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Leiding: Prof. Dr Ir A. van Wijngaarden

SOME THEOREMS ON ASYMPTOTIC EXPANSIONS AND DOUBLE INTEGRALS

by

J. Berghuis.

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Introduction.

The author has the intention to write his thesis on the subject "The calculation of asymptotic residus for double integrals". He therefore needs a number of theorems such as theorems on the expansion of simple integrals, theorems on the existence and the transformation of double integrals, theorems on the solutions of some equations, etc. These theorems are collected in this paper. Numbers in square brackets refer to the list of references at the end.

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CHAPTER I

Some Expansions of Simple Integrals.

1. Definitions.

The expression "the function $\varphi(x)$ can be expanded in ascending powers of x for small positive values of x" means, that for small positive values of x the function $\varphi(x)$ can be represented by a series of the form

$$\varphi(x)=\sum_{h=0}^{\infty} B_h\, x^{b_h}, \tag{1.1}$$

where the exponents b_h are mutually different real numbers, increasing indefinitely with h. If $x \to 0$ the function $\varphi(x)$ tends to zero if and only if all exponents b_h are positive.

The expression "the function $\varphi(x)$ can be expanded in the interval (a,a') in the neighbourhood of the point $x = a$ "means,

A1. if $\underline{a}$ is finite and if $a' > a$: for small positive values of $x-a$, the function $\varphi(x)$ can be expanded in ascending powers of $x-a$;

A2. if $\underline{a}$ is finite and if $a' < a$: for small positive values of $a-x$ the functions $\varphi(x)$ can be expanded in ascending powers of $a-x$;

B. if $a = \infty$ and if $a' < a$: for small positive values of x^{-1} the function $\varphi(x)$ can be expanded in ascending powers of x^{-1};

C. if $a = -\infty$ and if $a' > a$: for small positive values of $(-x)^{-1}$ the function $\varphi(x)$ can be expanded in ascending powers of $(-x)^{-1}$;

D. if $a' = a$, the expression does not contain an assertion about the function $\varphi(x)$, whatsoever.

If $\varphi(x)$ does not only depend of x, but also of another parameter y and if the relation (1.1) holds uniformly in y, one says that for small positive values of x the function $\varphi(x)$ can be expanded uniformly in y.

The expression "the function $\varphi(x)$ can be expanded asymptotically in ascending powers of x for small positive values of x" means, that it is possible to find a formal series

$$\sum_{h=0}^{\infty} B_h\, x^{b_h},$$

where the coefficients B_h and the exponents b_h are independent of x and the exponents b_h are mutually different real numbers, increasing indefinitely with h, such that for each fixed integer $H \geqslant 0$ the relation

$$\varphi(x) = \sum_{h=0}^{H-1} B_h x^{b_h} + O(x^{q_H}) \tag{1.2}$$

holds, where q_H ($H= 0,1,\ldots$) are suitably chosen real numbers, independent of x, increasing indefinitely with H.

The expression "the function $\varphi(x)$ can be expanded asymptotically in the interval (a,a') in the neighbourhood of $x = a$" means,

A1. if $\underline{a}$ is finite and if $a' > a$: for small positive values of $x-a$ the function $\varphi(x)$ can be expanded asymptotically in ascending powers of $x-a$;

A2. if $\underline{a}$ is finite and if $a' < a$: for small positive values of $a-x$ the function $\varphi(x)$ can be expanded asymptotically in ascending powers of $a-x$;

B. if $a = \infty$ and if $a' < a$: for small positive values of x^{-1} the function $\varphi(x)$ can be expanded asymptotically in ascending powers of x^{-1};

C. if $a = -\infty$ and if $a' > a$: for small positive values of $(-x)^{-1}$ the function $\varphi(x)$ can be expanded asymptotically in ascending powers of $(-x)^{-1}$;

D. if $a = a'$: the expression does not contain an assertion about the function $\varphi(x)$, whatsoever.

If $\varphi(x)$ does not only depend of x, but also of an other parameter y, and the relations (1.2) hold uniformly in y, one says that for small positive values of x the function $\varphi(x)$ can be expanded asymptotically uniformly in y.

2. Some simple properties.

Theorem 1. Suppose that for small positive values of x the function $\varphi(x)$ can be expanded asymptotically in ascending powers of x so that $\varphi(x)$ can be represented asymptotically by the formal series

$$\sum_{h=0}^{\infty} B_h x^{b_h};$$

One assumes the exponents b_h to increase steadily. Then it is always possible to choose the exponent q_H used in formula (1.2) equal to b_H.

Proof: Using (1.2) with $H = K$ one finds

$$\varphi(x) = \sum_{h=0}^{K-1} B_h x^{b_h} + O(x^{q_K}).$$

Since q_K increases indefinitely with K, one can choose K so large that $q_K \geqslant b_H$.

From
$$B_h\, x^{b_h} = O(x^{b_H}) \qquad (H \leqslant h < K)$$

it follows that
$$\varphi(x) = \sum_{h=0}^{H-1} B_h\, x^{b_h} + O(x^{b_H}).$$

An analogous theorem holds if $\varphi(x)$ can be expanded in ascending powers of x for small positive values of x; this theorem is not given here.

Theorem 2: Be β a fixed real number;be, for small positive values of u, $\xi(u)$ a real function that tends to a limit α if u tends to zero, that, for small positive values of u, is either continually $> \alpha$ or continually $< \alpha$.

For each small positive value of u and for each x of the closed interval $(\xi(u), \beta)$ the function $\varphi(x,u)$ is supposed to be expandable asymptotically in ascending powers of u uniformly in x.

It is assumed, further, that it is possible to find a positive number a with the following property: if α is finite, then $u^a(\xi(u)-\alpha)$ tends to a finite limit $\neq 0$ if $u \to 0$; if α is infinite, then $u^{-a}\,\xi(u)$ tends to a finite limit $\neq 0$ if $u \to 0$.

Finally it is assumed that one can find real numbers b_h $(h = 0,1,2,\ldots)$ with the following property:

if α is finite, the function $\varphi_h(x)(x-\alpha)^{b_h}$ tends to a finite limit $\neq 0$, if x tends to α from the same side as $\xi(u)$ does.

if α is infinite, the function $\varphi_h(x)\,.\,x^{-b_h}$ tends to a finite limit $\neq 0$, if x tends to α from the same side as $\xi(u)$ does.

Under these conditions $ab_h + c_h$ increases indefinitely if h increases indefinitely.

Proof: For each x of the interval $(\xi(u), \beta)$ holds uniformly in x:
$$\varphi(x,u) = \sum_{h=0}^{H-1} u^{c_h}\,\varphi_h(x) + O(u^{q_H})$$

and
$$\varphi(x,u) = \sum_{h=0}^{H} u^{c_h}\,\varphi_h(x) + O(u^{q_{H+1}}),$$

where the exponents q_H increase indefinitely with H and $q_{H+1} \geqslant q_H$. Hence for each x of the interval $(\xi(u), \beta)$ holds uniformly in x
$$u^{c_H}\,\varphi_H(x) = O(u^{q_H}),$$

or with $x = \xi(u)$
$$u^{c_H}\,\varphi_H(\xi(u)) = O(u^{q_H}).$$

In the case that α is finite, one has

$$|\xi(u)-\alpha|^{-b_H} = O(\varphi_H(x))$$

and

$$u^{-a} = O(\xi(u)-\alpha).$$

So one finds

$$u^{ab_H+c_H} = O(|\xi(u)-\alpha|^{-b_H}\, u^{c_H})$$

$$= O(\varphi_H(\xi(u))\, u^{c_H})$$

$$= O(u^{q_H}).$$

From this one deduces $q_H \leqslant a\, b_H + c_H$, so that $ab_H + c_H$ increases indefinitely with H.

The proof is analogous in the case that α is infinite.

3. Expansion of an integral.

Theorem 3. Suppose that the integral

$$J(u)= \int_{\xi(u)}^{\eta(u)} \varphi(x)dx$$

exists for small positive values of u. It is assumed that $\xi(u)$ and $\eta(u)$ are real functions of u such that $\eta(u) \geqslant \xi(u)$, and that these functions can be expanded asymptotically in ascending powers of u for small positive values of u, and that $\xi(u)$ resp. $\eta(u)$ tend to the limit α resp. β if $u \longrightarrow 0$. Finally it is assumed that $\varphi(x)$ can be expanded asymptotically in the interval $(\xi(u),\alpha)$ in the neighbourhood of $x = \alpha$, and also that $\varphi(x)$ can be expanded asymptotically in the interval $(\eta(u),\beta)$ in the neighbourhood of $x = \beta$.

Under these conditions there exists a constant C such that $J(u) - C \log u$ can be expanded asymptotically for small positive values of u.

Remark: Under these conditions there exists, therefore, a real constant κ such that

$$J(u)= O(u^{-\kappa})$$

for small positive values of u.

Proof: In the case that α is finite, the function $\varphi(x)$ possesses a formal expansion of the form:

$$\sum_{h=0}^{\infty} B_h\, |x-\alpha|^{b_h}, \tag{3.1}$$

where $|x-\alpha| = x-\alpha$ if $\xi(u) \geqslant \alpha$, and

$$|x-\alpha| = \alpha - x \text{ if } \xi(u) \leq \alpha .$$

Be

$$\psi(x) = \sum_h{}' \; B_h |x-\alpha|^{b_h},$$

where the dash denotes that the summation is extended only over those values of the index h, for which $b_h < -1$.

A number B is defined as follows:

if in the expansions (3.1) a term occurs with exponent $b_h = -1$, then B is equal to the corresponding coefficient B_h; if such a term does not occur, then B is equal to zero.

Finally the function r(x) is defined by

$$\varphi(x) = \psi(x) + B\,|x-\alpha|^{-1} + r(x),$$

so that for small positive values of $|x-\alpha|$ the function r(x) can be expanded asymptotically.

In the case that α is infinite, the function $\varphi(x)$ has a formal expansion of the form

$$\sum_{h=0}^{\infty} B_h |x|^{-b_h}, \tag{3.2}$$

where $|x| = x$ if $\alpha = \infty$, and $|x| = -x$ if $\alpha = -\infty$.

Be

$$\psi(x) = \sum_h{}' \; B_h |x|^{-b_h},$$

where the dash denotes, that the summation is extended only over those values of the index h, for which $b_h < 1$.

A number B is defined as follows:

If in the expansion (3.2) a term occurs with exponent $b_h = 1$, then B is equal to the corresponding coefficient B_h; if such a term does not occur, then B is equal to zero.

Finally the function r(x) is defined by

$$\varphi(x) = \psi(x) + B|x|^{-1} + r(x),$$

so that for small positive values of $|x|^{-1}$ the function r(x) can be expanded asymptotically in ascending powers of $|x|^{-1}$.

If α is finite and if $\xi(u)$ is not identically equal to α, then there exists an uniquely determined positive number <u>a</u> such that for small positive values of u the function $u^{-a}(\alpha - \xi(u))$ can be expanded asymptotically in ascending powers of u. In this expansion all exponents are ≥ 0, and the first term is equal to a constant $\neq 0$.

If α is infinite and if $\xi(u)$ is not identically equal to α, then there exists an uniquely determined positive number <u>a</u> such that for small positive values of u the function $u^a \; \xi(u)$ can be expanded asymptotically in ascending powers of u. In this expansion all exponents are ≥ 0, and the first term is equal to a constant $\neq 0$.

In the proof the following cases are distinguished:
I: It is possible to determine a finite fixed number A, such that for sufficiently small values of u the number A lies in the interval $(\xi(u),\eta(u))$. In that case one can split up J(u) as follows:

$$J(u)=\int_{\xi(u)}^{A}\varphi(x)\,dx+\int_{A}^{\eta(u)}\varphi(x)dx.$$

The demonstration is quite the same for both integrals and is given only for the first one. Three subcases are distinguished.
Ia. Be $\alpha = A$. Since

$$\int_{\xi(u)}^{\alpha}\varphi(x)dx=\int_{\xi(u)}^{A}\varphi(x)dx$$

exists, there do not occur in the expansion of $\varphi(x)$ in ascending powers of α -x, terms with exponents ≤ -1, so that

$$\varphi(x)=\sum_{h=0}^{H-1}B_h(\alpha-x)^{b_h}+O((\alpha-x)^{b_H}),$$

where each exponent $b_h > -1$. Consequently

$$\int_{\xi(u)}^{\alpha}\varphi(x)dx=-\sum_{h=0}^{H-1}\frac{B_h}{b_h+1}(\alpha-\xi(u))^{b_h+1}+O((\alpha-\xi(u))^{b_H+1}).$$

From the fact that the function $u^{-a}(\alpha-\xi(u))$ in which the constant a is positive, can be expanded asymptotically in ascending powers of u, for small positive values of u, the same property follows for the integral in question.
Ib. Be $-\infty<\alpha<A$. One has

$$\int_{\xi(u)}^{A}\varphi(x)dx=\int_{\xi(u)}^{A}\psi(x)dx+B\int_{\xi(u)}^{A}(\alpha-x)^{-1}dx+\int_{\alpha}^{A}r(x)dx-\int_{\alpha}^{\xi(u)}r(x)dx$$

$$=\sum{}'\frac{B_h}{b_h+1}(A-\alpha)^{b_h+1}+B\log(A-\alpha)+\int_{\alpha}^{A}r(x)dx$$

$$-\sum{}'\frac{B_h}{b_h+1}(\xi(u)-\alpha)^{b_h+1}-B\log(\xi(u)-\alpha)-\int_{\alpha}^{\xi(u)}r(x)dx.$$

The first three terms of the right hand side are independent of u; in the same way as above one sees that after addition of aBlog u, the remaining terms can be expanded asymptotically for small positive values of u.
Ic. Be $\alpha= -\infty$. Without loss of generality one may suppose A to be negative, for if A is not negative, one can consider the integral

$$\int_{A'}^{A}\varphi(x)dx,$$

where A' denotes a finite negative number, as a constant independent of u.

Again one has

$$\int_{\xi(u)}^{A} \varphi(x)dx = \int_{\xi(u)}^{A} \psi(x)dx + B\int_{\xi(u)}^{A} |x|^{-1}\, dx + \int_{-\infty}^{A} r(x)dx - \int_{-\infty}^{\xi(u)} r(x)dx$$

$$= -\sum{}' \frac{B_h}{-b_h+1} A^{-b_h+1} - B\log|A| + \int_{-\infty}^{A} r(x)dx + \sum{}' \frac{B_h}{-b_h+1} (\xi(u))^{-b_h+1}$$

$$+ B\log|\xi(u)| - \int_{-\infty}^{\xi(u)} r(x)dx.$$

If it is proved that for small positive values of u the integral $\int_{-\infty}^{\xi(u)} r(x)dx$ can be expanded asymptotically in ascending powers of u, then it is proved that the right hand side can be written as the sum of -a B log u and a function that can be expanded asymptotically for small positive values of u.

The function r(x) can be written in the form

$$r(x) = \sum_{h}^{H-1} B_h |x|^{-b_h} + O(|x|^{-b_H}),$$

where the summation is extended over those values of h for which $b_h > 1$, and where b_H increases indefinitely with H. So

$$\int_{-\infty}^{\xi(u)} r(x)dx = -\sum_{h}^{H-1} B_h(\xi(u))^{-b_h+1} + O(\{\xi(u)\}^{-b_H+1}),$$

and by this the desired result is obtained, since $u^a\varphi(u)$ can be expanded asymptotically for small positive values of u in ascending powers of u, where the first term is a constant.

<u>II</u>. It is not possible to determine a finite fixed number A such that for sufficiently small positive values of u the number A lies in the interval $(\xi(u), \eta(u))$. Hence $\alpha = \beta$.

Moreover, one knows:

if α is finite then either $\alpha \leq \xi(u) \leq \eta(u)$ or $\alpha \geq \xi(u) \geq \eta(u)$ holds and further

if $\alpha = -\infty$ then $\alpha \geq \xi(u) \geq \eta(u)$ holds and finally

if $\alpha = \infty$ then $\alpha \leq \xi(u) \leq \eta(u)$ holds.

The proof can be given in the same way as above.

4. <u>A generalisation</u>.

One can give a generalisation of the theorem mentioned in the preceding section:

<u>Theorem 4</u>. Suppose that for every small positive value of u the

integral

$$J(u)=\int_{\xi(u)}^{\eta(u)} \varphi(x,u)dx$$

exists. Suppose further that $\xi(u)$ and $\eta(u)$ are real functions for positive values of u that can be expanded asymptotically in ascending powers of u for small positive values of u. Their respective limits are indicated by α and β. For every x of the closed interval $(\xi(u),\eta(u))$ and for small positive values of u the function $\varphi(x,u)$ is supposed to be expanded asymptotically uniformly in x by means of

$$\varphi(x,u)\sim\sum_{h=0} u^{c_h} \varphi_h(x).$$

One assumes that the coefficients $\varphi_h(x)$ are integrable in the closed interval $(\xi(u),\eta(u))$ for small positive values of u and that the exponents c_h increase indefinitely with h.

Finally it is supposed that the functions $\varphi_h(x)$ can be expanded asymptotically in the interval $(\xi(u),\alpha)$ in the neighbourhood of $x = \alpha$ and also in the interval $(\eta(u), \beta)$ in the neighbourhood of $x = \beta$.

Assertion: i: If $\xi(u)$ and $\eta(u)$ are finite for small positive values of u, then one can write J(u) in the form

$$J_1(u) \log u + J_2(u),$$

where $J_1(u)$ and $J_2(u)$ can be expanded asymptotically for small positive values of u.

ii: The assertion above also holds in the case that $\eta(u)=\infty$ for small positive values of u, for every fixed integer $H \geqslant 0$ holds

$$\varphi(x,u)=\sum_{h=0}^{H-1} u^{c_h} \varphi_h(x) + O(u^{c_H} \Phi_H(x)),$$

where $\Phi_H(x)$ denotes a suitably chosen positive function of x that is integrable in the closed interval $(\xi(u),\infty)$.

iii: The assertion i, also holds in the case that $\xi(u)= -\infty$ for small positive values of u, if for every fixed integer $H \geqslant 0$ holds

$$\varphi(x,u)=\sum_{h=0}^{H-1} u^{c_h} \varphi_h(x)+ O(u^{c_H} \Phi_H(x)),$$

where $\Phi_H(x)$ denotes a suitably chosen positive function of x that is integrable in the closed interval $(-\infty, \eta(u))$.

iv. An analogous result holds if at the same time $\xi(u)=-\infty$ and $\eta(u)=\infty$.

Proof: The four assertions are proved as follows:

i: Uniformly in x holds

$$\varphi(x,u)=\sum_{h=0}^{H-1} u^{c_h}\varphi_h(x)+O(u^{q_H}).$$

Inserting this expansion into the integralrepresentation of $J(u)$, one gets

$$J(u)=\sum_{h=0}^{H-1} u^{c_h}\int_{\xi(u)}^{\eta(u)}\varphi_h(x)dx+\int_{\xi(u)}^{\eta(u)}O(u^{q_H})dx$$

For every $h \geqslant 0$ the integrals $\int_{\xi(u)}^{\eta(u)}\varphi_h(x)dx$ satisfy the conditions of theorem 3 and, therefore, they can be expanded asymptotically for small positive values of u with the exception of a logarithmical term. The remainder term can be written as

$$O\left\{|\eta(u)-\xi(u)|\; u^{q_H}\right\}.$$

It is always possible to determine a real number γ such that $u^{-\gamma}|\eta(u)-\xi(u)|$ tends to a finite limit $\neq 0$, if u tends to zero. The remainderterm is, therefore, of the same order as $u^{q_H-\gamma}$, and $q_H-\gamma$ increases indefinitely with H.

So $J(u)$ can be written in this case in the form

$$J_1(u)\log u+J_2(u),$$

where $J_1(u)$ and $J_2(u)$ can be expanded asymptotically for small positive values of u.

ii. If $\alpha<\infty$, one can split up $J(u)$ as follows:

$$J(u)=I(u)+K(u),\quad I(u)=\int_{\xi(u)}^{A}\varphi(x,u)dx,\quad K(u)=\int_{A}^{\infty}\varphi(x,u)dx,$$

where A denotes a finite real number independent of u such that for sufficiently small positive values of u $\xi(u)<A$. The assertion for $I(u)$ has been proved already under i. The assertion for $K(u)$ holds in virtue of:

$$K(u)=\sum_{h=0}^{H-1} u^{c_h}\int_{A}^{\infty}\varphi_h(x)dx+O(u^{c_H}\int_{A}^{\infty}\Phi_H(x)dx)$$

$$=\sum_{h=0}^{H-1} u^{c_h}\int_{A}^{\infty}\varphi_h(x)dx+O(u^{c_H}).$$

Therefore, the assertion holds also for $J(u)$ itself.

ii,b. If $\alpha=\infty$ the assertion follows from

$$J(u)=\sum_{h=0}^{H-1} u^{c_h}\int_{\xi(u)}^{\infty}\varphi_h(x)dx+O(u^{c_h}\int_{\xi(u)}^{\infty}\Phi_H(x)dx).$$

In the cases iii and iv the proof can be given in a way analogous to the one used for the second case.

5. A more complicated integral.

Finally there will be given an expansion of a more complicated integral. This expansion is needed for the calculation of asymptotic residues along critical lines.

Theorem 5: Let $J(u)$ be an integral of the form

$$J(u)=\int_{\xi(u)}^{\eta(u)} \chi\left(\frac{u}{x-\alpha}\right) \varphi(x)dx,$$

that exists for small positive values of u. Again $\xi(u)$ and $\eta(u)$ are real functions for positive values of u, that can be expanded asymptotically for small positive values of u. Their respective limits are called α and β, where α is a finite real number. For sufficiently small positive values of u holds $\eta(u) \geqslant \xi(u) > \alpha$. Put $\lambda(u)= u\left\{\eta(u)-\alpha\right\}^{-1}$ and $\tau(u)= u\left\{\xi(u)-\alpha\right\}^{-1}$. Both functions of u are, therefore, positive and for sufficiently small positive values of u holds $\lambda(u) \leqslant \tau(u)$. The respective limits of $\lambda(u)$ and $\tau(u)$ for $u \longrightarrow 0$ are indicated by γ and δ. These limits are $\geqslant 0$ or ∞.

Assertion: It is possible to write $J(u)$ as follows

$$J(u)= J_I(u) \log u + J_{II}(u),$$

where $J_I(u)$ and $J_{II}(u)$ can be expanded asymptotically for small positive values of u, if the following conditions are satisfied:

i. The function $\varphi(x)$ can be expanded asymptotically in the interval $(\alpha,\xi(u))$ in the neighbourhood of $x=\alpha$ and in the interval $(\beta\ \eta(u))$ in the neighbourhood of $x=\beta$.

ii. The function $\chi(v)$ can be expanded asymptotically for small positive values of v. Moreover, it can be expanded asymptotically in the interval $(\gamma,\lambda(u))$ in the neighbourhood of γ and in the interval $(\delta,\tau(u))$ in the neighbourhood of δ.

iii. If $\alpha=\beta$, then the integral

$$\int_{\lambda(u)}^{\tau(u)} \left|\chi(v)\right| dv$$

exists for small positive values of u and there exists a positive number $\theta< 1$, independent of u, with the property that $u^{-\theta}(\eta(u)-\alpha)$ is finite for small positive values of u.

iv. If $\alpha<\beta$, then the integrals

$$\int_{\xi(u)}^{\eta(u)} |\varphi(x)|\,dx \quad \text{and} \quad \int_{\lambda(u)}^{\tau(u)} |\chi(v)|\,dv$$

exists for small positive values of u.

Remark: One gets an analogous result if the condition $\eta(u)\geqslant \xi(u)>\alpha$ for small positive values of u is replaced by $\alpha>\xi(u)\geqslant\eta(u)$ for small positive values of u; in that case one has to replace $x-\alpha$ by $\alpha-x$.

Proof: I. Be first considered the case $\alpha=\beta$: Suppose that for small positive values of $x-\alpha$ the function $\varphi(x)$ can be represented asymptotically by

$$\varphi(x)=\sum_{h=0}^{H-1} B_h(x-\alpha)^{b_h}+O((x-\alpha)^{b_H}).$$

Then one finds for J(u):

$$J(u)=\sum_{h=0}^{H-1} B_h\, I_h(u)+O(I_H'(u));$$

where

$$I_h(u)=\int_{\xi(u)}^{\eta(u)} \chi\left(\frac{u}{x-\alpha}\right)(x-\alpha)^{b_h}\,dx,$$

which integral can be written by means of the substitution $v=u.(x-\alpha)^{-1}$ in the form

$$u^{b_h+1}\int_{\lambda(u)}^{\tau(u)} \chi(v)\, v^{-b_h-2}\,dv.$$

With the aid of theorem 3 this integral can be expanded asymptotically for small positive values of u, with the exception of a logarithmical term.

Further

$$I_H'(u)=\int_{\xi(u)}^{\eta(u)} \left|\chi\left(\frac{u}{x-\alpha}\right)\right| (x-\alpha)^{b_H}\,dx,$$

that is equal to

$$u^{b_H+1}\int_{\lambda(u)}^{\tau(u)} |\chi(v)|\; v^{-b_H-2}\,dv. \tag{5.1}$$

One can distinguish now the two cases:

Ia. Be $\gamma>0$. For sufficiently small positive values of u the inequality $\lambda(u)>\frac{\gamma}{2}$ holds, so that in virtue of $\tau(u)\geqslant\lambda(u)$ holds:

$$\int_{\lambda(u)}^{\tau(u)} |\chi(v)|\, v^{-b_H-2}\,dv = O\int_{\lambda(u)}^{\tau(u)} |\chi(v)|\,dv.$$

From the remark, added to theorem 3 and applied with $\varphi(x) = |\chi(x)|$, it follows that there exists a real constant κ such that the last integral is of the order $u^{-\kappa}$. The integral in (5.1) is also of the same order as $u^{b_H+1-\kappa}$, where the exponent $b_H+1-\kappa$ increases indefinitely with H, so that this integral tends asymptotically to zero for $H \to \infty$. The case Ia ($\alpha = \beta, \gamma > 0$) is, therefore, dealt with completely.

Ib. Be $\gamma = 0$. According to condition 3 the expression

$$u^{-\theta}\ (\eta(u) - \alpha) = \frac{u^{1-\theta}}{\lambda(u)}$$

is bounded for small positive values of u. So in the interval $\lambda(u) \leq v \leq \tau(u)$ holds

$$u^{1-\theta}\ v^{-1} \leq \frac{u^{1-\theta}}{\lambda(u)}$$

and, therefore, this is also bounded, so that the integral in (5.1) is at most of the order

$$u^{\theta b_H + 2\theta - 1} \int_{\lambda(u)}^{\tau(u)} |\chi(v)|\, dv.$$

Applying again the remark of theorem 3, one finds that the last integral is of the order $u^{-\kappa}$, so that (5.1) is at most of the order $u^{\theta b_H + 2\theta - 1 - \kappa}$, where the exponent $\theta b_H + 2\theta - 1 - \kappa$ increases indefinitely with H. This is the proof in the case $\alpha = \beta, \gamma = 0$, so that the case I, where $\alpha = \beta$, is dealt with completely.

II. Suppose $\alpha < \beta$. One can choose in the open interval (α, β) a finite real number A independent of u; for sufficiently small positive values of u holds $\xi(u) < A < \eta(u)$.

Under IIa the theorem is proved with $\xi(u)$ replaced by A and under IIb with $\eta(u)$ replaced by A. Addition of the two derived results gives the desired theorem.

IIa. In the interval $A \leq x \leq \eta(u)$ the quantity $\frac{u}{x-\alpha}$ is a small positive number, if u represents a small positive number, so that for small positive values of u the function $\chi(\frac{u}{x-\alpha})$ can be expanded asymptotically uniformly in x in ascending powers of u:

$$\chi\left(\frac{u}{x-\alpha}\right) \doteq \sum_{h=0}^{H-1} A_h \frac{u^{a_h}}{(x-\alpha)^{a_h}} + O\left(\frac{u^{a_H}}{(x-\alpha)^{a_H}}\right).$$

Using theorem 4, applied with

$$\varphi(x,u) = \chi\left(\frac{u}{x-\alpha}\right) \quad \varphi(x)$$

the integral taken from A to $\eta(u)$ can be written in the form

$$I_i(u).\log u + I_{ii}(u),$$

where $I_i(u)$ and $I_{ii}(u)$ can be expanded asymptotically for small positive values of u.

<u>IIb</u>. Finally one deals with the integral

$$I(u)=\int_{\xi(u)}^{A} \chi\left(\frac{u}{x-\alpha}\right) \varphi(x)dx,$$

where A denotes a finite number larger than α.

For small positive values of v holds

$$\chi(v)=\sum_{h=0}^{H-1} A_h v^{a_h} + O(v^{a_H})$$

and for small positive values of $x-\alpha$ holds

$$\varphi(x)=\sum_{h=0}^{H-1} B_h(x-\alpha)^{b_h} + O((x-\alpha)^{b_H}),$$

where a_h and b_h increase indefinitely with h. Now, put

$$\chi(v)= \chi_o(v) +\sum\nolimits_1 A_h v^{a_h} \tag{5.2}$$

and

$$\varphi(x)= \varphi_o(x)+\sum\nolimits_2 B_h(x-\alpha)^{b_h}, \tag{5.3}$$

where $\sum_1$ and $\sum_2$ are extended only over those values of the index h, for which the corresponding exponents a_h and b_h are negative. The function $\varphi_o(x)$ tends to a finite limit as x tends to α, so that one can choose A such that $\varphi(x)$ is bounded in the closed interval (α, A). It is clear that the functions $\chi_o(v)$ and $\varphi_o(x)$ can be expanded asymptotically for small positive values of v and $x-\alpha$, where the exponents occurring in these expansions are $\geqslant 0$.

One puts

$$\begin{aligned} I(u)= &\sum\nolimits_1 A_h u^{a_h} \int_{\xi(u)}^{A} \varphi(x) (x-\alpha)^{-a_h} dx \\ &+\sum\nolimits_2 B_h u^{b_h+1} \int_{\lambda(u)}^{\tau(u)} \chi_o(v)v^{-b_h-2} dv + I_o(u). \end{aligned} \tag{5.4}$$

According to theorem 3 the integrals occurring in the right hand side of (5.4) can be expanded asymptotically with the exception of a logarithmical term for small positive values of u.

$I_o(u)$ represents the integral

$$I_o(u)=\int_{\xi(u)}^{A} \chi_o\left(\frac{u}{x-\alpha}\right) \varphi_o(x)dx . \tag{5.5}$$

The function $\chi_o(v)$ tends to a finite limit, as $v \longrightarrow 0$, that is denoted by $\chi_o(0)$. Now the asymptotic expansions of $\chi_o(v)$ and $\varphi_o(x)$ are written in the form:

$$\chi_o(v) = \chi_o(o) + \sum_{h=1}^{H-1} E_h\, v^{e_h} + O(v^{e_H}) \tag{5.6}$$

and

$$\varphi_o(x) = \sum_{h=0}^{H-1} F_h(x-\alpha)^{f_h} + O((x-\alpha)^{f_H}); \tag{5.7}$$

the exponents e_h are larger than zero, the exponents $f_h \geqslant 0$ and e_h and f_h increase indefinitely with h.

One puts

$$\varphi_o(x) = \psi_o(x) + \sum\nolimits_3 F_h(x-\alpha)^{f_h},$$

where $\sum_3$ is extended only over those values of h for which the corresponding exponent $f_h \leqslant e_1$.

In this case $\lambda(u)$ takes the simple form $\frac{u}{A-\alpha}$.

According to the fourth condition

$$\int_{\lambda(u)}^{\tau(u)} \left|\chi(v)\right| dv$$

exists, so that in virtue of relation (5.2) also

$$\int_{\lambda(u)}^{\tau(u)} \left|\chi_o(v)\right| dv$$

exists.

By means of the remark added to theorem 3, there appears to exist a number $\kappa \geqslant 0$ such that

$$\int_{\lambda(u)}^{\tau(u)} \left|\chi_o(v)\right| dv = O(u^{-\kappa}) \tag{5.8}$$

From the fact that $\varphi_o(x)$ is bounded in the closed interval (α, A), it follows

$$\int_{\xi(u)}^{A} \left|\chi_o\left(\tfrac{u}{x-\alpha}\right) \varphi_o(x)\right| dx = O\left(u \int_{\lambda(u)}^{\tau(u)} \left|\chi_o(v)\right| v^{-2}\, dv\right). \tag{5.9}$$

Since $\chi_o(v)$ tends to a finite limit as v tends to zero, it is possible to determine a positive number v_o such that $\chi_o(v)$ is bounded in the interval $0 < v < v_o$. For values of u for which $\tau(u) \leqslant v_o$, the right hand side of (5.9) is at most of the order

$$u \int_{\lambda(u)}^{v_o} v^{-2}\, dv \leqslant \frac{u}{\lambda(u)} = A - \alpha;$$

For values of u having $\tau(u) > v_o$, the right hand side of (5.9) is at most of the order

$$u \int_{\lambda(u)}^{v_o} v^{-2} dv + u \int_{v_o}^{\tau(u)} \left|\chi_o(v)\right| dv \leq A-\alpha + u \int_{\lambda(u)}^{\tau(u)} \left|\chi_o(v)\right| dv .$$

From (5.8) and $K \geq 0$ one finds in all cases

$$\int_{\xi(u)}^{A} \left|\chi_o\left(\frac{u}{x-\alpha}\right) \varphi_o(x)\right| dx = O(u^{-K}) .$$

The method, by which the integral $I_o(u)$ is handled, is the following one. One writes

$$I_o(u) = \sum\nolimits_3 F_h \, u^{f_h+1} \int_{\lambda(u)}^{\tau(u)} \chi_o(v) v^{-f_h-2} dv + R_o(u), \qquad (5.10)$$

where

$$R_o(u) = \int_{\xi(u)}^{A} \chi_o\left(\frac{u}{x-\alpha}\right) \psi_o(x) dx.$$

The integrals occurring in (5.10) can be expanded asymptotically with the exception of a logarithmical term for small positive values of u. Now, put

$$R_o(u) = \chi_o(0) \int_{\xi(u)}^{A} \psi_o(x) dx + u^{e_1} I_1(u),$$

where the integral can be expanded again asymptotically for small positive values of u in virtue of theorem 3.

So one writes $I_1(u)$ in the form:

$$I_1(u) = \int_{\xi(u)}^{A} \chi_1\left(\frac{u}{x-\alpha}\right) \varphi_1(x) dx$$

where $\chi_1(v) = \left\{\chi_o(v) - \chi_o(v)\right\} v^{-e_1}$ and $\varphi_1(x) = \psi_o(x) . (x-\alpha)^{-e_1}$.

One deals with the integral $I_1(u)$ in the same way as with $I_o(u)$. This is possible since the following properties hold:

i: For small positive values of v the function $\chi_1(v)$ can be expanded asymptotically in ascending powers of v, just as $\chi_o(v)$, and $\chi_1(v)$ tends to a finite limit as v approaches zero; that limit is denoted by $\chi_1(0)$.

ii: For small positive values of $x-\alpha$ the function $\varphi_1(x)$ can be expanded asymptotically in ascending powers of $x-\alpha$, just as $\varphi_o(x)$, and $\varphi_1(x)$ is bounded in an interval (α, A).

iii. One has $e_1 > 0$;

iv. One has

$$\int_{\xi(u)}^{A} \left| \chi_1\left(\frac{u}{x-\alpha}\right) \varphi_1(x) \right| dx = O(u^{-\kappa}) \tag{5.11}$$

It will be shown that in this formula the same exponent κ occurs as in formula (5.8). Since $\varphi_1(x)$ is bounded in the interval, the left hand side is at most of the order

$$\int_{\xi(u)}^{A} \left| \chi_1\left(\frac{u}{x-\alpha}\right) \right| dx = u \int_{\lambda(u)}^{\tau(u)} \left| \chi_1(v) \right| v^{-2}\, dv \; .$$

If $\delta = 0$, the right hand side of this equality is at most of the order 1, since $\chi(v)$ tends to a finite limit as v approaches zero. Therefore, the last mentioned integral is at most of the order $u^{-\kappa}$ and (5.11) is proved in this case.

If $\delta > 0$, it is possible to choose a positive number B smaller than δ . Just as is done in the case $\delta = 0$ one proves that the contribution of the integral

$$u \int_{\lambda(u)}^{B} \left| \chi_1(v) \right| v^{-2}\, dv$$

is at most of the order $u^{-\kappa}$.

In the interval $B < v < \tau(u)$ holds

$$\frac{\chi_1(v)}{v^2} = \frac{\chi_0(v) - \chi_0(0)}{v^{2+e_1}} = O\left(\left|\chi_0(v)\right| + \frac{1}{v^2}\right),$$

so that the contribution of the interval $B \leq v \leq \tau(u)$ is at most of the order

$$\int_{B}^{\tau(u)} \left|\chi_0(v)\right| dv + \int_{B}^{\infty} \frac{dv}{v^2} = O(u^{-\kappa}) + O(1) = O(u^{-\kappa}),$$

according to (5.8). So (5.11) is proved for all cases.

In the preceding pages the integral $I_0(u)$ has been written as the sum of $u^{e_1}\, I_1(u)$ and a function that can be expanded in the desired way. It has been shown that $I_0(u)$ and $I_1(u)$ are both of the order $u^{-\kappa}$, and that $I_1(u)$ satisfies the same conditions as $I_0(u)$ does.

Using the process of induction and introducing

$$\chi_2(v) = (\chi_1(v) - \chi_1(v))\, v^{e_1 - e_2},$$

one finds that $I_1(u)$ is equal to the sum of $u^{e_2 - e_1}\, I_2(u)$ and a function that can be expanded in the desired way. So $I_0(u)$ is equal to the sum of $u^{e_2}\, I_2(u)$ and a function that can be expanded in the desired way. Proceeding in this way one finds that for every natural number h, $I_0(u)$ is equal to a term $u^{e_h}\, I_h(u)$ augmented by a function that can be expanded in the desired way. Here the integrals $I_o(u), I_1(u), \ldots, I_h(u)$ are at most of the order $u^{-\varkappa}$, where $\varkappa$ is the same number for all integrals so that

$$u^{e_h} I_h(u) = O(u^{e_h - \varkappa}).$$

Since e_h increases indefinitely with h, the proof is given in the case IIb and the proof of theorem 5 has been completed.

CHAPTER II

Theorems concerning equations.

1. Introduction.

In this chapter several theorems are given concerning the solutions of certain equations; especially the behaviour of the solutions for special values of the occurring parameter is considered.

2. About the uniqueness of the solutions.

Theorem 6. Be α a positive number. Suppose that in the interval $y_0 - \alpha \leq y \leq y_0 + \alpha$, the real function $f(y)$ is continuously differentiable with $f'(y) \neq 0$ and that $f(y_0) = x_0$.

It is possible to choose a positive number β so small, that there is one and only one continuous function $y(x)$ that satisfies in the interval $x_0 \leq x \leq x_0 + \beta$, the relations $y_0 - \alpha \leq y(x) \leq y_0 + \alpha$ and $f(y(x)) = x$.

Proof: Be m the minimum value taken by $|f'(y)|$ on the closed interval $y_0 - \alpha \leq y \leq y_0 + \alpha$; one chooses $\beta > 0$ such that $m\alpha > \beta$.

Be ξ a number such that $x_0 < \xi \leq x_0 + \beta$. The values of the function $f(y) - x$ are compared at the points $(\xi, y_0 + \alpha)$ and $(\xi, y_0 - \alpha)$ by making use of the mean-value theorem:

$$f(y_0 + \alpha) - \xi = x_0 - \xi + \alpha f'(y + \delta\alpha),$$

$$f(y_0 - \alpha) - \xi = x_0 - \xi - \alpha f'(y - \delta_1\alpha),$$

where δ and δ_1 denote suitably chosen positive numbers less than one.

There holds, however,

$$\xi - x_0 \leq \beta < m\alpha \leq |\alpha f'(y + \delta\alpha)|$$

and

$$\xi - x_0 \leq \beta < m\alpha \leq |\alpha f'(y - \delta_1\alpha)|.$$

From these inequalities it is seen that $f(y_0 + \alpha) - \xi$ and $f_0(y - \alpha) - \xi$ have opposite signs and since $f'(y) \neq 0$ throughout the interval $y_0 - \alpha \leq y \leq y_0 + \alpha$, it follows that in the mentioned interval there exists one and only one value of y such that $f(y) - \xi = 0$. So for every x in the interval $x_0 \leq x \leq x_0 + \beta$ one can find one and only one value of y satisfying $f(y) = x$.

For an analogous theorem see [1].

3. <u>About the expansibility of the solutions.</u>

<u>Theorem 7</u>: If the exponents $\alpha_0 \neq 0, \alpha_1, \dots, \alpha_H$ are real, the equation

$$P^{\alpha_0} + \sum_{h=1}^{H} \lambda_h P^{\alpha_h} = 1 \qquad (7.1)$$

has one and only one solution $P = P(\lambda_1, \dots, \lambda_H)$ which is analytical in the neighbourhood of the origin $\lambda_1 = \lambda_2 = \dots = \lambda_4 = 0$ and takes the value 1 at the origin. That means that this solution can be expanded for sufficiently small values of $|\lambda_1|, \dots, |\lambda_4|$ in an absolutely convergent powerseries with respect to $\lambda_1, \lambda_2, \dots, \lambda_H$.

<u>Proof</u>: In the equation (7.1) is substituted $P = 1+x$, so that this equation takes the form

$$F(x, \lambda_1, \dots, \lambda_H) = 0$$

and the derivative $\frac{\partial F}{\partial x} = \alpha_0 \neq 0$ at $x = \lambda_1 = \dots = \lambda_H = 0$.

One has to prove that under the said conditions the equation $F(x, \lambda_1, \dots, \lambda_H) = 0$ has one and only one solution that is analytic in the neighbourhood of the point $x = \lambda_1 = \dots = \lambda_H = 0$ and that vanishes at that point. This is a special case of the following theorem. (See [2]):

If the functions

$$F_j(w_1, \dots, w_k; z_1, \dots, z_l), \qquad j = 1, \dots, k$$

are analytic functions of (k+l) variables in the neighbourhood of the origin, if $F_j(0;0)=0$ and if

$$\frac{\partial(F_1, \dots, F_k)}{\partial(w_1, \dots, w_k)} \neq 0 \quad \text{for } (w)=(z)=(0),$$

then the equations

$$F_j(w_1, \dots, w_k, z_1, \dots, z_k) = 0 \qquad j = 1, \dots, k$$

have an unique solution

$$w_j = w_j(z_1, \dots, z_l)$$

vanishing for $(z)=(0)$ and analytic in the neighbourhood of the origin.

4. <u>A generalization.</u>

<u>Theorem 8</u>. The equation

$$p^{\alpha_0} + \sum_{h=1}^{H} c_h p^{\alpha_h} = u,$$

where $\alpha_h > \alpha_0 > 0$ for $h = 1, \dots, H$ and H denotes a fixed natural number, has for small positive values of u one and only one continuous solution p(u) that tends to zero as u tends to zero; this solution can be expanded for small positive values of x in an absolutely convergent series of ascending powers of u; the exponents occurring in this expansion are all positive, but not necessarily integers.

Remark: For positive values of u and for real values of α the quantity u^{α} is supposed to be positive.

Proof: One puts $\alpha = \alpha_o$

$$\beta_h = \alpha_h - \alpha, \text{ so } \beta_h > 0 \text{ for } h \geq 1 \text{ and}$$
$$u^{1/\alpha} = x,$$

so that the given equation is transformed into $f(p)=x$, where

$$f(p)=p(1 + \sum_{h=1}^{H} c_h\, p^{\beta_h})^{1/\alpha}.$$

If the positive number p_o is chosen small enough the function $f(p)$ is continuously differentiable in the interval $0 \leq p \leq p_o$ with $f'(p) \neq 0$. Indeed,

$$f'(p) = P^{1/\alpha} + \frac{1}{\alpha}\, P^{-1+\frac{1}{\alpha}} \sum_{h=1}^{H} c_h \beta_h\, p^{\beta_h},$$

where

$$P = 1 + \sum_{h=1}^{H} c_h\, p^{\beta_h},$$

is approximately equal to one, so $\neq 0$.

According to theorem 6 the equation $f(p)= x$ and also the given equation for $x \geq 0$ have in the neighbourhood of the origin one and only one continuous solution $p = p(x)$ that tends to zero as x tends to zero. In order to show that this solution can be expanded for small positive values of u, one applies theorem 7 to the equation equivalent with the originally given equation:

$$y^{\alpha_o} + \sum_{h=1}^{H} c_h \lambda_h\, y^{\alpha_h} = 1,$$

where

$$y = p\, u^{-\frac{1}{\alpha_o}} \quad \text{and} \quad \lambda_h = u^{\alpha_o^{-1}(\alpha_h - \alpha_o)}, \quad (h \geq 1).$$

Since $\alpha_h > \alpha_o > 0$ for $h \geq 1$ the quantity λ_h tends to zero as u tends to zero. The conditions of theorem 7 are satisfied, so that the equation has a solution $y = y(\lambda_1,\ldots,\lambda_H)$ that can be expanded in an absolutely convergent power series with respect to $\lambda_1,\ldots,\lambda_H$ in the neighbourhood of the origin $\lambda_1 = \lambda_2 = \ldots = \lambda_H = 0$, and that is equal to one at the origin.

In this manner one obtains an absolutely convergent expansion in ascending powers of u for the solution $p(u)$ uniquely determined following the arguments mentioned above. So the proof is given.

5. Final theorem.

Theorem 9: Suppose that for small positive values of p the function f(p) is positive and can be expanded asymptotically in ascending powers of p. Suppose further that f(p) tends to zero as p tends to zero, and that the derivative f'(p) exists for small positive values of p; moreover, f'(p) be continuous and satisfy the relation

$$|f'(p)|^{-1} = O((p)^{\rho}) \tag{9.1}$$

where ρ denotes a suitably chosen real number.

Assertion: The equation f(p)= u has for small positive values of u one and only one continuous solution p = p(u) that tends to zero as u tends to zero. This solution p(u) can be expanded asymptotically for small positive values of u.

Proof: In a manner similar to those of the preceding theorem, it follows from theorem 6 that the equation

$$f(p)= u$$

has one and only one continuous solution p(u) that tends to zero with u.

Be

$$f(p) \sim \sum_{h=0}^{\infty} c_h\, p^{\alpha_h}$$

the asymptotic expansion of f(p) for small positive values of p, so that the exponents α_h increase indefinitely with h.

Without loss of generality one may assume that the exponents increase with h. Then the coefficient c_0 is positive.

According to theorem 8 the equation

$$f_k(p)= \sum_{h=0}^{k-1} c_h\, p^{\alpha_h} = u,$$

where k is a natural number, has one and only one continuous solution $p_k(u)$, that tends to zero with u; this solution can be expanded for small positive values of u in an absolutely convergent series of ascending powers of u. This series starts with the term bu^{β}, where b and β are positive numbers. For small positive values of u the quantity $p_k(u)$ is a small positive number so that $f(p_k(u))$ exists. One writes

$$f(p_k(u))-f(p(u))=f(p_k(u))-u = f_k(p_k(u))+O\left\{(p_k(u))^{\alpha_k}\right\} -u$$

$$= O\left\{(p_k(u))^{\alpha_k}\right\}= O(u^{\beta\alpha_k}),$$

since $p_k(u)$ is of the order u^{β}.

Because of the fact that $f'(p)$ exists and is continuous for small positive values of p, it is possible to write down the identity:

$$f(p_k(u))-f(p(u))= \int_{p(u)}^{p_k(u)} f'(v)dv. \tag{9.2}$$

Since $p(u)$ and $p_k(u)$ are both positive and are of the order u^{β}, the variable of integration v is of the same order in the interval $(p(u),p_k(u))$, so that the integral is of the order $u^{-\beta\rho}(p_k(u)-p(u))$, according to (9.1). From (9.2) one deduces

$$u^{-\beta\rho}\ p_k(u)-p(u) \quad = O(u^{\beta\alpha_k}),$$

so that

$$p(u)= p_k(u)+ O(u^{\beta\alpha_k+\beta\rho}).$$

But α_k increases indefinitely with k. The function $p_k(u)$ has been expanded in an absolutely convergent series of ascending powers of u that represents at the same time the asymptotic expansion of $p_k(u)$. In this way the solution $p(u)$ has been expanded asymptotically in ascending powers of u for small positive values of u.

6. A generalisation.

Theorem 10. The equation

$$p^{\alpha_0} + \sum_{h=1}^{H} c_h\ p^{\alpha_h} = u$$

where $\alpha_0 > \alpha_h > 0$ for $h = 1,\ldots,H$ and where H denotes a fixed natural number, has for large positive values of u one and only one continuous solution $p(u)$ that tends to infinity with u; this solution can be expanded for large positive values of u in an absolutely convergent series of descending powers of u.

The proof is given in a manner similar to that of theorem 8.

CHAPTER III

Theorems concerning double integrals.

1. Introduction.

In this chapter the definition is given for double integrals in the improper sense. Some theorems are given on the existence of double integrals in improper sense and also some theorems on the possibility of using transformation.

The theorems 11 and 13 are proved in H. von Mangoldt, Einführung in die Höhere Mathematik, 1948, and, therefore, they are given here without proof.

2. General theorems.

Theorem 11. If the function $f(x,y)$ is continuous on the closed bounded set B, then $f(x,y)$ is integrable on B. For the proof see [3].

Theorem 12. If the function $f(x,y)$ is continuous on the closed bounded set B, if M represents the maximum value of $|f(x,y)|$ on B and if, finally, I represents the area of the closed bounded set B, then holds

$$\left| \iint_B f(x,y)dx\,dy \right| \leqslant MI.$$

Proof: According to the mean value theorem (confer [4]) holds

$$\iint_B f(x,y)dx\,dy = I\mu ,$$

where μ indicates a suitably chosen number lying between the upper and lower limits of $f(x,y)$ on B (the boundaries inclosed).

From this the proposed inequality follows immediately.

3. First transformation theorem.

Theorem 13. Let B' be a bounded measurable set of the (u,v) plane. On a set B_1' containing B' and his boundary as interior points, are given the functions $x = \varphi(u,v)$ and $y = \psi(u,v)$. These functions are continuously differentiable with respect to u and v in the interior of B_1', while the Jacobian

$$J(u,v) = \begin{vmatrix} \varphi_u(u,v) & \psi_u(u,v) \\ \varphi_v(u,v) & \psi_v(u,v) \end{vmatrix}$$

is supposed to be either always positive or always negative in the interior of B'.

The image of B' in the (x,y) plane is indicated by B; it is supposed that two different interior points of B' correspond to different points of B. So the set B has also an area A and this is equal to

$$A = \iint_{B'} J(u,v)\, du\, dv.$$

If on B is given a function f(x,y), integrable on B, then also the function $f(\varphi(u,v),\psi(u,v))\,|J(u,v)|$ is integrable on B' and there holds

$$\iint_{B} f(x,y)dx\, dy = \iint_{B'} f(\varphi(u,v),\psi(u,v))\,|J(u,v)|\, du\, dv.$$

For the proof see [5] .

4. Some definitions.

In this section the following two definitions are given:

Definition I. Let f(x,y) be continuous in a certain region G of the (x,y) plane; then according to theorem 11 the integral

$$\iint_{D} f(x,y)dx\, dy$$

exists for every closed bounded subset D of G. Be R the boundary of G. Now, consider infinitely many of such subsets $D_1, D_2, \ldots$ of G such that the distance of each boundary point of D_m to R tends to zero as n increases indefinitely. If for every choice of such subsets $D_1, D_2, \ldots$ the integral

$$\iint_{D_n} f(x,y)dx\, dy$$

tends to a finite limit as n tends to infinity, one says that the function f(x,y) is integrable (in improper sense) on G and one puts

$$\iint_{G} f(x,y)dx\, dy = I.$$

Definition II. One says that a plane pointset has an external measure less than or equal to μ if that point set can be covered by a finite number of squares having a total area smaller than or equal to μ .

5. On the area in the neighbourhood of a continuous arc.

Lemma 1. Be R a plane continuous arc with a finite length L. The points lying in the plane of the continuous arc R and having a distance $\leq$ t to that arc, form for every positive value of $t \leq 1$ a set of the external measure $\leq 4(L+1)t$.

Proof: Suppose that the arc R lies in the (x,y) plane. Let this plane be covered by fitting squares that have sides with a length t and that are parallel to the coordinate axes. It is sufficient to prove that at most $4(1+L.t^{-1})$ such fitting squares have a point in common with the arc R, for in that case the number of squares is less than or equal to $4t^{-1}(1+L)$, so that the total area of these squares is at most equal to $4t(1+L)$. In proving this one may suppose that there occur at least five of the mentioned squares. In considering 5 different squares it is always possible to find among them two squares having a distance larger than or equal to t. An arc of R that needs 5 squares to be covered, contains, therefore, at least two points having a distance larger than t, so that the length of the arc is also larger than t. An arc of R with length less than or equal to t, can be covered, therefore, by four or less of the mentioned squares. Since the arc R has a total length L, R can be divided into less than $1+L.t^{-1}$ parts, each having a length $\leq t$, so that the whole arc can be covered by less than $4(1+L.t^{-1})$ of the fitting squares.

6. On the existence of an integral.

Theorem 14: Be f(x,y) continuous in the bounded region G, with the property that

$$|f(x,y)| \leq Ca^{-1+\delta},$$

where C and δ denote fixed positive numbers and a indicates the distance of the point (x,y) to the boundary of G. (It is clear that a depends on x and y).

Suppose that the boundary of G can be divided in a finite number of continuous arcs each satisfying the following condition: the arc has either a finite length or f(x,y)= 0 at each point (x,y) of G in the neighbourhood of that arc. Under these conditions the double integral

$$\iint_G f(x,y)dx\, dy$$

exists.

Proof: First step. The (x,y) plane is covered first by fitting directed squares V_m with sides equal to 2^{-m}, where m denotes a natural number. Be Q_m the set formed by those squares V_m that belong entirely to the interior of G and that have a distance to the boundary R of G that is larger than 2^{-m}. The set Q_m is closed and bounded and the function f(x,y) is continuous at each point of Q_m, so that the integral

$$I(m)= \iint_{Q_m} f(x,y)dx\, dy$$

exists according to theorem 11.

First it will be shown that this integral tends to a finite limit I as m increases to infinity. Therefore one remarks that I(m+1)-I(m) can be written as the integral with integrand f(x,y) extended over all squares V_{m+1}, that contribute to I(m+1), whereas the squares V_m containing V_{m+1} do not contribute to I(m). These squares V_m, therefore, have either at least one point in common with the boundary or with the exterior of G or they contain at least one point that has a distance to R less than or equal to 2^{-m}. Each point of the considered squares V_m and therefore each point of the corresponding squares V_{m+1} has a distance to R being at most equal to 2^{-m} augmented by the diagonal of those squares V_m, so that this distance is at most equal to $2^{-m}(1+2^{\frac{1}{2}})$. According to the preceding lemma, applied with $t = 2^{-m}(1+2^{\frac{1}{2}})$, for every arc with a finite length L the total area of the squares V_{m+1} coming into consideration is at most equal to

$$2^{-m+2}(1+2^{\frac{1}{2}})(L+1).$$

Since the number of continuous arcs of finite length is finite and f(x,y)= 0 in the neighbourhood of all other arcs, the total area of the squares V_{m+1} coming into consideration is at most equal to $C_1\ 2^{-m+2}$, where C_1 denotes a suitably chosen positive constant.

According to the definition, each of the mentioned squares V_{m+1} has a distance to R larger than 2^{-m-1} and so holds in those squares:

$$|f(x,y)| < C(2^{-m-1})^{-1+\delta}.$$

From this it follows:

$$|I(m+1)-I(m)| < CC_1\ 2^{3-\delta}\ .2^{-m\delta},$$

so that the series $\sum_{m=1}^{\infty} I(m+1)-I(m)$

converges (even absolutely) and, therefore, I(m) tends to a finite limit as m increases to infinity.

<u>Remark</u>: In the preceding lines f(x,y) may be replaced by its absolute value, since this value is also continuous. Thus, one finds also that

$$\iint_{Q_m} |f(x,y)|\ dx\ dy$$

tends to a finite limit if m increases to infinity.

<u>Second step</u>: One shows further that

$$\iint_{E} f(x,y)\ dx\ dy$$

tends to the above introduced limit I if E runs through a series of bounded closed subsets of G with the property that the distance of each boundary point of E to R tends to zero.

It is sufficient to prove that for every positive ε and every closed bounded subset E of G, of which each boundary point has a sufficiently small distance to R, the inequality

$$\left| \iint_E f(x,y)dx\,dy - I \right| < \varepsilon \tag{14.1}$$

holds.

According to the first step of the proof and the remark added to that step the following properties hold for each sufficiently large natural number m:

$$\text{i,} \qquad \left| \iint_{Q_m} f(x,y)\,dx\,dy - I \right| < \varepsilon/2 \tag{14.2}$$

$$\text{ii,} \qquad \left| \iint_{Q_M} |f(x,y)|\,dx\,dy - \iint_{Q_m} |f(x,y)|\,dx\,dy \right| < \varepsilon/2,$$

for every integer $M > m$. (The last inequality is the result of the application of the criterion of Cauchy to the result of the remark added to the first step). Let m be a natural number with these two properties. The proof is given, if one can prove (14.1) for each closed bounded subset E of G, the boundary points of which have a distance to R less than 2^{-m}. In that case Q_m is a subset of E, while it is possible to find a natural number $M \geq m$ such that E is a subset of Q_M and then holds:

$$\left| \iint_E f(x,y)dx\,dy - \iint_{Q_m} f(x,y)dx\,dy \right| = \left| \iint_{E-Q_m} f(x,y)dx\,dy \right| \leq$$

$$\iint_{Q_M-Q_m} \left| f(x,y)dx\,dy \right| \leq \varepsilon/2,$$

so that (14.1) follows from (14.2).

7. Second transformation theorem.

Theorem 15. Suppose the conditions of theorem 14 are satisfied. Be given in the region D of the (u,v) plane two continuously differentiable functions

$$x = \varphi(u,v) \quad \text{and} \quad y = \psi(u,v),$$

such that the Jacobian

$$J(u,v) = \begin{vmatrix} \varphi_u(u,v) & \psi_u(u,v) \\ \varphi_v(u,v) & \psi_v(u,v) \end{vmatrix}$$

is always positive or always negative on D; the image of D in the (x,y) plane is denoted by G. The transformation is supposed to be one to one correspondant. Finally one assumes that if the distance of the point (x,y) of G to the boundary R' of G tends to zero, also the distance of the corresponding point (u,v) of D to the boundary R of D tends to zero. Under these conditions the integrals

$$\iint_G f(x,y)dx\,dy \quad \text{and} \quad \iint_D f(\varphi(u,v),\psi(u,v))\,|J(u,v)|\,du\,dv$$

exist in improper sense and are equal.

Proof: Consider infinitely many regions E_n' belonging with their boundary to D' such that the distance of every boundary point of E_n' to the boundary of D' tends to zero as n increases to infinity.

Applying theorem 14 one finds

$$\iint_{E_n'} f(x,y)dx\,dy$$

tends to a finite limit as n increases to infinity.

From this it follows that f(x,y) is integrable on D' and that I is equal to the integral of that function f(x,y) extended over D'.

According to the transformation theorem 13 holds

$$\iint_{E_n'} f(x,y)dx\,dy = \iint_{E_n} f(\varphi(u,v),\psi(u,v))\,|J(u,v)|\,du\,dv, \qquad (15.1)$$

where E_n denotes the region corresponding with E_n'. Each boundary point of E_n' has the property that its distance to R' tends to zero as n increases to infinity. From this it follows that the distance of the corresponding point (u,v) to the boundary R of D tends also to zero. Since the integral in (15.1) tends to a finite limit I, if n increases to infinity, the function $f(\varphi(u,v),\psi(u,v))\,|J(u,v)|$ is integrable on D, and I is equal to the integral of this function extended over this region D. So the proof is complete.

7. On the existence of an integral, II.

Theorem 16. Be the function f(x,y) continuous on G, with the exception of a finite number of points $z_1,\dots,z_r$, with the property that

$$|f(x,y)| \leqslant C(a_0^{-1+\delta_0} + a_1^{-2+\delta_1} + \dots + a_r^{-2+\delta_r}),$$

where C, $\delta_0,\delta_1,\dots,\delta_r$ denote fixed positive numbers, a_0 denotes the distance of the point (x,y) to the boundary R of G and $a_1,a_2,\dots,a_r$ denote the distances of the point (x,y) to the respective points $z_1,\dots,z_r$. (It is clear that $a_0,\dots,a_r$ depend on x and y).

Suppose that the boundary R of G can be divided into a finite number of continuous arcs, each satisfying the following condition: the arc has either a finite length or f(x,y)= 0 at each point (x,y) of G in the neighbourhood of that arc.

Then the integral

$$\iint_G f(x,y)dx\,dy$$

exists in improper sense.

Proof: First step: Be the (x,y) plane covered by fitting directed squares V_m with sides equal to 2^{-m}, where m denotes a natural number. Be Q_m the set formed by those squares V_m that belong entirely to the interior of G. These squares have a distance larger than 2^{-m} to the boundary R of G and they have a distance larger than $2^{-\frac{m}{2}}$ to each of the points $z_1,\dots,z_r$. The set Q_m is bounded and closed and the function f(x,y) is continuous on Q_m, so that the integral

$$I(m) = \iint_{Q_m} f(x,y)dx\,dy$$

exists according to theorem 12.

First it is shown that this integral tends to a finite limit I as m increases to infinity. Therefore, one remarks that $I(m+1)-I(m)$ can be written as the integral with integrand f(x,y) extended over those squares V_{m+1}, that contribute to $I(m+1)$ whereas the squares V_m containing V_{m+1} do not contribute to $I(m)$. These squares V_m, therefore, have either at least one point in common with the boundary or with the interior of G or they contain at least one point that has a distance $\leq 2^{-m}$ to the boundary R or that has a distance $\leq 2^{-\frac{m}{2}}$ to a least one of the points $z_1,\dots,z_r$.

Each point of the considered squares V_m and also each point of the corresponding squares V_{m+1} has a distance to the boundary R being at most equal to 2^{-m} augmented by the diagonal of that square V_m (so that this distance is at most equal to $2^{-m}(1+2^{\frac{1}{2}})$) or that point has a distance $\leq 2^{-\frac{m}{2}}(1+2^{\frac{1}{2}})$ to at least one of the points $z_1,\dots,z_r$.

One applies lemma 1 to each continuous arc of the boundary R with finite length choosing $t = 2^{-m}(1+2^{\frac{1}{2}})$. Then one finds that the points that have to the boundary R a distance $2^{-m}(1+2^{\frac{1}{2}})$, form a set of which the external measure is at most of the order 2^{-m}. It is clear that the points having a distance $2^{-\frac{m}{2}}(1+2^{\frac{1}{2}})$ to at least one of the points $z_1,\dots,z_r$, form a set with external measure also of the order 2^{-m}.

Since each of the mentioned squares V_{m+1} have according to their definition a distance larger than 2^{-m-1} to R and a distance larger than $2^{-\frac{m+1}{2}}$ to each of the points $z_1,\dots,z_r$, the function f(x,y) is at most of the order

$$(2^{-(m+1)})^{-1+\delta}$$

at each point of such a square, so that I(m+1)-I(m) is at most of the order $2^{-(m+1)\delta}$. Therefore the series

$$\sum_{m=1}^{\infty} I(m+1)-I(m)$$

converges (even absolutely) and I(m) tends to a finite limit as m approaches infinity.

Remark: In the preceding lines f(x,y) may be replaced by its absolute value, since this value is continuous too. Hence, one finds that also

$$\iint_{Q_m} |f(x,y)| \, dx \, dy$$

tends to a finite limit if m increases to infinity.

Second step: One shows further that

$$\iint_E f(x,y)dx \, dy$$

tends to the above introduced limit I if E runs through a series of bounded closed subsets of G with the property that the distance of each boundary point of E to R or to at least one of the points $z_1,\ldots,z_r$ tends to zero.

It is sufficient to prove that for every positive ε and every closed bounded subset E of G, of which each boundary point has a sufficiently small distance to R or to at least one of the points $z_1,\ldots,z_r$ the inequality

$$\left| \iint_E f(x,y)dx \, dy - I \right| < \varepsilon \tag{16.1}$$

holds. According to the first step of the proof and the remark added to that step the following properties hold for each sufficiently large natural number m:

$$\text{i,} \quad \left| \iint_{Q_m} f(x,y)dx \, dy - I \right| < \varepsilon/2; \tag{16.2}$$

$$\text{ii,} \quad \left| \iint_{Q_M} |f(x,y)| dx \, dy - \iint_{Q_m} |f(x,y)| \, dx \, dy \right| < \varepsilon/2,$$

for every integer M > m. (The last inequality is the result of the application of the criterion of Cauchy to the result of the remark added to the first step). Be m a natural number with these two properties. The proof is given if one can prove (16.1) for each closed bounded subset E of G the boundary points of which have a distance to R smaller than 2^{-m} or have a distance to at least one of the points $z_1,\ldots,z_r$ smaller than $2^{-\frac{m}{2}}$. In that case Q_m is a subset of E, while it is possible to find a natural number $M \geqslant m$ such that E is a subset of Q_M, and then holds:

$$\left| \iint\limits_E f(x,y)dx\,dy - \iint\limits_{Q_m} f(x,y)dx\,dy \right| = \left| \iint\limits_{E-Q_m} f(x,y)dx\,dy \right| \leqslant$$

$$\iint\limits_{Q_M-Q_m} \left| f(x,y) \right| dx\,dy \leqslant \frac{\varepsilon}{2}$$

so that (16.1) follows from (16.2).

8. Third transformation theorem.

Theorem 17. Suppose that the conditions of theorem 16 are satisfied. Let be defined in a certain region D of the (u,v) plane the functions

$$x = \varphi(u,v) \quad ; \quad y = \psi(u,v).$$

These functions are continuously differentiable in D such that their Jacobian

$$J(u,v) = \begin{vmatrix} \varphi_u(u,v) & \psi_u(u,v) \\ \varphi_v(u,v) & \psi_v(u,v) \end{vmatrix}$$

is either always positive or always negative on D; these conditions do not need to be satisfied on the boundary R of D.

Be G the image of D in the (x,y) plane. It is assumed that between the points of G and D is an one to one correspondance. Then the integrals

$$\iint\limits_G f(x,y)dx\,dy \quad \text{and} \quad \iint\limits_D f(\varphi(u,v), \psi(u,v)) \left| J(u,v) \right| du\,dv$$

exist in improper sense and are equal.

Proof: Consider infinitely many regions E_n' belonging with their boundary to G and containing none of the points $z_1, z_2, \ldots, z_r$, such that the distance of each boundary point of E_n' to the boundary R' or to at least one of the points $z_1, z_2, \ldots, z_r$ tends to zero as n increases to infinity.

According to theorem 16 the integral

$$\iint\limits_{E_n'} f(x,y)dx\,dy$$

tends to a finite limit I. Therefore f(x,y) is integrable on G, and I is the integral of that function on G.

According to the first transformation-theorem:

$$\iint_{E_n'} f(x,y)dx\,dy = \iint_{E_n} f(\varphi(u,v),\psi(u,v))\left|J(u,v)\right|du\,dv, \tag{17.1}$$

where E_n denotes the region of the (u,v) plane corresponding with E_n'. Each boundary point of E_n' has the property that its distance to R' or to at least one of the points $z_1,\dots,z_r$ tends to zero as n increases infinitely.

From this it follows that the distance of the corresponding point (u,v) to the boundary R of D or to at least one of the points $z_1^*,\dots,z_r^*$ tends also to zero; here z_1^*, etc. denote the image of z_1 in the (u,v) plane. Since the integrals in (17.1) tend to a finite limit I, if n increases to infinity, the function $f(\varphi(u,v),\psi(u,v))\left|J(u,v)\right|$ is integrable on D, and I is equal to the integral of this function extended over the region D.

CHAPTER IV

Several theorems.

1. Introduction.

In this chapter three theorems are collected. The first theorem deals with the length of a certain curve. A lemma used in its proof is given first. The two other theorems deal with the analytic continuation of the Beta-function.

2. A lemma.

Lemma 2. If in the interval $0 < x < x_0$

$$y = \sum_{h=0}^{\infty} c_h x^{\alpha_h}, \tag{18.1}$$

where the exponents α_h increase to infinity with h, then in this interval the function y is differentiable with respect to x and one has

$$\frac{dy}{dx} = \sum_{h=0}^{\infty} c_h \alpha_h x^{\alpha_h - 1}. \tag{18.2}$$

Proof: Without loss of generality one may suppose that each exponent α_h is larger than one, since in the other case one needs only to treat separately a finite number of terms.

Be x_1 an arbitrary point between x and x_0. Since the series (18.1) converges for $x = x_1$, all terms of that series are bounded at $x = x_1$, so

$$|c_h x_1^{\alpha_h}| < C,$$

Dirichlet

where C denotes a fixed positive number.

From this it follows that the series mentioned in (18.2) converges for each x between 0 and x_1, and even uniformly in the interval $0 \leqslant x \leqslant x_2$ if x_2 lies between 0 and x_1.

The sum $\psi(x)$ of that series is therefore a continuous function of x and the series can be integrated term by term, so that one finds

$$\int_0^x \psi(t)dt = \sum_{h=0}^{\infty} c_h x^{\alpha_h} = y$$

at each x of the interval $0 \leqslant x \leqslant x_2$.

Since the integrand $\psi(t)$ is continuous, the left hand side is differentiable with respect to x, so that one gets

$$\frac{dy}{dx} = \psi(x) = \sum_{h=0}^{\infty} c_h \alpha_h x^{\alpha_h - 1}.$$

This result holds for all x between 0 and x_2, where x_2 may represent an arbitrarily chosen number between 0 and x_o so that the relation holds at each x between 0 and x_o.

3. The length of a certain curve.

Theorem 18: If in the interval $0 < x < x_o$ holds

$$\varphi(x) = \sum_{h=0}^{\infty} c_h\, x^{\alpha_h},$$

where each exponent α_h is larger than or equal to zero and α_h increases indefinitely with h and the coefficients c_h denote real numbers, then the curve defined by the equation $y = \varphi(x)$ has a finite length in the interval $0 < x < x_o$.

Proof: Choose a number x_1 between 0 and x_o. According to the preceding lemma

$$\varphi'(x) = \sum_{h=0}^{\infty} c_h \alpha_h\, x^{\alpha_h - 1}$$

exists at each point x between x_1 and x_o and therefore the integrals

$$\int_{x_1}^{x_o} \left\{1+(\varphi'(x))^2\right\}^{\frac{1}{2}} dx = \int_{x_1}^{x_o} \left\{1 + \left(\sum_{h=0}^{\infty} c_h \alpha_h\, x^{\alpha_h - 1}\right)^2\right\}^{\frac{1}{2}} dx \qquad (18.3)$$

exist. If α denotes the smallest positive exponent occurring in the expansion of $\varphi(x)$, then for small positive values of x the integrand is at most of the order $(1+ x^{\alpha-1})$, where $\alpha - 1 > -1$.

The integrals in (18.3) tend therefore to a finite limit as x_1 tends to zero and this limit is the length of the curve according to its definition.

4. Analytic continuation of the Beta-function.

Theorem 19. Let the function $r(w,\alpha,\beta)$ be defined by the relation

$$r(w,\alpha,\beta) = \frac{w^{\alpha}}{(1+w)^{\beta}} - P_N(w),$$

where α and β may be real or complex numbers but such that $\mathrm{Re}\,\beta > 0$, $\mathrm{Re}\,\alpha > -1$, and that $\mathrm{Re}\,\beta - \mathrm{Re}\,\alpha$ is not an integer ≤ 1; furthermore, $P_N(w)$ is the truncated binomial expansion of $\frac{w^{\alpha}}{(1+w)^{\beta}}$, i.e.

$$P_N(w) = w^{\alpha-\beta} \sum_{k=0}^{N} \binom{-\beta}{k} w^{-k}, \text{ if } \mathrm{Re}\,\beta < \mathrm{Re}\,\alpha + 1,$$

where N is the integer ≥ 0 such that $\mathrm{Re}\,\alpha \leq \mathrm{Re}\,\beta + N < \mathrm{Re}\,\alpha + 1$

and $P_N(w) = 0$ if $\mathrm{Re}\,\beta > \mathrm{Re}\,\alpha + 1$.

Under these conditions $r(w,\alpha,\beta)$ is integrable in the interval $0 \leqslant w \leqslant \infty$, and

$$\int_0^\infty r(w,\alpha,\beta)dw = B(\alpha+1,\beta-\alpha-1), \text{ where } B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

<u>Proof</u>: For $\mathrm{Re}\,\beta > \mathrm{Re}\,\alpha+1$ it is well known that

$$\int_0^\infty r(w,\alpha,\beta)dw = \int_0^\infty \frac{w^\alpha}{(1+w)^\beta} dw = B(\alpha+1,\beta-\alpha-1).$$

The function on the right hand side satisfies the recurrence relation

$$B(p,q) = \frac{p+q}{q} B(p,q+1).$$

By partial integration one gets

$$\int_0^H w^{\alpha-\beta} \left\{(1+w^{-1})^{-\beta} - \sum_{k=0}^N \binom{-\beta}{k} w^{-k}\right\} dw$$

$$= \frac{w^{\alpha-\beta+1}}{-\beta+\alpha+1} \left\{(1+w^{-1})^{\beta} - \sum_{k=0}^N \binom{-\beta}{k} w^{-k}\right\} \Bigg|_0^H +$$

$$+ \frac{\beta}{\beta-\alpha-1} \int_0^H w^{\alpha-\beta-1} \left\{(1+w^{-1})^{-\beta-1} - \sum_{k=0}^{N-1} \binom{-\beta-1}{k} w^{-k}\right\} dw.$$

Here

$$(1+w^{-1})^{-\beta} - \sum_{h=0}^N \binom{-\beta}{k} w^{-k}$$

is for large positive values of w at most of the order w^{-N-1} and therefore of smaller order than $w^{\mathrm{Re}\beta-\mathrm{Re}\alpha-1}$; the function under consideration is for small positive values of w at most of the order w^{-N} and therefore of smaller order than $w^{\mathrm{Re}\beta-\mathrm{Re}\alpha-1}$. Consequently the first term in the right hand side of the identity obtained above tends to zero as $H \to \infty$. Therefore

$$\int_0^\infty r(w,\alpha,\beta)dw = \frac{\beta}{\beta-\alpha-1} \int_0^\infty r(w,\alpha,\beta+1)dw, \tag{19.1}$$

assuming that the integral on the right hand side exists. That is true in the case that $\mathrm{Re}\,\beta > \mathrm{Re}\,\alpha$, so that according to (19.1) the integral of $r(w,\alpha,\beta)$ from 0 to ∞ exists also if $\mathrm{Re}\,\beta > \mathrm{Re}\,\alpha$.

Applying (19.1) with $\beta-1$ instead of with β one finds that

$$\int_0^\infty r(w,\alpha,\beta)dw$$

exists if $\mathrm{Re}\,\beta > \mathrm{Re}\,\alpha-1$. Continuing in this way one gets the result that the mentioned integral exists for each choice of α and β, provided, of course, that $\mathrm{Re}\,\beta - \mathrm{Re}\,\alpha$ is not an integer $\leqslant 1$.

Since this integral satisfies the same recurrence relation as $B(\alpha+1,\beta-\alpha-1)$ and since it is equal to $B(\alpha+1,\beta-\alpha-1)$ for $\mathrm{Re}\,\beta > \mathrm{Re}\,\alpha+1$ one has for $\mathrm{Re}\,\beta > 0$, $\mathrm{Re}\,\alpha > -1$ and $\mathrm{Re}\,\beta - \mathrm{Re}\,\alpha$ is not an integer ≤ 1 the result

$$\int_0^\infty r(w,\alpha,\beta)\,dw = B(\alpha+1,\beta-\alpha-1).$$

In an analogous way the following proposition is proved:

<u>Theorem 20</u>: Be $r(w,\alpha,\beta)$ the function defined by

$$r(w)= \frac{w^{\alpha}}{(1+w)^{\beta}} - Q_N(w),$$

where $\mathrm{Re}\,\beta > 0$, $\mathrm{Re}(\beta-\alpha-1) > 0$ and where $\mathrm{Re}\,\alpha$ is not an integer < 0: we choose

$$Q_N(w)= \sum_{k=0}^{N} \binom{-\beta}{k} w^{\alpha+k} , \text{ if } \mathrm{Re}\,\alpha < -1,$$

where N denotes the integer ≥ 0 defined by

$$-2 < \mathrm{Re}\,\alpha + N < -1,$$

and $Q_N(w)= 0$ if $\mathrm{Re}\,\alpha > -1$.

Then the function $r(w,\alpha,\beta)$ is integrable from zero to infinity and

$$\int_0^\infty r(w,\alpha,\beta)\,dw = B(\alpha+1,\beta-\alpha-1).$$

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