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MATHEMATICS

A TRANSFORMATION OF FORMAL SERIES. I

BY

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1. *Introduction.* It is well known that the computation of a function $f(z)$ of a real or complex variable z presents often difficulties when z is neither small nor large. For small values of z one often has at hand suitable expansions into ascending powers of z, whereas asymptotic expansions form an appropriate means to compute the function for large values of z. Even in the most favourable case, viz. that $f(z)$ is an entire function so that the ascending series is convergent everywhere, this ascending series is practically of little use unless z is relatively small. On the other hand, the asymptotic series is nearly always divergent. It represents a class of functions rather than a particular one, and even in the most favourable case, viz. that a suitable estimate of the remainderterm is available, it can only be used for large values of the argument. The larger the required accuracy is the more difficult it is to bridge the gap between small and large z. Often one has to look for other means for moderately large values of z. Even if these are available, one pays in losing the uniformity of computation, an argument that weighs heavily especially in automatic computing.

This paper is concerned with a very general transformation of formal, i.e. not necessarily convergent series containing a complex variable z. To such a formal series one can attribute a class of functions of z, and it is shown that when certain conditions are fulfilled the transform considered, in general again a formal series, is a convergent series, the sum of which is in a certain region of the complex z-plane, e.g. a halfplane, a well defined function in that class.

The transform is a series, each term of which is the product of a coefficient c_k and a function $s_k(z)$. The coefficients c_k do not depend on z but only on the particular formal series under consideration. The functions $s_k(z)$, called associates of a standardfunction $s(z)$ do depend on z but not on the particular formal series under consideration. The definition of the coefficients c_k is wholly constructive. In order to compute a finite number of them one has only to perform a finite number of elementary arithmetical operations. The associates of the standardfunction that take the place of

z^k in an ascending series or z^{-k} in a descending series have to be computed once and for all for various values of z.

There is an infinite variety of standardfunctions possible and the character and power of a transformation depends on the choice. For instance, one particular choice yields the well-known transformation of Euler. Another very interesting and much more powerful special transformation is investigated in more detail. Only very recently the author discovered that this particular case has been dealt with already by J. Ser [1]), but as far as he can conceive, only in a formal way. This particular transformation and in general all the transformations under consideration are here shown to be fully legitimate analytical tools. That they form a practical tool, moreover, in the computation of special functions is shown by some worked out examples. These examples are restricted since so far a standardfunction and its associates have only been tabulated for real values of z by the Computation Department of the Mathematical Centre at Amsterdam. As soon as more material is available for imaginary and complex values of z, much more interesting functions can be investigated.

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2. *The standardfunction and its associates.* Let two functions $S(t)$ and $e(z, t)$ be chosen of which nothing more is required than that the following three conditions are satisfied.

i) $S(t)$ is analytic for $t \geqslant 0$, and $S^{(k)}(0) \neq 0$ for all $k \geqslant 0$.

ii) There exists a region D of the complex z-plane in which holds for all $k \geqslant 0$

$$\int_0^\infty t^k\, e(z, t)\, dt < \infty, \quad \int_0^\infty t^k\, S^{(k)}(t)\, e(z, t) < \infty.$$

iii) There exists a subregion D^* of D in which holds for all $k \geqslant 0$

$$\int_0^\infty t^k\, |\, e(z, t)|\, dt < \infty, \quad \int_0^\infty t^k\, S^{(k)}(t)\, e(z, t) < \infty.$$

If one writes for the sake of abbreviation

$$S_k(t) = \frac{(-)^k}{k!}\, t^k S^{(k)}(t)\,,$$

whence $S_0(t) = S(t)$,
then a "standardfunction" $s(z) = s_0(z)$ and its "associates" $s_k(z)$ are defined in D by

$$s_k(z) = \int_0^\infty S_k(t)\, e(z, t)\, dt, \qquad k \geqslant 0.$$

One may attribute to $s(z)$ in D a formal series S that is obtained by

[1]) J. Ser, Bull. Sci. Math., 2e Ser., **60**, 199–202 (1936), **61**, 74–81 (1937), **62**, 171–182 (1938).

expanding $S(t)$ into a series of ascending powers of t and then interchanging the order of integration and summation, thus

$$s(z) \backsim S = \sum_{k=0}^{\infty} \frac{S^{(k)}(0)}{k!} \int_0^{\infty} t^k \, e(z,t) \, dt.$$

Apparently, the terms of this series exist, but it is not at all required that the series S should converge.

3. *The transformation.* Let be given a function $f(z)$ that in the region D is defined by the convergent integralrepresentation

$$f(z) = \int_0^{\infty} F(t) \, e(z,t) \, dt,$$

in which, moreover, the function $F(t)$ satisfies the condition

$$F(t) \text{ is analytic for } t \geqslant 0.$$

Again, a formal series F may be attributed to $f(z)$ in D that is obtained by expanding $F(t)$ into a series of ascending powers of t and then interchanging the order of integration and summation, thus

$$f(z) \backsim F = \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!} \int_0^{\infty} t^k \, e(z,t) \, dt.$$

Again, although the terms of this series exist, it is not required that the series F should converge. On the contrary, the most interesting situation arises if F is a divergent series so that it is not possible to sum it in the ordinary sense.

Now a transformation T is applied to the series F that is based on the comparison of the two series F and S. To that end one first introduces the ratio r_k between the terms of the two series, thus

$$r_k = \frac{F^{(k)}(0)}{S^{(k)}(0)}, \qquad k \geqslant 0.$$

Since $S^{(k)}(0) \neq 0$, this ratio r_k exists for all $k \geqslant 0$. Next the leading differences of the sequence r_k, provided with alternating signs, are introduced, thus

$$c_k = (-)^k \Delta^k r_0 = \sum_{h=0}^{k} (-)^h \binom{k}{h} r_h.$$

With these notations the transform TF of F is defined as follows

$$TF = \sum_{k=0}^{\infty} c_k \, s_k(z).$$

This transform is, therefore, also a series which may or may not converge. Whatever be the case, it should be realised that the functions $s_k(z)$ do not depend on the particular function $f(z)$ considered, whereas the calculation of a finite number of the coefficients c_k that do not depend on z involves only a finite number of elementary arithmetical operations.

The main object of this paper is, however, to show that under certain circumstances the transform TF converges and that its sum is $f(z)$.

4. *Sufficient conditions for convergence of TF.* A sufficient set of conditions for the convergence of TF is given by the following theorem, the proof of which reveals at the same time the background of the transformation.

Theorem 1. If using the notations and the conditions of sections 2 and 3, the series

$$U(t) = \sum_{k=0}^{\infty} |c_k S_k(t)|$$

is uniformly convergent for each positive t_0 in the neighbourhood of the interval $0 \leqslant t \leqslant t_0$, then for all z in D for which

$$\int_0^{\infty} U(t) \, |e(z, t)| \, dt < \infty$$

the transform TF of F is convergent, and its sum is $f(z)$.

Proof. The conditions are sufficient to insure that

$$TF = \sum_{k=0}^{\infty} c_k \, s_k(z) = \sum_{k=0}^{\infty} c_k \int_0^{\infty} S_k(t) \, e(z, t) \, dt = \int_0^{\infty} V(t) \, e(z, t) \, dt,$$

where the series

$$V(t) = \sum_{k=0}^{\infty} c_k S_k(t)$$

is uniformly convergent for each positive t_0 over the interval $0 \leqslant t \leqslant t_0$. Since each term $c_k S_k(t)$ is analytic for $t \geqslant 0$, also $V(t)$ is analytic for $t \geqslant 0$.

Hence it follows for $m \geqslant 0$

$$S_k^{(m)}(t) = \frac{(-)^k}{k!} \sum_{n=0}^{m} \binom{m}{n} \left(\frac{d^n t^k}{dt^n}\right) S^{(k+m-n)}(t),$$

whence

$$S_k^{(m)}(0) = \begin{cases} 0 & , \text{ if } 0 \leqslant m < k, \\ (-)^k \binom{m}{k} S^{(m)}(0), & \text{if } 0 \leqslant k \leqslant m. \end{cases}$$

Hence,

$$\begin{aligned} V^{(m)}(0) &= S^{(m)}(0) \sum_{k=0}^{m} (-)^k \binom{m}{k} c_k \\ &= S^{(m)}(0) \sum_{k=0}^{m} (-)^k \binom{m}{k} \sum_{h=0}^{k} (-)^h \binom{k}{h} r_h \\ &= S^{(m)}(0) \sum_{h=0}^{m} r_h \sum_{k=h}^{m} (-)^{k-h} \binom{m}{k} \binom{k}{h} \\ &= S^{(m)}(0) \sum_{h=0}^{m} r_h \binom{m}{h} \sum_{k=h}^{m} (-)^{k-h} \binom{m-h}{k-h} \\ &= S^{(m)}(0) \sum_{h=0}^{m} r_h \binom{m}{h} \sum_{l=0}^{m-h} (-)^l \binom{m-h}{l} \\ &= S^{(m)}(0) \sum_{h=0}^{m} r_h \binom{m}{h} (1-1)^{m-h} \\ &= S^{(m)}(0) \, r_m = F^{(m)}(0). \end{aligned}$$

Since both $V(t)$ and $F(t)$ are analytic for $t \geqslant 0$, they are identical for $t \geqslant 0$. If therefore, z lies in D, then

$$TF = \int_0^\infty F(t)\, e(z, t)\, dt = f(z),$$

what proves the theorem.

A weaker theorem, the conditions of which, however, lend themselves better to verification, is the following one.

Theorem 2. If using the notations and conditions of sections 2 and 3, the following three conditions are satisfied

i) $S(t)$ is not only analytic for $t \geqslant 0$ but also throughout the halfplane $\operatorname{Re} t > 0$;

ii) One can choose two fixed nonnegative numbers A and p, such that in the neighbourhood of $t = 0$ and throughout the halfplane $\operatorname{Re} t > 0$ holds $|S(t)| \leqslant A(1 + |t|)^p$;

iii) One can choose two fixed nonnegative numbers B and q, such that for all $k \geqslant 0$ holds $|c_k| \leqslant B(1 + k)^q$;

then for all z in D^* the transform TF of F is convergent, and its sum is $f(z)$.

Proof. From conditions i and ii it follows that there exists a positive number δ, such that $S(w)$ is analytic, and satisfies $|S(w)| \leqslant A(1 + |w|)^p$ if $|w| \leqslant 2\delta$ and also if $\operatorname{Re} w > 0$. The circle C with centre $t + \varepsilon$ ($t \geqslant 0$, $|\varepsilon| \leqslant \delta$) and with radius $(t^2 + \delta^2)^{\frac{1}{2}}$ lies completely in this region, and the origin $w = 0$ lies either on C or inside C. Hence, the maximum value of $|w|$ on C is at most equal to the diameter of C, whence on C, $|w| \leqslant 2(t^2 + \delta^2)^{\frac{1}{2}}$. From this it follows

$$|S^{(k)}(t + \varepsilon)| = \left| \frac{k!}{2\pi i} \int_C S(w)\, (w - t - \varepsilon)^{-k-1}\, dw \right| \leqslant k!\, A\, \{1 + 2(t^2 + \delta^2)^{\frac{1}{2}}\}^p \, (t^2 + \delta^2)^{-k/2}$$

whence

$$|S_k(t + \varepsilon)| \leqslant A\{1 + 2(t^2 + \delta^2)^{\frac{1}{2}}\}^p \{t(t^2 + \delta^2)^{-\frac{1}{2}}\}^k.$$

Moreover, it is no restriction to suppose q to be an integer. Then

$$(1 + k)^q \leqslant (1 + k)(2 + k) \ldots (q + k) = q!\, \frac{(q+1)(q+2)\ldots(q+k)}{k!} = (-)^k\, q! \binom{-q-1}{k},$$

whence according to condition iii

$$|c_k S_k(t + \varepsilon)| \leqslant q!\, AB\, (-)^k \binom{-q-1}{k} \{1 + 2(t^2 + \delta^2)^{\frac{1}{2}}\}^p \{t(t^2 + \delta^2)^{-\frac{1}{2}}\}^k.$$

Introducing the series $U(t)$ from Theorem 1, one has for $0 \leqslant t \leqslant t$

$$\begin{aligned}
U(t+\varepsilon) &= \sum_{k=0}^{\infty} |c_k S_k(t+\varepsilon)| \\
&\leqslant q!\,AB\,\{1+2(t^2+\delta^2)^{\frac{1}{2}}\}^p \sum_{k=0}^{\infty}(-)^k \binom{-q-1}{k} \{t(t^2+\delta^2)^{-\frac{1}{2}}\}^k \\
&= q!\,AB\,\{1+2(t^2+\delta^2)^{\frac{1}{2}}\}^p \{1-t(t^2+\delta^2)^{-\frac{1}{2}}\}^{-q-1} \\
&= q!\,AB\,\{1+2(t^2+\delta^2)^{\frac{1}{2}}\}^p (t^2+\delta^2)^{(q+1)/2} \{(t^2+\delta^2)^{\frac{1}{2}}-t\}^{-(q+1)} \\
&= q!\,AB\,\{1+2(t^2+\delta^2)^{\frac{1}{2}}\}^p (t^2+\delta^2)^{(q+1)/2} \delta^{-2(q+1)} \{(t^2+\delta^2)^{\frac{1}{2}}+t\}^{q+1} \\
&\leqslant q!\,AB\,(2\delta^{-2})^{q+1} \{1+2(t^2+\delta^2)^{\frac{1}{2}}\}^p (t^2+\delta^2)^{q+1} \\
&\leqslant q!\,AB\,(2\delta^{-2})^{q+1} \{1+2(t_0^2+\delta^2)^{\frac{1}{2}}\}^p (t_0^2+\delta^2)^{q+1},
\end{aligned}$$

so that the series $U(t)$ is uniformly convergent for each positive t_0 in the neighbourhood of the interval $0 \leqslant t \leqslant t_0$.

Moreover,

$$\int_0^{\infty} \{1+2(t^2+\delta^2)^{\frac{1}{2}}\}^p (t^2+\delta^2)^{q+1} |e(z,t)|\,dt$$

$$< (1+2\sqrt{2}\,\delta)^p \int_0^{\delta} (t^2+\delta^2)^{q+1} |e(z,t)|\,dt + \int_{\delta}^{\infty} (1+2\sqrt{2}\,t)^p (t^2+\delta^2)^{q+1} |e(z,t)|\,dt.$$

If z lies in D^* (and therefore in D), the above integrals exist even when the intervals of integration viz. $(0, \delta)$ resp. (δ, ∞) are replaced by the interval $(0, \infty)$. For this z holds therefore

$$\int_0^{\infty} U(t+\varepsilon)\,|e(z,t)|\,dt < \infty,$$

so that all conditions of Theorem 1 are satisfied. This proves Theorem 2.

5. *On a special class of functions.* In the following sections a special class of functions will be shown to be of fundamental importance for the subject under consideration. It is defined as follows.

Definition 1. A function $G(t)$ of the complex variable t is said to be a T-function if it satisfies the following two conditions:

i) $G(t)$ is analytic in the halfplane $\operatorname{Re} t > -\frac{1}{2}$;

ii) one can choose two fixed nonnegative numbers A and p such that throughout the halfplane $\operatorname{Re} t > -\frac{1}{2}$ one has

$$|G(t)| \leqslant A\left(1-\left|\frac{t}{1+t}\right|\right)^{-p} = A\left(\frac{|1+t|}{|1+t|-|t|}\right)^p.$$

These T-functions form a closed family with respect to addition, multiplication, differentiation and integration, as is shown by the following theorems.

Theorem 3. If $G_1(t)$ and $G_2(t)$ are both T-functions then also their sum $G_1(t)+G_2(t)$ and their product $G_1(t)\,G_2(t)$ are both T-functions.

Proof. One knows that in the halfplane $\operatorname{Re} t > -\frac{1}{2}$

$$|G_1(t)| \leqslant A_1 \left(\frac{|1+t|}{|1+t|-|t|}\right)^{p_1}, \qquad |G_2(t)| \leqslant A_2 \left(\frac{|1+t|}{|1+t|-|t|}\right)^{p_2}.$$

Hence,

$$|G_1(t)+G_2(t)| \leqslant |G_1(t)| + |G_2(t)| \leqslant A \left(\frac{|1+t|}{|1+t|-|t|}\right)^{p},$$

where $A = A_1 + A_2$ and p is the greatest of the two numbers p_1 and p_2.

Similarly,

$$|G_1(t)\, G_2(t)| = |G_1(t)|\, |G_2(t)| \leqslant A \left(\frac{|1+t|}{|1+t|-|t|}\right)^{p},$$

where $A = A_1 A_2$ and $p = p_1 + p_2$.

Moreover, $G_1(t) + G_2(t)$ and $G_1(t)\, G_2(t)$ are both analytic in the halfplane $\operatorname{Re} t > -\frac{1}{2}$, whence they are both T-functions.

Since a constant and t are both T-functions it follows by repeated application of theorem 3 that each polynomial in t is a T-function.

Theorem 4. If $G(t)$ is a T-function having a zero at t_0 of order $\geqslant k$ then $(t-t_0)^{-k} G(t)$ is a T-function.

Proof. If $\operatorname{Re} t_0 \leqslant -\frac{1}{2}$, then $(t-t_0)^{-k}$ is itself a T-function, whence the proof follows from theorem 3. If $\operatorname{Re} t_0 > -\frac{1}{2}$ then $G(t)$ is analytic, in t_0, and due to its zero of order $\geqslant k$, $(t-t_0)^{-k} G(t)$ is analytic, whence bounded, within a circle of sufficiently small radius δ and centre t_0. Outside and on that circle $|(t-t_0)^{-k}| \leqslant \delta^{-k}$, and $(t-t_0)^{-k}$ is analytic. Hence, $(t-t_0)^{-k} G(t)$ is analytic in the halfplane $\operatorname{Re} t > -\frac{1}{2}$, and it satisfies conditions of the type mentioned in definition 1, sub ii, both inside and outside the circle, which may be combined into one condition valid in the complete halfplane.

Theorem 5. If $G(t)$ is a T-function then its derivative $G'(t)$ is a T-function.

Proof. Consider the circle C in the complex τ-plane, having its centre at t ($\operatorname{Re} t > -\frac{1}{2}$), and with radius ϱ, where

$$\varrho = \frac{|1+t|-|t|}{4} = \frac{1}{4}\,\frac{|1+t|^2-|t|^2}{|1+t|+|t|} = \frac{2\operatorname{Re} t+1}{4(|1+t|+|t|)} \leqslant \tfrac{1}{2}\left(\operatorname{Re} t + \tfrac{1}{2}\right),$$

so that C lies entirely within the halfplane $Re\, \tau > -\frac{1}{2}$ in which $G(\tau)$ is analytic. Hence, $G'(t) = (2\pi i)^{-1} \int_C G(\tau)\, (\tau - t)^{-2}\, d\tau$.

On C is $|\tau - t| = \varrho$, $|\tau| \leqslant |t| + \varrho$, $|1+\tau| \geqslant |1+t| - \varrho$, whence on C

$$|G(\tau)| \leqslant A \left(1 - \left|\frac{\tau}{1+\tau}\right|\right)^{-p} \leqslant A \left(1 - \frac{|t|+\varrho}{|1+t|-\varrho}\right)^{-p} = A \left(\frac{|1+t|-|t|-2\varrho}{|1+t|-\varrho}\right)^{-p}$$

$$\leqslant A \left(\frac{|1+t|}{|1+t|-|t|-2\varrho}\right)^{p} = 2^p A \left(\frac{|1+t|}{|1+t|-|t|}\right)^{p}.$$

Hence,

$$|G'(t)| \leqslant \frac{2^{p+2} A}{|1+t|} \left(\frac{|1+t|}{|1+t|-|t|}\right)^{p+1} \leqslant 2^{p+3} A \left(\frac{|1+t|}{|1+t|-|t|}\right)^{p+1},$$

and since $G'(t)$ is, of course, analytic in the halfplane $\operatorname{Re} t > -\frac{1}{2}$, it is a T-function.

Theorem 6. If $G(t)$ is a T-function, then the integral $\int_a^t G(\tau)d\tau$, where a, t and the path of integration ly entirely within the halfplane $\operatorname{Re} t > -\frac{1}{2}$, is a T-function.

Proof. One may choose as path of integration the straight line L_1 joining a and the origin $\tau = 0$ and the straight line L_2 joining the origin and t. The integral over L_1 is a constant, independent of t, thus a T-function. The length of L_2 is $|t|$, and the maximum value of $|\tau(1+\tau)^{-1}|$ along L_2 is $|t(1+t)^{-1}|$. Hence,

$$\left|\int_0^t G(\tau)\,d\tau\right| \leqslant A\,|t|\left(\frac{|1+t|}{|1+t|-|t|}\right)^p \leqslant A\,(|1+t|-|t|)\left(\frac{|1+t|}{|1+t|-|t|}\right)^{p+1}$$
$$\leqslant A\left(\frac{|1+t|}{|1+t|-|t|}\right)^{p+1},$$

and, of course, it is analytic, whence a T-function. The complete integral is the sum of two T-functions, and consequently a T-function.

By repeated application of the theorems 5 and 6 it follows that all derivatives and all repeated integrals of a T-function are T-functions.

6. *The order of magnitude of the coefficients* c_k. The conditions of theorem 2 clearly are of two different types. The first and second condition are restrictions on $S(t)$ only. Since $S(t)$ is, according to section 2, a function that one may choose at his own convenience out of a very general class of functions, it only means that the choice of $S(t)$ is somewhat more limited, still leaving considerable freedom however. On the other hand, the third condition requires something about the order of magnitude of the coefficients c_k for large values of k. The coefficients c_k are, apart from the signs, the leading differences of the sequence of the coefficients r_k that depend again on both the given function $F(t)$ and the chosen function $S(t)$. Of course, one can compute arbitrarily many coefficients c_k, but it is by no means obvious how to give in general an estimate of $|c_k|$ for large values of k. If, however, one has sufficient information about the generating function of the sequence r_k, then the following two theorems yield the required information about the differences c_k.

Theorem 7. Let the two functions $G(t)$ and $H(t)$ of the complex variable t be connected by the two equivalent relations

$$G(t) = (1+t)^{-1}\,H\{t(1+t)^{-1}\},$$
$$H(t) = (1-t)^{-1}\,G\{t(1-t)^{-1}\}.$$

If one of both functions is analytic in $t = 0$ then so is the other. If in that case they are represented in the neighbourhood of $t = 0$ by the series

$$G(t) = \sum_{k=0}^{\infty} (-)^k\, r_k\, t^k,$$

$$H(t) = \sum_{k=0}^{\infty} c_k\, t^k,$$

then

$$c_k = (-)^k\, \Delta^k\, r_0.$$

Proof. First of all, if one puts for a moment $u = t(1+t)^{-1}$ then $t = u(1-u)^{-1}$ and $(1+t)^{-1} = 1-u$, whence from $G(t) = (1+t)^{-1} H\{t(1+t)^{-1}\}$ it follows that $H(u) = (1-u)^{-1} G\{u(1-u)^{-1}\}$ so that the given relations between $G(t)$ and $H(t)$ are indeed equivalent. Since $(1+t)^{-1}$ and $(1-t)^{-1}$ are analytic in the neighbourhood of $t = 0$, if one of both functions is analytic in $t = 0$ then so is the other, and the following derivation is seen to be legitimate.

$$
\begin{aligned}
H(t) &= \sum_{k=0}^{\infty} (-)^k r_k t^k (1-t)^{-k-1} = \sum_{k=0}^{\infty} (-)^k r_k \sum_{j=0}^{\infty} (-)^j \binom{-k-1}{j} t^{k+j} \\
&= \sum_{k=0}^{\infty} (-)^k r_k \sum_{j=0}^{\infty} \binom{k+j}{k} t^{k+j} = \sum_{h=0}^{\infty} t^h \sum_{k=0}^{h} (-)^k \binom{h}{k} r_k \\
&= \sum_{h=0}^{\infty} (-)^h \Delta^h r_0 t^h = \sum_{h=0}^{\infty} c_h t^h.
\end{aligned}
$$

In many simple cases this theorem yields an easy means to give a closed expression for c_k, from which not only its behaviour for large values of k is known, but that also renders good service for the computation. In other less simple cases it provides often at least suitable recurrence relations between successive c_k's. Anyhow, the behaviour of c_k for large values of k as far as it is of importance for the subject under consideration, follows from the following theorem.

Theorem 8. A necessary and sufficient condition in order that one can choose two fixed nonnegative numbers B and q such that $c_k = (-)^k \Delta^k r_0$ satisfies for all $k \geqslant 0$ the equation

$$
|c_k| \leqslant B(1+k)^q,
$$

is that there is a T-function $G(t)$ that has in the neighbourhood of $t = 0$ the expansion

$$
G(t) = \sum_{k=0}^{\infty} (-)^k r_k t^k.
$$

Proof. First it will be shown that the condition is necessary. Using the notations of theorem 7 it follows from $|c_k| \leqslant B(1+k)^q$ that for $|t| < 1$ the series $H(t) = \sum_{k=0}^{\infty} c_k t^k$ is convergent. Hence $H(t)$ is analytic for $|t| < 1$ and consequently the function $G(t) = (1+t)^{-1} H\{t(1+t)^{-1}\}$ is analytic for values of t such that $(1+t)^{-1}$ is analytic and $|t(1+t)^{-1}| < 1$, i.e. in the halfplane $\operatorname{Re} t > -\frac{1}{2}$. Moreover this function $G(t) = \sum_{k=0}^{\infty} (-)^k r_k t^k$ in the neighbourhood of $t = 0$ according to theorem 7. Now in the halfplane $\operatorname{Re} t > -\frac{1}{2}$, where $|1+t| \geqslant \frac{1}{2}$,

$$
\begin{aligned}
|G(t)| &= |1+t|^{-1} |H\{t(1+t)^{-1}\}| \leqslant 2B \sum_{k=0}^{\infty} (1+k)^q \left|\frac{t}{1+t}\right|^k \\
&\leqslant 2B \sum_{k=0}^{\infty} (1+k)(2+k)\dots(q+k) \left|\frac{t}{1+t}\right|^k = 2Bq! \sum_{k=0}^{\infty} \binom{k+q}{k} \left|\frac{t}{1+t}\right|^k \\
&= 2Bq! \sum_{k=0}^{\infty} (-)^k \binom{-q-1}{k} \left|\frac{t}{1+t}\right|^k = 2Bq! \left(1 - \left|\frac{t}{1+t}\right|\right)^{-(q+1)}
\end{aligned}
$$

and choosing $A = 2Bq!$ and $p = 1 + q$ it follows $|G(t)| \leqslant A(1 - |t/1+t|)^{-p}$, whence it is seen that $G(t)$ is a T-function.

In order to prove that the condition is sufficient one starts with the T-function $G(t)$ that has in the neighbourhood of $t = 0$ the expansion

$$G(t) = \sum_{h=0}^{\infty} (-)^h r_h t^h = \sum_{h=0}^{\infty} c_h t^h (1+t)^{-h-1},$$

whence

$$(1+t)^k G(t) = \sum_{h=0}^{k-1} c_h t^h (1+t)^{k-h-1} + c_k t^k (1+t)^{-1} + \sum_{h=k+1}^{\infty} c_h t^h (1+t)^{k-h-1}.$$

In the first sum the highest power of t after developing the polynomial $(1+t)^{k-h-1}$, where $h \leqslant k-1$, is t^{k-1}. In the second sum the lowest power of t after expansion of the rational function $(1+t)^{k-h-1}$, where $h \geqslant k+1$, in ascending powers of t, is t^{k+1}. Hence only the separate term contributes to a term containing t^k, and it follows that c_k is the coefficient of t^k in the expansion of $(1+t)^k G(t)$ in ascending powers of t.

Hence, if C is any simple closed contour in the complex t-plane around the origin $t = 0$ and lying entirely within the halfplane $\operatorname{Re} t > -\frac{1}{2}$, where $G(t)$ is analytic since it is a T-function, then

$$c_k = \frac{1}{2\pi i} \int_C (1+t)^k t^{-k-1} G(t)\, dt.$$

Now for C the circle is chosen on which

$$\left|\frac{1+t}{t}\right| = \cosh 2\varepsilon_k,$$

where ε_k is a positive number depending on k which will be specified later on. The equation of C in coordinates is

$$(\operatorname{Re} t - \sinh^{-2} 2\varepsilon_k)^2 + (\operatorname{Im} t)^2 = (\cosh 2\varepsilon_k \sinh^{-2} 2\varepsilon_k)^2.$$

The point on C nearest to the origin $t = 0$ has, therefore, the coordinates

$$\operatorname{Re} t = \sinh^{-2} 2\varepsilon_k(1 - \cosh 2\varepsilon_k) = -\tfrac{1}{2}\cosh^{-2} \varepsilon_k, \quad \operatorname{Im} t = 0,$$

whence on C

$$|t|^{-1} \leqslant 2\cosh^2 \varepsilon_k.$$

Moreover, the radius of C is $\cosh 2\varepsilon_k \sinh^{-2} 2\varepsilon_k$, whence

$$|c_k| \leqslant \cosh^k 2\varepsilon_k \cdot 2\cosh^2 \varepsilon_k\, A(1 - \cosh^{-1} 2\varepsilon_k)^{-p} \cosh 2\varepsilon_k \sinh^{-2} 2\varepsilon_k =$$
$$= 2^{-(1+p)} A \cosh^{1+k} 2\varepsilon_k \cosh^p 2\varepsilon_k \sinh^{-2(1+p)} \varepsilon_k.$$

Now from the Moivre's theorem

$$\cosh nx = \cosh^n x + \binom{n}{2} \cosh^{n-2} \sinh^2 x + \binom{n}{4} \cosh^{n-4} \sinh^4 x + \ldots,$$

it follows $\cosh^n x \leqslant \cosh nx$. Hence, if one chooses $2\varepsilon_k = (1+k)^{-1}$ then

$$\cosh^{1+k} 2\varepsilon_k = \cosh^{1+k} \frac{1}{1+k} \leqslant \cosh 1,$$

$$\cosh^p 2\varepsilon_k = \cosh^p \frac{1}{1+k} \leqslant \cosh^p 1,$$

$$\sinh^{-2(1+p)} \varepsilon_k = \sinh^{-2(1+p)} \frac{1}{2(1+k)} < 2^{2(1+p)} (1+k)^{2(1+p)},$$

whence

$$|c_k| \leqslant (2 \cosh 1)^{1+p} A (1+k)^{2(1+p)}.$$

Choosing $B = (2 \cosh 1)^{1+p} A$ and $q = 2(1+p)$, one has therefore $|c_k| \leqslant B(1+k)^q$, what completes the proof.

7. *The generating function* $G(t)$. In the last section it has become obvious how important it is to know the analytic character of the generating function $G(t)$ that in the neighbourhood of $t = 0$ is given by the series

$$G(t) = \sum_{k=0}^{\infty} (-)^k r_k t^k = \sum_{k=0}^{\infty} (-)^k \frac{F^{(k)}(0)}{S^{(k)}(0)} t^k.$$

If $S(t)$ once has been chosen the transformation from $F(t)$ to $G(t)$ is of a rather obscure character that renders it difficult to derive in general the properties of $G(t)$ from those of $F(t)$. However, there is a class of functions $S(t)$ for which just enough information about $G(t)$ can be drawn from the properties of $F(t)$ as is necessary for the subject under consideration. Rather then to describe this class in general only some examples will be given.

The simplest example is provided by $S(t) = (1+t)^{-1}$. Then $S^{(k)}(0) = (-)^k k!$, whence $G(t) = F(t)$. Again, taking $S(t) = (1+t)^{-2} = -(d/dt)(1+t)^{-1}$, one has $S^{(k)}(0) = (-)^k (k+1)!$, whence

$$G(t) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} F^{(k)}(0) t^k = t^{-1} \int_0^t F(\tau)\, d\tau.$$

More generally, if $S(t) = (-d/dt)^n (1+t)^{-1}$, where $n \geqslant 0$, then $S^{(k)}(0) = (-)^k (k+n)!$, whence

$$G(t) = \sum_{k=0}^{\infty} \frac{1}{(k+n)!} F^{(k)}(0) t^k = t^{-n} \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \dots \int_0^{\tau_2} F(\tau_1)\, d\tau_1.$$

Conversely, if one takes with non-vanishing a_0, $S(t) = a_0 - \log(1+t) = a_0 - \int_0^t (1+\tau)^{-1} d\tau$, then $S(0) = a_0$, and for $k > 0$, $S^{(k)}(0) = (-)^k (k-1)!$, whence

$$G(t) = \frac{F(0)}{a_0} + \sum_{k=1}^{\infty} \frac{1}{(k-1)!} F^{(k)}(0) t^k = \frac{F(0)}{a_0} + t F'(t).$$

More generally again, if one takes with non-vanishing $a_0, a_1, \dots, a_{n-1}$,

$$S(t) = a_0 - a_1 t + \dots + (-)^{n-1} a_{n-1} t^{n-1} + (-)^n \int_0^t d\tau_n \int_0^{\tau_n} d\tau_{n-1} \dots \int_0^{\tau_2} (1+\tau_1)^{-1} d\tau_1,$$

then for $0 \leqslant k \leqslant n-1$, $S^{(k)}(0) = (-)^k k!\, a_k$, and for $k \geqslant n$, $S^{(k)}(0) = (-)^k (k-n)!$, whence

$$G(t) = \frac{F(0)}{a_0} + \frac{F'(0)}{1!\, a_1} t + \dots \frac{F^{(n-1)}(0)}{(n-1)!\, a_{n-1}} t^{n-1} + \sum_{k=n}^{\infty} \frac{1}{(k-n)!} F^{(k)}(0)\, t$$

$$= \frac{F(0)}{a_0} + \frac{F'(0)}{1!\, a_1} t + \dots \frac{F^{(n-1)}(0)}{(n-1)!\, a_{n-1}} t^{n-1} + t^n F^{(n)}(t).$$

In other cases, a little manipulation may yield similar results. For instance, if one takes

$$S(t) = 1 - \sum_{k=1}^{\infty} (-)^k k^{-2} t^k, \text{ one has } S(0) = 1, \text{ and for } k \geqslant 1,$$

$S^{(k)}(0) = (-)^k k^{-1} (k-1)!$, whence

$$G(t) = F(0) + \sum_{k=1}^{\infty} \frac{kF^{(k)}(0)}{(k-1)!} t^k = F(0) + \sum_{k=1}^{\infty} \left\{ \frac{1}{(k-1)!} + \frac{1}{(k-2)!} \right\} F^{(k)}(0)\, t^k =$$
$$= F(0) + t\,F'(t) + t^2 F''(t).$$

Now, in all these and similar cases, $G(t)$ is expressed as the sum of a polynomial in t and derivatives and repeated integrals of $F(t)$, the last ones being divided by as many factors t as integrations from 0 onwards have been performed. From the theorems of section 5, it then follows precisely that in all these cases $G(t)$ is a T-function if $F(t)$ is a T-function, so that in order to settle the problem of the convergence one has only to consider the function $F(t)$ and can forget about $G(t)$.

In order to show the results so far obtained, a special but important theorem follows. It should be remembered, however, from what has been said just now, that with minor modifications the theorem not only holds for the special function $S(t)$ mentioned but for all functions $S(t)$ of the class treated above.

Theorem 9. If using the notations of section 2 and 3, and with $\operatorname{Re} \alpha > 0$, one has

i) $S(t) = (1 + \alpha t)^{-1}$,
ii) $F(t/\alpha)$ is a T-function;

then for all z in D^* the transform TF of F is convergent, and its sum is $f(z)$.

Proof. The only singular point of $S(t)$ is $t = -\alpha^{-1}$, and since $\operatorname{Re} \alpha > 0$, $S(t)$ is analytic if $\operatorname{Re} t > 0$ and in the neighbourhood of $t = 0$. Since moreover $|S(t)|$ is bounded if $\operatorname{Re} t > 0$, $S(t)$ satisfies the first and second condition of Theorem 2. Next, $S^{(k)}(0) = (-)^k k!\, \alpha^k$, whence

$$G(t) = \sum_{k=0}^{\infty} (-)^k \frac{F^{(k)}(0)}{S^{(k)}(0)} t^k = \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!} \alpha^{-k} t^k = F\left(\frac{t}{\alpha}\right).$$

Hence $G(t)$ is a T-function, whence in virtue of theorem 8 it follows that also the third condition of theorem 2 is satisfied, whence theorem 2 provides the required result.

MATHEMATICS

A TRANSFORMATION OF FORMAL SERIES. II

BY

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(Communication MR 13 of the Computation Department of the Mathematical Centre at Amsterdam)

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8. *A special choice of* $e(z, t)$. So far, hardly any limitations have been put onto the function $e(z, t)$. Actually, in the theorems in the preceding sections, it only appears in that z should lie in the region D or D^*. Now a special choice is made for $e(z, t)$, viz.

$$e(z, t) = ze^{-zt}. \tag{8.1}$$

This special case of the transformation is so important that it will be kept to in the following sections. Apparently the theory is from now on more closely related to that of the Laplace transform. First of all for all z in the halfplane $\operatorname{Re} z > 0$, and for all $k \geqslant 0$

$$z \int_0^\infty t^k e^{-zt}\, dt < \infty \;,\; |z| \int_0^\infty t^k \,|e^{-zt}|\, dt < \infty,$$

whence in many cases, viz. if $S^{(k)}(t)$ and $F(t)$ behave properly, the regions D and D^* are simply the halfplane $\operatorname{Re} z > 0$.

Next, the formal series F for $f(z) = z \int_0^\infty F(t)\, e^{-zt}\, dt$ is now

$$f(z) \sim F = \sum_{k=0}^{\infty} F^{(k)}(0)\, z^{-k}, \tag{8.2}$$

and from the theory of the Laplace transform one knows that under rather liberal conditions on $f(z)$ this series is an asymptotic series in the proper sense. Hence in many cases the transform under consideration can be used to sum asymptotic series to a special function out of the whole class of functions that have that asymptotic expansion, viz. to the one that is representable as a Laplace-integral of a certain type.

The particular form of $e(z, t)$ makes it possible to derive many properties from the standardfunction $s(z)$ and its associates $s_k(z)$. If it is supposed that for all z in D also $\lim_{t \to \infty} S^{(k)}(t)\, t^h\, e^{-zt} = 0$ for all $k \geqslant 0$, then it follows by repeated partial integration of

$$s_k(z) = \frac{(-)^k}{k!}\, z \int_0^\infty S^{(k)}(t)\, t^k\, e^{-zt}\, dt$$

that

$$s_k(z) = \frac{1}{k!}\, z \int_0^\infty S(t) \left(\frac{\partial}{\partial t}\right)^k (t^k\, e^{-zt})\, dt.$$

Introducing the Laguerre polynomial $L_k(x)$, defined by

$$L_k(x) = e^x \left(\frac{d}{dx}\right)^k (x^k e^{-x}) = k! \sum_{h=0}^{k} (-)^h \binom{k}{h} \frac{x^h}{h!}, \tag{8.3}$$

one can write, therefore

$$s_k(z) = \frac{1}{k!} z \int_0^\infty S(t)\, L_k(zt)\, e^{-zt}\, dt. \tag{8.4}$$

On the other hand if for a certain value of k, K say, holds

$$s^{(k)}(z) = (-)^k \int_0^\infty S(t)\, (zt-k)\, t^{k-1} e^{-zt}\, dt, \tag{8.5}$$

then

$$s^{(K+1)}(z) = (-)^K \int_0^\infty S(t)\, (-zt^2 + Kt + t)\, t^{K-1} e^{-zt}\, dt =$$

$$= (-)^{K+1} \int_0^\infty S(t)\, (zt-K-1)\, t^K e^{-zt}\, dt,$$

whence (8.5) also holds for $k = K + 1$. Since it holds for $k = 0$, it holds for all $k \geqslant 0$. Now consider for $k > 0$ the following sum

$$\sum_{h=1}^{k} \binom{k-1}{h-1} \frac{1}{h!} s^{(h)}(z)\, z^h = \int_0^\infty S(t) \left\{ \sum_{h=1}^{k} (-)^h \binom{k-1}{h-1} \frac{1}{h!} (zt-h)\, z^h t^{h-1} \right\} e^{-zt}\, dt$$

$$= z \int_0^\infty S(t) \left\{ \sum_{h=1}^{k} (-)^h \binom{k-1}{h-1} \frac{1}{h!} (zt)^h + \sum_{h=1}^{k} (-)^{h-1} \binom{k-1}{h-1} \frac{1}{(h-1)!} (zt)^{h-1} \right\} e^{-zt}\, dt$$

$$= z \int_0^\infty S(t) \left\{ \sum_{h=1}^{k} (-)^h \binom{k-1}{h-1} \frac{1}{h!} (zt)^h + \sum_{h=0}^{k-1} (-)^h \binom{k-1}{h} \frac{1}{h!} (zt)^h \right\} e^{-zt}\, dt$$

$$= z \int_0^\infty S(t) \left\{ \sum_{h=0}^{k} (-)^h \binom{k}{h} \frac{1}{h!} (zt)^h \right\} e^{-zt}\, dt = \frac{1}{k!} z \int_0^\infty S(t)\, L_k(zt)\, e^{-zt}\, dt.$$

Comparing this with (8.4) one finds

$$\begin{cases} s_0(z) = s(z), \\ s_k(z) = \displaystyle\sum_{h=1}^{k} \binom{k-1}{h-1} \frac{1}{h!} s^{(h)}(z)\, z^h, \qquad k > 0. \end{cases} \tag{8.6}$$

The very first application of the transformation under consideration will be the computation of the standardfunction itself. Indeed let be

$$F(t) = S\{(1-w)t\}, \quad |w| < 1.$$

Then for real w, and $\operatorname{Re} z > 0$, one has

$$f(z) = z \int_0^\infty S\{(1-w)\, t\}\, e^{-zt}\, dt = \frac{z}{1-w} \int_0^\infty S(t)\, e^{-\frac{z}{1-w}}\, dt = s\left(\frac{z}{1-w}\right),$$

and by analytic continuation it holds also for complex values of w. Also, if $z(1-w)^{-1}$ lies outside the region $D^* = D$, viz. the halfplane $\operatorname{Re} z > 0$ in which $s(z)$ is defined originally, then the formula yields the analytic continuation of $s(z)$.

Now, $F^{(k)}(0) = (1-w)^k S^{(k)}(0)$, whence $r_k = (1-w)^k$ and

$$c_k = (-)^k \Delta^k r_0 = w^k.$$

Hence, if $S(t)$ satisfies the conditions of theorem 2, and if $|w| \leqslant 1$, $w \neq 1$, then $|c_k| \leqslant 1$ for all $k \geqslant 0$ so that also the third condition is fulfilled, one has

$$s\left(\frac{z}{1-w}\right) = \sum_{k=0}^{\infty} w^k s_k(z), \qquad |w| \leqslant 1, w \neq 1, \operatorname{Re} z > 0. \tag{8.7}$$

Moreover if $\lim_{z\to\infty} s(z) = s(\infty)$ exists then $s(\infty) = \lim_{w\to 1} \sum_{k=0}^{\infty} w^k s_k(z)$, and since $\sum_{k=0}^{\infty} s_k(z)$ is convergent, indeed even $\sum_{k=0}^{\infty} (1+k)^q s_k(z)$ is convergent, it follows from Abel's theorem that in that case

$$s(\infty) = \sum_{k=0}^{\infty} s_k(z). \tag{8.8}$$

The formulae (8.7) and (8.8) are of great importance for the computation of the standardfunction. As soon as $s_k(z)$, $k \geqslant 0$ is known for a particular value of z then formula (8.7) yields an easy means to compute $s(z)$ in a considerable domain. Moreover formula (8.8) yields a simple and efficient check on the sequence $s_k(z)$ itself. At last formula (8.7) reveals the true character of the associates $s_k(z)$ if $e(z, t) = ze^{-zt}$. They are simply the coefficients in the expansion in ascending powers of w of the function $s\{z(1-w)^{-1}\}$. From this remark also (8.6) may be derived.

9. *Special transformations.* A complete transformation is defined by giving the two functions $S(t)$ and $e(z, t)$. A familiar result is obtained by choosing

$$S(t) = e^{-t} \ , \ e(z, t) = ze^{-zt}. \tag{9.1}$$

One has

$$s_k(z) = \frac{z}{k!} \int_0^{\infty} t^k \cdot e^{-(z+1)t}\, dt = z(z+1)^{-k-1},$$

and

$$S^{(k)}(0) = (-)^k.$$

One sees that this transformation is nothing else than the general Euler transformation. Often one applies this transformation directly to the complete terms of the series, here $F^{(k)}(0) z^{-k}$ rather than to the coefficients $F^{(k)}(0)$ only. This means that effectively one takes $z = 1$. Then $s_k(1) = 2^{-k-1}$, what yields the ordinary Euler transformation.

A much more powerful transformation is defined by

$$S(t) = (1+t)^{-1} \ , \ e(z, t) = ze^{-zt}. \tag{9.2}$$

One has

$$S^{(k)}(0) = (-)^k k! \tag{9.3}$$

The corresponding standardfunction is then

$$s(z) = z \int_0^{\infty} (1+t)^{-1} e^{-zt}\, dt = -ze^z Ei(-z). \tag{9.4}$$

and its associates are

$$s_k(z) = z \int_0^\infty (1 + t)^{-k-1} t^k e^{-zt}\, dt, \qquad k \geqslant 0. \tag{9.5}$$

In the following sections examples are given of the application of this particular transformation. Here and there numerical examples are given for real values of z. The required values of the functions $s_k(z)$ have been taken from the tables computed by the Computation Department of the Mathematical Centre. These tables together with a more detailed analysis of the properties of the functions $s_k(z)$ will form the subject of a separate paper. Here only a few remarks will be made that do not require much analysis and reveal some salient features.

From (9.5) it follows directly that for positive real values of z, the associates $s_k(z)$ decrease steadily to zero when k increases indefinitely. Moreover $s(\infty) = 1$, so that according to (8.8) one has

$$\sum_{k=0}^{\infty} s_k(z) = 1. \tag{9.6}$$

Hence, for $z > 0$ all $s_k(z)$ are numerically less than unity.

With fixed $z > 0$ the convergence with respect to k is rather poor. On the other hand, if z increases, the initial convergence with respect to k becomes better and better, so that the transform TF is never inferior to the asymptotic series F.

10. *Computation of the error integral.* The first example will be the computation of the error integral

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt. \tag{10.1}$$

First of all, this function must be brought into a suitable form. To this end, one writes

$$\Phi(z) = 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt = 1 - \frac{1}{\sqrt{\pi}} \int_{z^2}^\infty e^{-u} u^{-\frac{1}{2}}\, du$$

$$= 1 - \frac{e^{-z^2}}{\sqrt{\pi}} \int_0^\infty (t^2 + z^2)^{-\frac{1}{2}} e^{-t}\, dt = 1 - \frac{e^{-z^2}}{z\sqrt{\pi}} z^2 \int_0^\infty (1 + t)^{-\frac{1}{2}} e^{-z^2 t}\, dt$$

whence

$$\Phi(z) = 1 - \frac{e^{-z^2}}{z\sqrt{\pi}} f(z^2), \tag{10.2}$$

where

$$f(z) = z \int_0^\infty (1 + t)^{-\frac{1}{2}} e^{-zt}\, dt. \tag{10.3}$$

Now, $f(z)$ has the appropriate form, but not yet the most advantageous one. Indeed, $F(t) = G(t) = (1 + t)^{-\frac{1}{2}}$, what is a T-function, but its only singularity is in $t = -1$. If instead one uses

$$f(z) = 2z \int_0^\infty (1 + 2t)^{-\frac{1}{2}} e^{-(2z)t}\, dt, \tag{10.4}$$

one has $F(t) = G(t) = (1 + 2t)^{-\frac{1}{2}}$, which is also a T-function, but now its singularity occurs in $t = -\frac{1}{2}$. The advantage is that in the transformation the standardfunction and its associates will have an argument twice as large as it would be in the original form, so that one can expect a much better convergence. Actually, in this particular case the advantage is even much more striking since it will appear that now one half of the coefficients vanish. These coefficients follow directly from the generating function $H(t)$, as follows

$$H(t) = \frac{1}{1-t} G\left(\frac{t}{1-t}\right) = \frac{1}{1-t}\left(1 + \frac{2t}{1-t}\right)^{-\frac{1}{2}} = (1-t^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} 2^{-2k} \binom{2k}{k} t^{2k}.$$

Hence, one has

$$c_{2k+1} = 0, \qquad c_{2k} = 2^{-2k}\binom{2k}{k} = O(k^{-\frac{1}{2}}).$$

Finally one obtains therefore the following expansion, valid if $|\arg z| < \pi/4$,

$$\Phi(z) = 1 - \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{k=0}^{\infty} 2^{-2k}\binom{2k}{k} s_{2k}(2z^2). \tag{10.5}$$

It should be realised that (10.5) is effectively the result of applying the transformation under consideration to the divergent asymptotic series F,

$$\Phi(z) \sim 1 - \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{k=0}^{\infty} (-)^k k!\, 2^{-2k}\binom{2k}{k} z^{-2k}. \tag{10.6}$$

For $z = 1$, say, one has from (10.6)

$$\Phi(1) \approx 1 - \frac{1}{e\sqrt{\pi}}(1 - 0.5 + 0.75 - 1.875 + 6.5625 \ldots),$$

whence it is obvious that not much information concerning the numerical value of $\Phi(1)$ is obtained. On the other hand, (10.5) yields for $z = 1$

$$\begin{aligned}\Phi(1) = 1 - \frac{10^{-12}}{e\sqrt{\pi}} & (722657233776 + 29300318218 + 4404491987 + \\ & + 1028700386 + 304550364 + 105029931 \\ & + 40365755 + 16836107 + 7489956 + 3511607 \\ & + 1720017 + 874359 + 458964 + 247765 + 137106 \\ & + 77564 + 44759 + 26297 + 15705 + 9520 + 5851 \\ & + 3642 + 2294 + 1461 + 940 + 611 + \ldots) \\ = 0.84270\ 07932. &\end{aligned}$$

By extrapolating the remainder of the series as a geometric series one gets $\Phi(1) = 0.84270\ 07929\ 52$ whereas actually $\Phi(1) = 0.84270\ 07929\ 50$.

11. *Computation of the generalised exponential integral.* The next example concerns the generalised exponential integral

$$E(a, z) = \int_0^z \frac{1-e^{-u}}{u}\, d\xi, \qquad u = (a^2 + \xi^2)^{\frac{1}{2}}. \tag{11.1}$$

This function is extensively tabulated for real values of a and z in "Tables of the Generalized Exponential-Integral functions", by the Staff of the Computation Laboratory, Harvard University, 1949.

These tables were computed by means of numerical integration. It will be shown that a very manageable expansion can be obtained by means of the transformation under consideration. To that end, one observes that the Besselfunction of the second kind

$$K_0(a) = \int_0^\infty \frac{e^{-u}}{u}\, d\xi, \tag{11.2}$$

whence it follows easily that

$$E(a, z) = \operatorname{arsinh} \frac{z}{a} - K_0(a) + \int_z^\infty \frac{e^{-u}}{u}\, d\xi. \tag{11.3}$$

Introducing for a moment $\zeta = (a^2 + z^2)^{\frac{1}{2}}$, $\alpha = a/\zeta$, $\beta = z/\zeta = (1-\alpha^2)^{\frac{1}{2}}$, one has

$$\int_z^\infty \frac{e^{-u}}{u}\, d\xi = \int_\zeta^\infty \frac{e^{-u}}{(u^2-a^2)^{\frac{1}{2}}}\, du = \int_1^\infty \frac{e^{-\zeta v}\, dv}{(v^2-\alpha^2)^{\frac{1}{2}}} = e^{-\zeta} \int_0^\infty \frac{e^{-\zeta t}\, dt}{\{(t+1)^2-\alpha^2\}^{\frac{1}{2}}}.$$

Hence

$$E(a, z) = \operatorname{arsinh} \frac{z}{a} - K_0(a) + \frac{e^{-\zeta}}{\zeta} f(\zeta),$$

where

$$f(\zeta) = \zeta e^{\zeta} \int_z^\infty \frac{e^{-u}}{u}\, d\xi = \zeta \int_0^\infty \frac{e^{-\zeta t}\, dt}{\{(t+1)^2-\alpha^2\}^{\frac{1}{2}}}.$$

This is of the required form, but $F(t) = \{(t+1)^2 - \alpha^2\}^{-\frac{1}{2}}$ is not a T-function if $\operatorname{Re} \alpha > \frac{1}{2}$, since its singular points are $t = -1 \pm \alpha$. The following transformation appears to be suitable.

$$f(\zeta) = \beta^{-1} \cdot \beta^2 \zeta \int_0^\infty (1 + 2t + \beta^2 t^2)^{-\frac{1}{2}}\, e^{-(\beta^2\zeta)t}\, dt.$$

Now $F(t) = (1 + 2t + \beta^2 t^2)^{-\frac{1}{2}}$ has singular points at

$$t = -(1 \pm \alpha)^{-1} = -\frac{(a^2+z^2)^{\frac{1}{2}}}{(a^2+z^2)^{\frac{1}{2}} \pm a},$$

so that, at least for real a and z, $F(t)$ is a T-function. Now,

$$H(t) = (1-t)^{-1} F\{t(1-t)^{-1}\} = \{(1-t)^2 + 2t(1-t) + \beta^2 t^2\}^{-\frac{1}{2}} =$$
$$= (1-\alpha^2 t^2)^{-\frac{1}{2}} = \sum_{k=0}^\infty c_k t^k,$$

whence

$$c_{2k+1} = 0, \qquad c_{2k} = 2^{-2k} \binom{2k}{k} \alpha^{-2k} = O(k^{-\frac{1}{2}} \alpha^{-2k}).$$

The complete expansion becomes, at least for $z > 0$, $a > 0$,

$$\left\{\begin{aligned} &E(a, z) = \operatorname{arsinh} \frac{z}{a} - K_0(a) + \\ &+ \frac{1}{z} e^{-\sqrt{a^2+z^2}} \sum_{k=0}^\infty 2^{-2k} \binom{2k}{k} \left(\frac{a}{\sqrt{a^2+z^2}}\right)^{2k} s_{2k}\left(\frac{z^2}{\sqrt{a^2+z^2}}\right). \end{aligned}\right. \tag{11.4}$$

As an example, take $a = 4$, $z = 3$, $\sqrt{a^2 + z^2} = 5$. Then

$$E(4,3) = \log 2 - K_0(4) + \tfrac{1}{3} e^{-5} \sum_{k=0}^{\infty} 2^{-2k} \binom{2k}{k} 0.64^k s_{2k}(1.8)$$

$$= 0.69314718056 - 0.01115967609 + 0.00224598233 \times 10^{-10}$$
$$(7046849141 + 202048531 + 20886204 + 3311564 + 660390$$
$$+ 152591 + 39136 + 10860 + 3206 + 995 + 322 + 108$$
$$+ 37 + 13 + 5 + 2) = 0.68362\,12237.$$

The Harvard-tables mentioned above give $E(4, 3) = 0.683621$ in accordance with the more accurate result just derived.

12. *Computation of the Besselfuction* $K_0(z)$. An example in which some more analytic manipulation is needed is provided by the Besselfunction of the second kind $K_0(z)$, that is known to be representable as follows

$$\sqrt{\frac{2}{\pi}}\, z e^{z^2} K_0(z^2) = z \int_0^{\infty} e^{-t^2/4} I_0(t^2/4)\, e^{-zt}\, dt, \tag{12.1}$$

in which expression $I_0(z)$ is the other Besselfunction of the second kind. The representation is of the required type, but $e^{-t^2/4} I_0(t^2/4)$ is not a T-function, what is directly seen from its behaviour for purely imaginary values of t. However, a wellknown artifice from the theory of the Laplace transformation yields

$$\sqrt{\frac{2}{\pi}} \sqrt{z}\, e^z K_0(z) = z \int_0^{\infty} \left\{ \frac{1}{\sqrt{\pi t}} \int_0^{\infty} e^{-(\tau^2/4t)-(\tau^2/16)} I_0\left(\frac{\tau^2}{16}\right) d\tau \right\} e^{-zt}\, dt$$

$$= z \int_0^{\infty} \left\{ \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-u-ut/4} I_0(ut/4)\, u^{-\frac{1}{2}}\, du \right\} e^{-zt}\, dt.$$

Now $F(t) = \pi^{-\frac{1}{2}} \int_0^{\infty} e^{-u-ut/4} I_0(ut/4)\, u^{-\frac{1}{2}}\, du$. Since $e^{-z} I_0(z) \sim (2\pi z)^{-\frac{1}{2}}$ i $|\arg z| < \pi$, it follows easily that $F(t)$ is analytic and uniformly bounded in the sector $|\arg t| \leqslant \pi - \varepsilon < \pi$, and also in the neighbourhood of $t = 0$. Moreover in the neighbourhood of $t = 0$ one has according to a theorem of Hardy

$$F(t) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \left\{ \sum_{k=0}^{\infty} (-)^k \frac{1}{k!} \binom{2k}{k} \left(\frac{t}{8}\right)^k u^{k-\frac{1}{2}} \right\} e^{-u}\, du$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} (-)^k \frac{(k-\frac{1}{2})!}{k!} \binom{2k}{k} \left(\frac{t}{8}\right)^k = \sum_{k=0}^{\infty} (-)^k \binom{2k}{k}^2 \left(\frac{t}{32}\right)^k,$$

which series is seen to converge if $|t| < 2$. Hence $F(t)$ is a T-function, but, of course, one can do better again by shifting its singularity to $t = -\frac{1}{2}$, using

$$\sqrt{\frac{2}{\pi}} \sqrt{z}\, e^z K_0(z) = (4z) \int_0^{\infty} F(t/4)\, e^{-(4z)t}\, dt = \sum_{k=0}^{\infty} c_k\, s_k(4z), \tag{12.2}$$

where

$$c_k = (-)^k \Delta^k r_0, \qquad r_k = 2^{-3k} \binom{2k}{k}^2. \tag{12.3}$$

A table of the first 16 coefficients c_k runs as follows

$c_0 = 1$	$c_1 = 0.5$
$c_2 = 0.5625$	$c_3 = 0.40625$
$c_4 = 0.44628\ 90625$	$c_5 = 0.35595\ 70312$
$c_6 = 0.38580\ 32227$	$c_7 = 0.32290\ 64941$
$c_8 = 0.34684\ 96799$	$c_9 = 0.29885\ 97155$
$c_{10} = 0.31892\ 36075$	$c_{11} = 0.28025\ 25535$
$c_{12} = 0.29706\ 00774$	$c_{13} = 0.26524\ 61753$
$c_{14} = 0.28048\ 81721$	$c_{15} = 0.25277\ 81178$

As an example the value of $K_0(4)$ that was needed in the example of section 11 will be calculated.

$$K_0(4) = (\pi/8)^{\frac{1}{2}}\, e^{-4} \sum_{k=0}^{\infty} c_k\, s_k\,(16)$$

$$\begin{aligned} &= 0.01147\,762458 \times 10^{-11}\,(94412965774 + 2510209076 \\ &\qquad + 274212880 + 26601865 \\ &\qquad + 4865234 + 755499 + 179937 \\ &\qquad + 36478 + 10288 + 2492 + 792 + 218 \\ &\qquad + 76 + 23 + 9 + 3 + 1 \ldots) \\ &= 0.01115\,967609. \end{aligned}$$

13. *Computation of an integral of Goodwin and Staton.* The last example of this type concerns the computation of the following function

$$f(z) = z \int_0^\infty \frac{e^{-\sigma^2} d\sigma}{z+\sigma}. \tag{13.1}$$

This function is of particular interest for the subject of summing asymptotic series. Indeed, GOODWIN and STATON [1]) who tabulated this integral for real values of z, showed that for moderately large values $f(z)$ can be computed by repeated application of the Euler transformation. Later on, van der Corput proved that this method is legitimate in this case.

However, it will be shown that again the much greater power of the transformation under consideration yields in one go a convergent series for $f(z)$.

To this end one transforms as follows:

$$f(z) = z \int_0^\infty e^{z\sigma}\, d\sigma\, \frac{e^{-(z+\sigma)\sigma}}{z+\sigma} = z \int_0^\infty e^{z\sigma}\, d\sigma \int_\sigma^\infty e^{-(z+\sigma)\tau}\, d\tau$$

$$= z \int_0^\infty d\sigma \int_\sigma^\infty e^{-z(\tau-\sigma)}\, e^{-\tau\sigma}\, d\tau.$$

Now, putting $\tau = u + t$, $\sigma = u - t$, one gets

$$f(z) = 2z \int_0^\infty \{e^{t^2} \int_t^\infty e^{-u^2}\, du\}\, e^{-(2z)t}\, dt. \tag{13.2}$$

Hence,

$$F(t) = e^{t^2} \int_t^\infty e^{-u^2}\, du = e^{t^2} \int_0^\infty e^{-(v+t)^2}\, dv = \int_0^\infty e^{-v^2-2vt}\, dv. \tag{13.3}$$

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[1]) The Quart. J. of Mech. and App. Math., I, 319–326 (1948).

Apparently $F(t)$ is an entire function, and if $\operatorname{Re} t > -\frac{1}{2}$, say, then

$$|F(t)| \leqslant \int_0^\infty e^{-v^2-2v \operatorname{Re} t}\, dv \leqslant \int_0^\infty e^{-v^2+v}\, dv$$

whence $|F(t)|$ is bounded and, so, $F(t)$ is a T-function. Hence the transformation will yield a convergent series, and it only remains to calculate the coefficients c_k.

One way is, of course, over the coefficients r_k, to be derived from the coefficients of the formal — here asymptotic — series F. From (13.3) it follows immediately

$$F^{(k)}(0) = (-)^k\, 2^k \int_0^\infty e^{-v^2} v^k\, dv = (-)^k\, 2^{k-1} \left(\frac{k-1}{2}\right)!\,,$$

whence

$$r_k = 2^{k-1} \left(\frac{k-1}{2}\right)!\, (k!)^{-1}, \tag{13.4}$$

from which c_k can be found by differencing.

The other way is over the function $H(t)$. From (13.3) one has

$$H(t) = \frac{1}{1-t}\, e^{\left(\frac{t}{1-t}\right)^2} \int_{t/1-t}^\infty e^{-u^2}\, du.$$

By differentiation one gets

$$H'(t) = (1-t)^{-1} H(t) + 2t(1-t)^{-3} H(t) - (1-t)^{-3},$$

whence

$$(1-t)^3 H'(t) - (1+t^2) H(t) = -1.$$

By continued differentiation one finds for $k > 0$

$$H^{(k+1)}(0) - (3k+1) H^{(k)}(0) + 3k(k-1) H^{(k-1)}(0) - k(k-1)^2 H^{(k-2)}(0) = 0$$

whence for $c_k = H^{(k)}(0)\,(k!)^{-1}$ the following recurrence relation results

$$(k+1)\, c_{k+1} = (3k+1)\, c_k - (3k-3)\, c_{k-1} + (k-1)\, c_{k-2},\ k > 0.$$

From this together with $c_0 = \frac{1}{2}\sqrt{\pi}$ and $c_1 = -1 + \frac{1}{2}\sqrt{\pi}$ the coefficients can readily be found, and using these c_k's one has if $\operatorname{Re} z > 0$

$$f(z) = \sum_{k=0}^\infty c_k\, s_k(2z). \tag{13.5}$$

The approximate numerical values of the first 38 coefficients are

c_0 =	0.88622 69254 52758	c_1 =	−0.11377 30745 47242
c_2 =	−0.22754 61490 94484	c_3 =	−0.12175 89648 55635
c_4 =	−0.01996 47257 7098	c_5 =	0.03073 01602 7890
c_6 =	0.04070 39023 8194	c_7 =	0.03037 11588 8834
c_8 =	0.01498 45267 928	c_9 =	0.00241 62388 708
c_{10} =	−0.00490 04683 537	c_{11} =	−0.00748 11115 764
c_{12} =	−0.00693 17795 23	c_{13} =	−0.00488 49470 95
c_{14} =	−0.00254 47971 34	c_{15} =	−0.00060 17652 10
c_{16} =	0.00067 56887 9	c_{17} =	0.00129 50722 4
c_{18} =	0.00140 45806 3	c_{19} =	0.00119 42084 1
c_{20} =	0.00083 64017	c_{21} =	0.00045 89360
c_{22} =	0.00013 96351	c_{23} =	−0.00008 66515

$c_{24} = -0.00021\ 6039$	$c_{25} = -0.00026\ 3210$
$c_{26} = -0.00025\ 1109$	$c_{27} = -0.00020\ 3623$
$c_{28} = -0.00014\ 122$	$c_{29} = -0.00007\ 896$
$c_{30} = -0.00002\ 626$	$c_{31} = 0.00001\ 241$
$c_{32} = 0.00003\ 63$	$c_{33} = 0.00004\ 70$
$c_{34} = 0.00004\ 75$	$c_{35} = 0.00004\ 10$
$c_{36} = 0.00003\ 1$	$c_{37} = 0.00001\ 9$

As an example one may choose $z = 1$. Then one has from the first 36 terms

$$\begin{aligned} f(1) = 10^{-12}\ (&640438298446 - 19110656897 - 13334349155 - \\ &- 3013982956 - 234491932 + 185509958 + 133990784 \\ &+ 57085563 + 16689529 + 1644416 - 2091463 - 2047407 \\ &- 1240354 - 581379 - 204535 - 33113 + 25771 + 34623 \\ &+ 26594 + 16165 + 8165 + 3257 + 726 - 332 - 615 \\ &- 560 - 401 - 246 - 130 - 55 - 14 + 5 + 12 + 12 \\ &+ 9 + 6 \ldots) = 0.6051336525. \end{aligned}$$

Of course, in practice, for such a low value of z an ascending expansion is just still preferable. The example, therefore, mainly serves to show that one can get by the method under consideration highly accurate results for small z in a fully legitimate way. Of course, and this remark holds in general, the advantage of the underlying asymptotic series, viz. the fast initial convergence for large values of z is fully preserved. Indeed, if one takes $z = 10$, the first 14 terms yield

$$\begin{aligned} f(10) = 10^{-12}\ (&845789197239 - 4754475334 - 771220009 - 46838510 \\ &- 1090815 + 281166 + 70887 + 11166 + 1267 + 51 \\ &- 27 - 11 - 3 - 1 \ldots) = 0.84021\ 59370\ 66, \end{aligned}$$

a result correct in all twelve decimals. Actually $z = 10$ is already high enough in order that one can get the same accuracy by means of the asymptotic series directly. Indeed from (13.4) it follows that this asymptotic series has the form

$$f(z) \sim \sum_{k=0}^{\infty} (-)^k\, 2^{k-1} \left(\frac{k-1}{2}\right)!\, (k!)^{-1}\, z^{-k}. \tag{13.6}$$

Taking $z = 10$, one has from the first 17 terms of the series

$$\begin{aligned} f(10) \approx 10^{-12}\ \{\tfrac{1}{2}\sqrt{\pi}\ &(1000000000000 + 5000000000 + 75000000 + \\ &+ 1875000 + 65625 + 2953 + 162 + 11 + 1) \\ &-\tfrac{1}{2}(100000000000 + 1000000000 + 20000000 + \\ &+ 600000 + 24000 + 1200 + 72 + 5)\} = \\ &= 0.84021\ 59370\ 66. \end{aligned}$$

Although somewhat more terms are needed (13.6) has for those values of z for which it is applicable the advantage that one does not need to know tabulated functions like the $s_k(z)$. But, of course the standard-function and its associates once being tabulated, (13.5) has the advantage of allowing a uniform and simple computational procedure for small, moderately large and large values of the argument without losing the advantage of fast convergence for large values of the argument.