{ij}} + \delta(B_i) \qquad (i = 1,2,\ldots,n) \qquad (**)$$

where $\forall i,j\ \overline{A_{ij}} \equiv \overline{B_{ij}}$, $\delta(A_i) \equiv \delta(B_i)$ and $\underline{A_{ij}} \equiv A_r$, $\underline{B_{ij}} \equiv B_r$ for some r, $1 \le r \le n$.

Furthermore, we shall show how to construct such sets of expressions. Next it will be shown that if A and B are equationally characterized by the sets of equations (*) and (**), respectively, then $\vdash_{FS1} A = B$.

A straightforward consequence of these two facts will be a completeness result for expressions of EXP:

$$\models A = B \Rightarrow \vdash_{FS1} A = B.$$

Together with the earlier obtained soundness of FS1, this yields the following main result:

$$\models A = B \Leftrightarrow \vdash_{FS1} A = B,$$

where $A \in EXP$ and $B \in EXP$.

5.1. <u>LEMMA</u>. *If* $\models A = B$ *and*

$$\vdash_{FS1} A = \sum_{j \in S(N)} \overline{A_j}\ \underline{A_j} + \delta(A)$$

$$\vdash_{FS1} B = \sum_{k \in S(N)} \overline{B_k}\ \underline{B_k} + \delta(B)$$

then for any $j \in S(N)$ *there exists* $k \in S(N)$ *(and vice versa) such that* (i) $\overline{A_j} \equiv \overline{B_k}$, (ii) $\models \underline{A_j} = \underline{B_k}$ *and* (iii) $\delta(A) \equiv \delta(B)$.

<u>PROOF</u>.

(i) Assume that for some $j \in S(N)$ there does not exist $k \in S(M)$ such that $A_j \equiv B_k$. We shall show that in this case $\not\models A = B$, i.e. there exists an interpretation $\phi 1_{M',\rho'}$, such that $\phi 1_{M',\rho'}(A) \neq \phi 1_{M',\rho'}(B)$. To prove this we introduce an auxiliary interpretation $\phi 1_{M',\rho'}$ as follows.

Let $\phi 1_{M,\rho}$ be an interpretation of FS1 for some M and ρ. Let $M' = M \cup X$ where $X = \{a' \mid a \in VAR\} \cup \{e'\}$.

Before defining ρ', we introduce some notations:
we use the notation $\alpha_{M'}$ to denote that $\alpha_{M'} \in M'^{\infty}$, i.e. $\alpha_{M'}$ contains both elements from M and X and yields $\alpha \in M^{\infty}$ after erasing all elements of X; furhtermore, we use the (ambiguous) notation α_{χ} to denote that $\alpha_{\chi} \in X^{\infty}$ and is obtained from $\alpha_{M'}$ by erasing all elements of M and supplementing all empty places (if there are any) in the constructions of the kind $< \ , \ >$ by the new element e'.

Now ρ' is defined as follows:
for all $a \in VAR$ and $\gamma \in M^{\infty}$ we have

$$\rho'(a)(\gamma_{M'}) = \{\beta\gamma_{\chi}a' \mid \beta \in \rho(a)(\gamma)\}.$$

The point of the interpretation $\phi 1_{M',\rho'}$ is that any resulting element $\gamma_{M'}$ contains the trace γ_{χ} of the expression which has been executed to obtain this result.

<u>Example</u>. Let $A \equiv (a * b)c$, $\alpha \in M^{\infty}$ for some M, and $\phi 1_{M,\rho}(a)(\alpha) = \beta_1$; $\phi 1_{M,\rho}(b)(\alpha) = \beta_2$ (and hence $\phi 1_{M,\rho}(a * b)(\alpha) = <\beta_1,\beta_2>$), $\phi 1_{M,\rho}(c)(<\beta_1,\beta_2> = \gamma$, where $\beta_1,\beta_2,\gamma \in M^{\infty}$. Then $M' = M \cup \{a',b',c'\} \cup \{e'\}$. Therefore, $\phi 1_{M',\rho'}(a)(\alpha) = \beta_1 a'$; $\phi 1_{M',\rho'}(b)(\alpha) = \beta_2 b'$; $\phi 1_{M',\rho'}(a * b)(\alpha) = <\beta_1 a',\beta_2 b'>$; $\phi 1_{M',\rho}(c)(<\beta_1 a',\beta_2 b'>) = \gamma<a',b'>c'$. Here $\beta_{\chi} \equiv <a',b'> \in X^{\infty}$, where $X = \{a',b',c'\} \cup \{e'\}$, and is obtained from $\beta_{M'} \equiv <\beta_1 a',\beta_2 b'> \in M'^{\infty}$ after erasing all elements of M. Obviously the element $\gamma_{\chi} \equiv <a',b'>c'$ corresponds in a unique way to the expression (a * b)c, which has been executed to obtain the resulting element $\gamma_{M'}$. End of example.

Thus, if $\overline{A_j} \neq \overline{B_k}$, then the traces of $\underline{\overline{A_j}}$ and $\overline{B_k}$, occurring in $\beta_{M'} \in \phi 1_{M',\phi'}(\overline{A_j})(\alpha_{M'})$ and $\gamma_{M'} \in \phi 1_{M',\phi'}(\overline{B_k})(\alpha_{M'})$ are syntactically different. Therefore, $\gamma_{M'} \neq \beta_{M'}$. Since the interpretation $\phi 1$ preserves this difference in all subsequent transformations of the elements of M'^{∞}, $\phi 1_{M',\rho'}(\overline{A_j}\ \underline{A_j})(\alpha_{M'}) \neq \phi 1_{M',\rho'}(\overline{B_k}\ \underline{B_k})(\alpha_{M'})$. Further, if for some $\overline{A_j}$ there does not exist k, such that $\overline{A_j} \equiv \overline{B_k}$, then $\phi 1_{M',\rho'}(A)(\alpha_{M'}) \neq \phi 1_{M',\rho'}(B)(\alpha_{M'})$, and, hence, $A \neq B$. But that contradicts the assumption.

(ii) Assume that $\overline{A_j} \equiv \overline{B_k}$ for some j and k and $\underline{A_j} \neq \underline{B_k}$. Then for some interpretation $\phi 1_{M,\rho}$ and $\alpha \in M^{\infty}$ there exists some β such that $\beta \in \phi 1_{M,\rho}(\underline{A_j})(\alpha)$ and $\beta \notin \phi 1_{M,\rho}(B_k)(\alpha)$ or vice versa. Let us take the first case. Then, we have also that $\beta_{M'} \in \phi 1_{M',\rho'}(\underline{A_j})(\alpha_{M'})$ and $\beta_{M'} \notin \phi 1_{M',\rho'}(\underline{B_k})(\alpha_{M'})$, and, hence, there exists some $\beta'_{M'}$ such that $\beta'_{M'} \in \phi 1_{M',\rho'}(\overline{A_j}\ \underline{A_j})(\gamma_{M'})$ and $\beta'_{M'} \notin \phi 1_{M',\rho'}(\overline{B_k}\ \underline{B_k})(\gamma_{M'})$ for some $\gamma_{M'} \in M'^{\infty}$.

Now, from $\overline{A_j} \neq \overline{B_i}$ $(i \neq k)$, it follows, by clause (i) of this Lemma, that $\beta'_{M'} \notin \phi 1_{M',\phi'}(\overline{B_i}\ \underline{B_i})$ for all $i \neq k$. Hence $\beta'_{M'} \notin \phi 1_{M',\rho'}(B)(\gamma_{M'})$ and, therefore, $A \neq B$, which contradicts the assumption.

(iii) Similar to the proof of (i). □

5.2. <u>LEMMA</u>. *If* $A \in EXP$, $B \in EXP$ *and* $\models A = B$ *then for* A *and* B *there are sets of expressions* $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots, B_n$, *such that* $A \equiv A_1$, $B \equiv B_1$, *and* (*) *and* (**) *(as in the introduction of Section 5) hold.*

<u>PROOF</u>. We will give a simultaneous construction of sets of equations as in (*) and (**) for A and B respectively.

Since, by Theorem 4.9, A and B are equationally characterized, there exist sets of expressions $\tilde{A} = \{A_1, A_2, \ldots, A_r\}$ and $\tilde{B} = \{B_1, B_2, \ldots, B_m\}$, such that $A \equiv A_1$, $B \equiv B_1$ and:

$$\vdash_{FS1} A_i = \sum_{j \in S(N)} \overline{A_{ij}}\ \underline{A_{ij}} + \delta(A_i) \qquad (i = 1,2,\ldots,r)$$

$$\vdash_{FS1} B_i = \sum_{k \in S(N)} \overline{B_{ik}}\ \underline{B_{ik}} + \delta(B_i) \qquad (i = 1,2,\ldots,m).$$

According to Lemma 5.1, we have:

$$\vdash_{FS1} A_1 = \sum_{j \in S(N)} \overline{A_{1j}}\ \underline{A_{1j}} + \delta(A_1)$$

$$\vdash_{FS1} B_1 = \sum_{k \in S(N)} \overline{B_{1k}}\ \underline{B_{1k}} + \delta(B_1)$$

where for any j there exists k (and vice versa) such that $\overline{A_{1j}} \equiv \overline{B_{1k}}$, $\models \underline{A_{1j}} = \underline{B_{1k}}$, $\delta(A_1) \equiv \delta(B_1)$, and moreover $\underline{A_{1j}} \equiv A_p \in \tilde{A}$ for some p, $1 \leq p \leq r$, $\underline{B_{1k}} \equiv B_q \in \tilde{B}$ for some q, $1 \leq q \leq m$.

Further, for any new pair of expressions A_p and B_q, since $\models A_p = B_q$, we have again by Lemma 5.1:

$$\vdash_{FS1} A_p = \sum_{j \in S(N)} \overline{A_{pj}}\ \underline{A_{pj}} + \delta(A_p)$$

$$\vdash_{FS1} B_q = \sum_{k\in S(N)} \overline{B_{qk}}\, \underline{B_{qk}} + \delta(B_q)$$

where for any j there exists k (and vice versa) such that $\overline{A_{pj}} \equiv \overline{B_{qk}}$, $\models \underline{A_{pj}} = \underline{B_{qk}}$, $\delta(A_p) \equiv \delta(B_q)$, and moreover $\underline{A_{pj}} \in \tilde{A}$, $\underline{B_{qk}} \in \tilde{B}$

We can continue this procedure for all new pairs until no new pairs appear. Since the number of all possible different pairs is finite (= n×m) this process is finite. Hence the result follows. □

5.3. LEMMA. *Let* A ∈ EXP *and* B ∈ EXP. *Suppose the following holds:*

$$\vdash_{FS1} A_i = \sum_{j\in S(N)} P_{ij} A_j + \delta(A_i) \qquad (i = 1,2,\dots,n)$$

$$\vdash_{FS1} B_i = \sum_{j\in S(N)} P_{ij} B_j + \delta(B_i) \qquad (i = 1,2,\dots,n)$$

where none of the expressions P_{ij} *possesses* ε ∈ IDEN, *and* $\delta(A_i \equiv \delta(B_i)$. *Then*

$$\vdash_{FS1} A_i = B_i \qquad (i = 1,2,\dots,n).$$

PROOF. Induction in n.

Basis: for n = 1 the proof immediately follows from the soundness of FS1.

Induction step: assume that the assertion holds for i = n-1. For i = n we have:

$$\vdash_{FS1} A_n = \sum_{j\in S(N)\&j\neq n} P_{nj} A_j [+ P_{nn} A_n] + \delta(A_n) \qquad (1)$$

$$\vdash_{FS1} B_n = \sum_{j\in S(N)\&j\neq n} P_{nj} B_j [+ P_{nn} A_n] + \delta(B_n) \qquad (2)$$

where the part of the expression inside [ ] may be absent.

Now we solve these equations for A_n and B_n and replace in the remaining equations for A_i and B_i (i = 1,2,...,n-1) all occurrences of A_n and B_n by their solutions. We obtain sets of n-1 equations, which satisfy the conditions of the Lemma, i.e. none of the expressions P_{ij} possesses ε ∈ IDEN. Thus, by the induction hypothesis:

$$\vdash_{FS1} A_i = B_i \qquad (i = 1,2,\dots,n-1).$$

Hence, by (1) and (2), $\vdash_{FS1} A_n = B_n$. □

5.4. <u>MAIN THEOREM</u>. *Let* $A \in EXP$, $B \in EXP$. *Then*:

(i) $\models A = B \Longleftrightarrow \vdash_{FS1} A = B$

(ii) $\models A = B$ *is decidable*.

<u>PROOF</u>. (i) Immediate from Lemma 5.2, Lemma 5.3 and Theorem 3.4.5.
(ii) Evident, since the proof of Lemma 5.2 provides also an algorithm for deciding the equivalence of two expressions. □

We conclude the paper with an example to demonstrate how semantical equivalence of two expressions can be decided.

5.5. <u>EXAMPLE</u>. Let $A \equiv (\{p+c\}d * k)n$ and $B \equiv (\{\{p\}+\{c\}\}d * k)n$. Check if A and B are equivalent or not.

$A \equiv A_1 =$ (by Proposition 4.2)
$(d * k)n +$
$(p\{p+c\}d * k)n +$
$(c\{p+c\}d * k)n$
$=$ (by Definition 4.3)
$(d \vee k)(\varepsilon \wedge \varepsilon)n +$
$(p \vee k)(\{p+c\}d \wedge \varepsilon)n$
$(c \vee k)(\{p+c\}d \wedge \varepsilon)n$
$= (d \vee k)A_2 +$
$(p \vee k)A_3 +$
$(c \vee k)A_4$

$A_2 = (\varepsilon \wedge \varepsilon)\varepsilon n$
$= (\varepsilon \wedge \varepsilon)A_5$

$A_3 \equiv (\{p+c\})d \wedge \varepsilon)n$
$=$ (by Proposition 4.2)
$(d \wedge \varepsilon)n +$
$(p\{p+c\}d \wedge \varepsilon)n +$
$(c\{p+c\}d \wedge \varepsilon)n$
$=$ (by Definition 4.3)
$(d \wedge \varepsilon)\varepsilon n +$
$(p \mid \varepsilon)(\{p+c\}d \wedge \varepsilon)n +$
$(c \mid \varepsilon)(\{p\ +\}d \wedge \varepsilon)n$

$B \equiv B_1 =$ (by Proposition 4.2 and Ax 1)
$(d * k)n +$
$(p\{p\}\{\{p\}+\{d\}\}d * k)n +$
$(c\{c\}\{\{p\}+\{c\}\}d * k)n$
$=$ (by Definition 4.3)
$(d \vee k)(\varepsilon \wedge \varepsilon)n +$
$(p \vee k)(\{p\}\{\{p\}+\{c\}\}d \wedge \varepsilon)n +$
$(c \vee k)(\{c\}\{\{p\}+\{c\}\}d \wedge k)n$
$= (d \vee k)B_2 +$
$(p \vee k)B_3 +$
$(c \vee k)B_4$

$B_2 = (\varepsilon \wedge \varepsilon)\varepsilon n$
$= (\varepsilon \wedge \varepsilon)B_5$

$B_3 \equiv (\{p\}\{\{p\}+\{c\}\}d \wedge \varepsilon)n$
$=$ (by Proposition 4.2 and Ax 1)
$(d \wedge \varepsilon)n +$
$(p\{p\}\{\{p\}+\{c\}\}d \wedge \varepsilon)n +$
$(c\{c\}\{\{p\}+\{c\}\}d \wedge \varepsilon)n$
$=$ (by Definition 4.3)
$(d \wedge \varepsilon)\varepsilon n +$
$(p \mid \varepsilon)(\{p\}\{\{p\}+\{c\}\}d \wedge \varepsilon)n +$
$(c \mid \varepsilon)(\{c\}\{\{p\}+\{c\}\}d \wedge \varepsilon)n$

Since the pairs of expressions εn and εn, $(\{p+c\}d \wedge \varepsilon)n$ and $(\{p\}\{\{p\}+\{c\}\}d \wedge \varepsilon)n$, $(\{p+c\}d \wedge \varepsilon)n$ and $(\{c\}\{\{p\}+\{c\}\}d \wedge \varepsilon)n$, are syntactically equivalent to the earlier obtained pairs A_5 and B_5, A_3 and B_3, A_4 and B_4, we have:

$$
\begin{aligned}
A_3 \quad &= (d \wedge \varepsilon)A_5 + \\
&\quad (p \mid \varepsilon)A_3 + \\
&\quad (c \mid \varepsilon)A_4 \\
A_4 \quad &\equiv (\{p+c\}d \wedge \varepsilon)n \\
&= \text{(by Proposition 4.2)} \\
&\quad (d \wedge \varepsilon)n + \\
&\quad (p\{p+c\}d \wedge \varepsilon)n + \\
&\quad (c\{p+c\}d \wedge \varepsilon)n \\
&= \text{(by Definition 4.3)} \\
&\quad (d \wedge \varepsilon)\varepsilon n + \\
&\quad (p \mid \varepsilon)(\{p+c\}d \wedge \varepsilon)n + \\
&\quad (c \mid \varepsilon)(\{p+c\}d \wedge \varepsilon)n
\end{aligned}
$$

$$
\begin{aligned}
B_3 \quad &= (d \wedge \varepsilon)B_5 + \\
&\quad (p \mid \varepsilon)B_3 + \\
&\quad (c \mid \varepsilon)B_4 \\
B_4 \quad &\equiv (\{c\}\{\{p\}+\{c\}\}d \wedge k)n \\
&= \text{(by Proposition 4.2 and Ax1)} \\
&\quad (d \wedge \varepsilon)n + \\
&\quad (p\{p\}\{\{p\}+\{c\}\}d \wedge \varepsilon)n + \\
&\quad (c\{c\}\{\{p\}+\{c\}\}d \wedge \varepsilon)n \\
&\quad \text{(by Definition 4.3)} \\
&\quad (d \wedge \varepsilon)\varepsilon n + \\
&\quad (p \mid \varepsilon)(\{p\}\{\{p\}+\{c\}\}d \wedge \varepsilon)n + \\
&\quad (c \mid \varepsilon)(\{c\}\{\{c\}+\{c\}\}d \wedge \varepsilon)n.
\end{aligned}
$$

Again, since all pairs of suffixes are syntactically equivalent to the earlier obtained pairs of expressions, we have:

$$
\begin{aligned}
A_4 \quad &= (d \wedge \varepsilon)A_5 + \\
&\quad (p \mid \varepsilon)A_3 + \\
&\quad (c \mid \varepsilon)A_4 \\
A_5 \quad &\equiv \varepsilon n = \text{(by Definition 4.3)} \\
&\quad n\varepsilon = nA_6 \\
A_6 \quad &\equiv \varepsilon
\end{aligned}
$$

$$
\begin{aligned}
B_4 \quad &= (d \wedge \varepsilon)B_5 + \\
&\quad (p \mid \varepsilon)B_3 + \\
&\quad (c \mid \varepsilon)B_4 \\
B_5 \quad &\equiv \varepsilon n = \text{(by Definition 4.3)} \\
&\quad n\varepsilon = nB_6 \\
B_6 \quad &\equiv \varepsilon
\end{aligned}
$$

Thus $\vdash_{FS1} A = B$, and hence $\models A = B$.

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