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POTENTIAL FLOW AROUND A BODY OF REVOLUTION.

by

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Part 1, Text.

POTENTIAL FLOW AROUND A BODY OF REVOLUTION.

1. The computation of the potential flow around a body of revolution is of some importance in practical hydro- and aerodynamics. A number of bodies are actually practically rotationally symmetric, e.g. airships, shells, torpedos, pitot-tubes, and so on. Other bodies are not, but as a first approximation they may often be treated as such as several authors have done. e.g. hulls of ships. In many cases, especially in those of elongated bodies, one can get a satisfactory picture of the actual - nonpotential - flow around the body by first computing the potential flow and then solving the boudary layer equations using the volocity distribution of the potential flow, and if necessary repeating the process after correction of the shape of the body on the basis of the displacement-thickness of the boundary layer.

Therefore, several authors have treated the computation of the velocity distribution in the case of potential flow. Of special importance is the method of von Kármán. Our purpose in this communication is to give a refinement to his method, and at the same time to procure a set of tables that may lighten the burden of computation considerably.

2. One way to compute the field of the potential flow is to solve the boundary value problem belonging to the partial differential equation that governs the flow, for instance, by means of relaxation methods Whereas result might be achieved in the case that the main velocity is parallel with the axis of revolution, this method becomes distinctly unattractive in the case that the flow comes in under a certain angle with the axis.

Another method is to insert in the field of the main stream a set of singularities (sources and sinks, and doublets) that distort the main flow in such a way that it surrounds an isolated part of the space of the same shape as that of the body of revolution that we consider. If the shape of the body is rather smooth, we can restrict ourselves to singularities along the axis (and, of course, only within the body).

As it is obvious that by superposition of two types of flow, i.e. in which the velocity at infinity is either parallel or perpendicular to the axis, we can obtain a field in which the velocity at infinity makes an arbitrary angle with the axis, it suffices to consider the two fields seperately. Von Kármán has shown that a distribution of sources and sinks in the first case and of doublets with their axes in the direction of the perpendicular flow in the second case indeed give rise to flow around a body of revolution. The problem is only to specify the distribution of the intensity of the sources or doublets so as to get the body with the desired shape.

3. Let us first consider the case of zero angle of incidence. The coordinates are x and r in the directions of the axis and radially respectively. Besides, we use polar coordiantes ρ, ϑ. Let the origin be at some place x_0 of the axis, then they are defined by

$$\rho^2 = (x-x_0)^2 + r^2 \quad \text{and} \quad \tan\vartheta = r/(x-x_0) \qquad (3,1)$$

The velocity be u, with components u_x and u_r. At infinity holds $u_x = V$ and $u_r = 0$, where V is the velocity of the undisturbed main stream. These velocities can be derived from a streamfunction by means of the formulae

This streamfunction consists of two parts, i.e. that belonging to the main stream and that to the system of sources. The first part is apparently $\frac{1}{2}Vr^2$. For an isolated source of intensity Q at $(x_0, 0)$ is the streamfunction

$$-\frac{Q}{4\pi}(1 + \cos\vartheta) = -\frac{Q}{4\pi}\left\{1 + \frac{x-x_0}{\sqrt{\{(x-x_0)^2 + r^2\}}}\right\} \qquad (3.2)$$

Now let us consider a section of the field made by a plane through the axis. The axis is a line along which $\Psi = 0$, and so is the contourline of the body. Also holds that at the contour the velocity is tangential to the contour. So, if a point of the contour is given as (x_c, r_c) then we can define the unknown intensity $f(x_o)$ of the sources by requiring that at each point (x_c, r_c) holds $\Psi = 0$ or $u_r/u_x = dr_c/dx_c$.
From the first observation we follow that

$$\tfrac{1}{2}\, V r_c^2 = \frac{1}{4\pi} \int_0^{\ell} f(x_o) \left\{ 1 + \frac{x_c - x_o}{\sqrt{\{(x_c - x_o)^2 + r_c^2\}}} \right\} dx_o \qquad (3,3)$$

If the length of the body is finite then the total strength of the sources vanishes, so $\int_0^{\ell} f(x_o)\, dx_o = 0$. So the integralequation simplifies into

$$2\pi V r_c^2 = \int_0^{\ell} f(x_o)(x_c - x_o) \left\{ (x_c - x_o)^2 + r_c^2 \right\}^{-1/2} dx_o \qquad (3.4)$$

Integration by parts gives if we define $g(x_o) = \int_0^{x_o} f(x)dx$, sothat apparently $g(0) = g(l) = 0$,

$$2\pi V r_c^2 = r_c^2 \int_0^{\ell} g(x_o) \left\{ f(x_c - x_o)^2 + r_c^2 \right\}^{-3/2} dx_o \qquad (3.5)$$

For all points except nose and tail is $r_c \neq 0$, so that

$$2\pi V = \int_0^{\ell} g(x_o) \left\{ (x_c - x_o)^2 + r_c^2 \right\}^{-3/2} dx_o \qquad (3.6)$$

If both r_c and dr_c/dx_c are very small one can derive an approximate solution by remarking that the integrand is then very large in the vicinity of $x_o = x_c$, so that this vicinity practically accounts for the complete value of the integral. If therefore $dg(x)/dx$ is small, we can replace in the integrand $g(x)$ by $g(x_c)$ so that

$$2\pi V = g(x_c) \int_0^{\ell} \left\{ (x_c - x_o)^2 + r^2 \right\}^{-3/2} dx_o \qquad (3.7)$$

or

$$g(x_c) = 2\pi V r_c^2 \left\{ \frac{\ell - x_c}{\sqrt{\{(\ell - x_c)^2 + r_c^2\}}} + \frac{x_c}{\sqrt{\{x_c^2 + r_c^2\}}} \right\}^{-1} \qquad (3.8)$$

For points not situated in the neighbourhood of nose or tail, the two terms within the braces are practically equal to unity, so that roughly holds $\quad g(x_c) \backsim \pi V r_c^2 \qquad (3.9)$

We see, by the way, that $dg(x)/dx$ is indeed small, because of the assumption made for r_c and dr_c/dx_c. This approximate solution is rather crude. In the case of a blunt body it fails completely and even in the case of a very slender body it shows some essential defects. If, for instance, we consider a very, say infinitely long body with a nose that is conical over some length then we find from (3.8) that α/x_c, equals some constant times r_c^2, so that the source-strength varies linearly with r_c and therefore with x_c. This is, however, impossible because it would give rise to a velocity that is infinite at $x_c = 0$ so that the nose could not be there.

4. A practical method of solving the integralequation as given by von Kármán is to replace the function f(x) by a discontinuous one, having a constant value over intervals of length a. This means that we consider a set of n independent sources of strength Q_i, each distributed uniformly over the length a, and situated side by side along the axis. If one takes n points on the contour, e.g. with the same absissae as those of the centers of the sources, one can compute the value of the streamfunction in these points expressed linearly in the unknowns Q_i and the main streamfunction, and by requiring this streamfunction to vanish we find n equations which we can solve for Q_i. These equations, moreover, are of a type that can be easily solved by iteration, so that we have a very efficient way of getting an approximate solution to our problem. To increase the accuracy we can only increase n, of course, at the cost of much more computational labour. For doubling n means first quadrupling the number of coefficients in the equations that have to be computed and roughly eight times as much work to solve the equations.

The refinement that we suggest now consists in placing again separate sources Q_i at a distance a but not of constant strength over a width a but with an intensity that is Q_i/a in the center and drops linearly towards both sides to become zero at a distance $\pm$ a from the center. Therefore the total width of these sources is 2a and they overlap one another. The superposition of these sources now gives automatically a continuous curve made up by straight lines, instead of a discontinuous curve. If there is a continuous solution to the integralequation then our solution is most likely an order more accurate than von Kármán's. If there is a discontinuous solution to the equation then the chance that the discontinuities of this solution and the artificial discontinuities of von Kármán's solution coincide is negligible as we have no indication where they should be. If, at last, there is no solution at all, our version is not worse than anything else. Apart from the computation of the coefficients of the equations there is not any difference between von Kármán's original method and our version of it.

5. The computation of the streamfunction of our "triangular" sources runs as follows. With the notations of fig. 1 the source-strength is distributed according to the law:

n equations

$$\sum_{i=1}^{n} C_{ik}\, z_i = \left(\frac{r_k}{a}\right)^2 \qquad (6.2)$$

This is the set of n equations which we can solve for z_i.

Of course we are not bound to use exactly n points on the contour above the sources. Indeed it is good policy in many cases to leave some points of this group out of consideration and te replace the missing conditions by other ones.

For instance, one might require the nose and the tail to be at the right spot. This cannot be done by means of the streamfunction as this function automatically vanishes on points of the axis. The condition is, however that the velocity in the axial direction vanishes, so

$$V + \sum_{i=1}^{n} \frac{Q_i}{2\pi a^2} u_{xin} = 0 \qquad (6.3)$$

or dimensionless

$$\sum_{i=1}^{n} u_{xin}\, z_i = 1 \qquad (6.4)$$

On the other hand, if the body extends at one side practically into infinity, that means that it passes into a cylinder, we know the total source-strength that must exist so that this cylinder has the right radius r_∞. This gives the relation

$$\frac{1}{2} Vr^2 - \sum_{i=1}^{n} \frac{Q_i}{2\pi} = 0. \qquad (6.5)$$

or dimensionless

$$2 \sum_{i=1}^{n} z_i = \left(\frac{r_\infty}{a}\right)^2 \qquad (6.6)$$

It is advisable to use these two relations, that is either (6.4) for both nose and tail or (6.4) for the nose and (6.6) for the parallel body. Otherwise one may get curious results. For instance, one might try to represent a sphere and find oneself left with a lemon-shaped body. If the body passes into a long cylinder one should use (6,6) and restrict oneself to a system of sources in that part of the body where the radius still varies appreciably. If one tries to raise the accuracy by putting sources a way up into the cylindrical body one may find for this tail-end of the source-distribution values of z_i of nearly constant amplitude but with alternating sign, what does not raise the accuracy but does raise the burden of computation.

Further one should not put the first source too near to the nose. By lack of better we advise to leave one source out and to put the first

source-center at a distance 2a from the nose. Apart from that, if we assume a source to be at a distance a from the nose and apply (6.4) then the unknown automatically vanishes, so we might as well leave it out directly. The same holds of course for the tail.
When we have found the quantities z_i, the components of the velocity can be determined from

$$u_x = V(1 + \sum_{i=1}^{n} u_{xik} z_i) \tag{6.7}$$

$$u_r = V \sum_{i=1}^{n} u_{rik} z_i \tag{6.8}$$

From this we can get the resulting velocity and the pressure distribution.

7. Next we give a quick survey of the corresponding problems in perpendicular flow. At infinity the velocity be W in the direction of the z-axis that is perpendicular to the x-axis. We assume a set of doublets with their axes also in the z-direction to be present at the axis of the body. We extend the polar coordinates also with a third one, i.e. φ that determines the meridional sections passing through the axis. The plane $\varphi = 0$ contains the x- and z-axes. In these coordinates the potential Φ of the doublet is (cf. von Kármán)

$$\Phi = -\frac{M}{4\pi} \frac{\sin\vartheta \cos\varphi}{\rho^2} \tag{7.1}$$

where M is the moment of the doublet. From this the components of the velocity may be computed from the relations

$$u_x = \frac{\partial\Phi}{\partial x}, \quad u_r = \frac{\partial\Phi}{\partial r}, \quad u_\varphi = \frac{1}{r}\frac{\partial\Phi}{\partial\varphi} \tag{7.2}$$

The analysis of von Kármán shows that such a system of doublets distorts the main stream in such a way that it surrounds a body of revolution. It is, therefore, sufficient for the determination of the intensities of the doublets along the axis to consider only the plane $\varphi = 0$ and to require there that the flow follows the contour.
We introduce again "Triangular" doublets, the intensity m of which again varies as

$$\begin{aligned} m &= \frac{M}{a}\left(1 - \left|\frac{\xi}{a}\right|\right) && \text{for } |\xi| < a \\ m &= 0 && \text{for } |\xi| > a \end{aligned} \tag{7.3}$$

The same process as before gives us

$$\Phi = -\frac{M\cos\varphi}{4\pi a^2} \delta_x^2 \frac{1}{\sin\vartheta} \tag{7.4}$$

From this we derive the components of the velocity

$$u_x = -\frac{M\cos\varphi}{4\pi a^3}\left(\frac{a}{r}\right)\delta_x^2\cos\vartheta = \frac{M\cos\varphi}{4\pi a^3}\,u'_{xik} \qquad (7.5)$$

$$u_r = \frac{M\cos\varphi}{4\pi a^3}\left(\frac{a}{r}\right)\delta_x^2\frac{\cos\vartheta}{\tan\vartheta} = \frac{M\cos\varphi}{4\pi a^3}\,u'_{rik} \qquad (7.6)$$

$$u_\varphi = \frac{M\sin\varphi}{4\pi a^3}\left(\frac{a}{r}\right)\delta_x^2\frac{1}{\sin\vartheta} = \frac{M\sin}{4\pi a^3}\,u'_{\varphi ik} \qquad (7.7)$$

the requirement that the flow follows the contour can be expressed as

$$\frac{W + \sum_{i=1}^{n}\frac{M_i}{4\pi a^3}u'_{rik}}{\sum_{i=1}^{n}\frac{M_i}{4\pi a^3}u'_{xik}} = \frac{dr_k}{dx_k} \qquad (7.8)$$

or if we put $z'_i = \frac{-M_i}{4\pi Wa^3}$: $\sum_{i=1}^{n}\left\{\frac{dr_k}{dx_k}u'_{xik} - u'_{rik}\right\} z'_i = 1$ (7.9)

Therefore, we get again a set of n linear equations in the unknowns z'_i. When we have determined the z'_i, we find the velocity at an arbitrary point of the surface from

$$u_x = -W\cos\varphi\sum_{i=1}^{n}u'_{xik}\,z'_i \qquad (7.10)$$

$$u_r = W\cos\varphi\left(1 - \sum_{i=1}^{n}u'_{rik}\,z'_i\right) \qquad (7.11)$$

$$u_\varphi = -W\sin\varphi\left(1 + \sum_{i=1}^{n}u'_{\varphi ik}\,z'_i\right) \qquad (7.12)$$

8. The actual calculation of the quantities c_{ik}, u_{xik}, and so on is not always easy in certain parts of the field, i.e. when the function which we have to form the second difference is very large, whereas the difference that we want to know may be small. It is, however possible to convert these expressions in many ways in second differences of other functions that are more manageable.

In practice we have only to do with values of x_i and of x_k that are multiples of a, say ia and ka. All the quantities now only depend on the ratio r_k/a and k-i. We have constructed a set of tables for them, ranging for r_k/a from 0 to 2 and for k-i from 0 to 9. In these tables one can read directly the quantities that one needs.

For negative values of k-i the following relations hold:

c_{ik}	=	2 - corresponding value for positive (k-i)
u_{xik}, u'_{xik}	=	- " " " " "
u_{rik}, u'_{rik}, u'_{ik}	=	" " " " "

In the construction of these tables continuous use has been made of the transformations mentioned above. So is, for instance, the expression $\delta_x^2 \ln \tan^2 \vartheta/2$ cumbersome for small r/a and large k-i as then ϑ is small. But if we use the identity

$$2\,\delta_x^2 \ln \tan^2 \vartheta/2 = \delta_x^2 \ln \cos \vartheta/(1+\cos \vartheta) - \ln \{1-1/(k-i)^2\} \qquad (8.1)$$

we get an expression for it without these troubles, but that does not work for k-i = 0 or 1.

Another identity is

$$\delta_x^2 \, \frac{1}{\sin \vartheta} = \delta_x^2 \tan \vartheta/2 \qquad (8.2)$$

from which we learn that the second difference that appears in the formula for u_{rik} is the same as that appears in the formula for $u'_{\varphi ik}$.

9. The work of von Kármán that is often mentioned in this communication is:

Th. v. Kármán, Berechnung der Druckverteilung an Luftschiffkörpern. Abhandlungen aus dem Aerodynamischen Institut an der Technischen Hochschule Aachen, Heft 6, 1927.