MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM
REKENAFDELING

THE ASYMPTOTIC EXPANSION OF A SPECIAL FUNCTION AND SOME RELATIONS WITH BESSEL FUNCTIONS

by

J.A. Zonneveld

and

J. Berghuis

MR 18

MATHEMATISCH CENTRUM
REKENAFDELING

1955

THE ASYMPTOTIC EXPANSION OF A SPECIAL FUNCTION AND SOME RELATIONS WITH BESSEL FUNCTIONS

MR 18

by

J.A. Zonneveld

and

J. Berghuis

1. Introduction.

In this report we give the asymptotic expansion for small positive values of t of the integral

$$I(t,x) = \frac{2}{\pi} \int_0^\infty f(t,u)\, g(x,u)\, du, \qquad (1.1)$$

where

$$f(t,u) = (1 - e^{-u^2 t}).u^{-2}$$

and

$$g(x,u) = \left\{ J_1(u).Y_0(xu) - Y_1(u).J_0(xu) \right\} . \left\{ J_1^2(u) + Y_1^2(u) \right\}^{-1},$$

we restrict ourselves to the cases $x > 1$ and $x = 1$.

The asymptotic expansion will be derived in two different manners and the answers will be compared. Then it is possible to prove that special integrals connected with Bessel functions are equal to zero.

In Chapter I the case $x > 1$ is treated, while in Chapter II we treat $x = 1$.

CHAPTER I

2. The asymptotic expansion.

The integral $I(t,x)$ is the solution of the partial differential equation

$$\frac{\partial^2 I}{\partial x^2} + \frac{1}{x}\frac{\partial I}{\partial x} - \frac{\partial I}{\partial t} = 0 \qquad (2.1)$$

satisfying the boundary conditions:

for $x > 1$ and $t=0$ the function I vanishes;
for $t > 0$ and $x=1$ the derivative $\frac{\partial I}{\partial x}$ equals one;
for $x \to \infty$ the function I vanishes.

The Laplace transformation is applied on the differential equation (2.1) and gives, if I^* denotes the Laplace transformation of I the ordinary differential equation

$$\frac{d^2 I^*}{dx^2} + \frac{1}{x}\frac{dI^*}{dx} - pI^* = 0 \qquad (2.2)$$

and the boundary conditions:

for $x=1$ we have $\frac{dI^*}{dx} = p^{-1}$;
for $x=\infty$ we have $I^* = 0$.

The solution satisfying (2.2) and the boundary conditions is

$$I^* = p^{-3/2}.K_0(xp^{\frac{1}{2}}).\left\{K_0'(p^{\frac{1}{2}})\right\}^{-1},$$

where K_0 is the imaginary Bessel function of the second kind and the accent means differentiation with respect to the argument.

If we apply the Mellin-transformation, we find the function $I(t,x)$ given in (1.1).

At the other hand it is possible to expand I^* for large positive values of p and by transforming back we get the asymptotic expansion.

For large positive values of p we have

$$I^* = x^{-\frac{1}{2}}.e^{-(x-1)p^{\frac{1}{2}}}\left\{p^{-3/2} - 8^{-1}(3+x^{-1})p^{-2} + 3.2^{-7}(11+2x^{-1}+3x^{-2}).p^{-5/2} + O(p^{-3})\right\}$$

And for small positive values of t we obtain the result

$$I = \frac{2.e^{-y^2/4t}}{\pi^{\frac{1}{2}}\, x^{\frac{1}{2}}}\left\{\frac{2t^{3/2}}{y^2} - \frac{3x^2+22x-1}{2xy^4}\, t^{5/2} + \frac{33x^4+228x^3+1758x^2-108x+9}{16x^2y^6}\, t^{7/2} + O(t^{9/2})\right\} \qquad (2.3)$$

where $y = x-1$.

The reader remarks that for small positive values of t the function $I(t,x)$ vanishes exponentially.

3. Another derivation of the asymptotic expansion.

The method shown in theorem 5 of MR 11 of the Computation Department of the Mathematical Centre can be used for the derivation of the asymptotic expansion of $I(t,x)$ for small positive values of t.

The function $f(t,u)$ possesses for small values of $|u|$ the expansion

$$f(t,u) = \sum_{n=0}^{\infty} b_n t^{n+1} u^{2n} \tag{3.1}$$

with $b_n = (-1)^n \{(n+1)!\}^{-1}$
and the function $g(x,u)$ possesses an asymptotic expansion for large positive values of u:

$$g(x,u) = \cos yu \{ a_0 + a_2 u^{-2} + a_4 u^{-4} + \dots \} + \\ + \sin yu \{ a_1 u^{-1} + a_3 u^{-3} + a_5 u^{-5} + \dots \} ;$$

here the coefficients a_i are defined by

$$\begin{aligned} a_0 &= x^{-\frac{1}{2}} \\ a_1 &= 2^{-3}.x^{-3/2}(3x+1) \\ a_2 &= -2^{-7}.x^{-5/2}(33x^2+6x+9) \\ a_3 &= -2^{-10}.x^{-7/2}(249x^3+33x^2+27x+75) \\ a_4 &= 2^{-15}.x^{-9/2}(9963x^4+996x^3+594x^2+900x+3675) \\ a_5 &= 2^{-18}.x^{-11/2}(116379x^5+13113x^4+4482x^3+450x^2+11025x+59535) \end{aligned}$$

Now we use the method of Theorem 5 of MR 11 (loc.cit.):
We write

$$I(t,x) = \frac{2a_0}{\pi} \int_0^{\infty} f(t,u) \cos yu \, du + \frac{2b_0}{\pi} \int_0^{\infty} \{ g(x,u) - a_0 \cos yu \} du \\ + I_1(t,x) \tag{3.2}$$

where

$$I_1(t,x) = \frac{2}{\pi} \int_0^{\infty} \{ f(t,u) - b_0 t \} \{ g(x,u) - a_0 \cos yu \} du.$$

Now we consider the first integral occurring in the righthand side of (3.2). Putting $v = u\, t^{\frac{1}{2}}$ we find

$$\int_0^{\infty} f(t,u) \cos yu \, du = \tfrac{1}{2} t^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1-e^{-v^2}}{v^2} \cos \beta v \, dv$$

where $\beta = y.t^{-\frac{1}{2}}$. For small positive values of t the parameter β is large positive. The function

$$\frac{1-e^{-v^2}}{v^2}$$

is analytic for each value of v ($-\infty < v \leq \infty$) and, therefore, each integral of the form

$$\int_{-\infty}^{\infty} \frac{1-e^{-v^2}}{v^2} \, e^{i/\beta\, v} \, dv$$

is asymptotically zero. ')

So we have

$$\int_0^{\infty} f(t,u) \cos y\, u \, du \sim 0$$

for small positive values of t.

Next we treat $I_1(t,x)$ by the same procedure

$$I_1(t,x) = \frac{2a_1}{\pi} \int_0^{\infty} \left\{ f(t,u) - b_0 t \right\} \frac{\sin y\, u}{u} \, du$$

$$+ \frac{2a_2}{\pi} \int_0^{\infty} \left\{ f(t,u) - b_0 t \right\} \frac{\cos y\, u}{u^2} \, du$$

$$+ \frac{2b_1}{\pi} t^2 \int_0^{\infty} u^2 g_2^*(x,u) \, du + I_2^*(t,x), \qquad (3.3)$$

where

$$I_2^*(t,x) = \frac{2}{\pi} \int_0^{\infty} f_2(t,u) \quad g_2^*(x,u) \, du$$

with

$$f_2(t,u) = f(t,u) - b_0 t - b_1 t^2 u^2$$

and

$$g_2^*(x,u) = g(x,u) - a_0 \cos y\, u - \frac{a_1}{u} \sin y\, u - \frac{a_2}{u^2} \cos y\, u.$$

The first two integrals of equation (3.3) are asymptotically equal to zero as can be shown in a similar manner as above.

Writing

$$I_2^*(t,x) = \frac{2a_3}{\pi} \int_0^{\infty} f_2(t,u) . \frac{\sin y\, u}{u^3} \, du + I_2(t,x)$$

we find that $I_2^*(t,x)$ is asymptotically equal to $I_2(t,x)$ and

$$I_2(t,x) = \frac{2}{\pi} t^2 \int_0^{\infty} f_2(v) . u^2 . g_2(x,u) \, du$$

with

$$f_2(v) = \frac{1-e^{-v^2} - b_0 v^2 - b_1 v^4}{v^4}$$

$$v = u t^{\frac{1}{2}}$$

$$g_2(x,u) = g_2^*(x,u) - \frac{a_3}{u^3} \sin y\, u.$$

The function $u^2 g_2(x,u)$ can be integrated absolutely, while the function $f_2(v)$ possesses maximum M independent of t.

Therefore we can conclude that $I_2(t,x)$ is of the order t^2. Using the procedure described above again we obtain

$$I(t,x) = \frac{2b_0}{\pi} \int_0^\infty \{g(x,u) - a_0 \cos y u\} du$$

$$+ \frac{2b_1 t^2}{\pi} \int_0^\infty u^2 g_2^*(x,u)\, du$$

$$+ \frac{2b_2 t^3}{\pi} \int_0^\infty u^4 g_3^*(x,u)\, du$$

$$+ R(t) + I_3(t,x) \tag{3.4}$$

where

$$g_3^*(x,u) = g_2(x,u) - \frac{a_4}{u^4} \cos \beta u$$

$$g_3(x,u) = g_3^*(x,u) - \frac{a_5}{u^5} \sin \beta u$$

$$f_3(t,u) = f_2(t,u) - b_2 t^3 u^4$$

$$I_3(t,x) = \int_0^\infty f_3(t,u).g_3(x,u)\, du$$

and the function $R(t)$ is asymptotically equal to zero. Furthermore we can show that $I_3(t,x)$ is of the order t^3. Using this procedure again and again we obtain an ordinary asymptotic expansion of $I(t,x)$ for small positive values of t.

4. Comparison.

In this section we compare the two asymptotic expansions obtained. Section 2 gives that for small positive values of t the function I(t,x) is asymptotically zero. Section 3 gives an asymptotic expansion of the form (3.4); that means this expansion has to be asymptotically zero and therefore, for $x > 1$ holds

$$\int_0^\infty \left\{ g(x,u) - a_0 \cos yu \right\} du = 0, \tag{4.1}$$

$$\int_0^\infty u^2 \, g_2^*(x,u) \, du = 0, \tag{4.2}$$

$$\int_0^\infty u^4 \, g_3^*(x,u) \, du = 0, \tag{4.3}$$

etc.

It seems that I(t,x) is the generating function of the coefficients mentioned in the relations (4.1), (4.2), (4.3) etc., and all these integralrepresentations have to vanish.

So in general we will find

$$\int_0^\infty u^{2k} \left\{ g(x,u) - \sum_{n=0}^{k} a_{2k} . u^{-2k} \cos yu - \sum_{n=0}^{k} a_{2k+1} . u^{-(2k+1)} \sin yu \right\} du = 0$$

a rather astonishing result.

The relations (4.1) and (4.2) were checked numerically for the values x=1.2, x=1.4 and x=2.

CHAPTER II

x = 1

5. The asymptotic expansion of I(t,1).

We can use the same method of section 2 of Chapter I and the result is that for small positive values of t we have:

$$I(t,1) = \frac{2}{\pi^{\frac{1}{2}}} t^{\frac{1}{2}} - \tfrac{1}{2}t + \frac{1}{2\pi^{\frac{1}{2}}} t^{3/2} + 0(t^2) \qquad (5.1)$$

Furthermore we remark that

$$g(1,u) = \frac{2}{\pi u \left| H_1^{(1),(2)}(u) \right|^2}$$

a function that can be expanded asymptotically for large positive values of u. A few terms are

$$1 - 3.8^{-1}.u^{-2} + 63.2^{-7}.u^{-4} + \ldots$$

For the rest of this Chapter we write g(u) in stead of g(1,u). Theorem 5 of MR 11 (loc.cit.) gives that there exists an asymptotic expansion for small positive values of t of the form:

$$\frac{2}{\pi} \int_0^\infty f(t,u)\, du + \frac{2}{\pi} t \int_0^\infty \left\{ g(u) - 1 \right\} du -$$

$$- \frac{3}{4\pi} \int_0^\infty u^{-2} \left\{ f(t,u) - t \right\} du + 0(t^2) \qquad (5.2)$$

We can evaluate all these integrals.

6. Evaluation of integrals.

a. $$\int_0^\infty f(t,u)\, du = t^{\frac{1}{2}} \int_0^\infty \frac{1-e^{-v^2}}{v^2}\, dv = \pi^{\frac{1}{2}} t^{\frac{1}{2}} \qquad (6.1)$$

b. $$\int_0^\infty u^{-2} \left\{ f(t,u) - t \right\} du = t^{3/2} \int_0^\infty v^{-4}(1-e^{-v^2}-v^2)dv = -\tfrac{2}{3}\pi^{\frac{1}{2}} t^{3/2} \qquad (6.2)$$

From (1.2), (2.1) and (2.2) we deduce

$$\int_0^\infty \left\{ g(u) - 1 \right\} du = -\frac{\pi}{4} \qquad (6.3)$$

This relation can be checked in another way.

We know (See Watson: A Treatise of the Theory of Bessel functions p. 76) that

$$\mathfrak{W}\left(H_1^{(1)}(z), H_1^{(2)}(z)\right) = -4i.(\pi z)^{-1},$$

or

$$\frac{\left\{H_1^{(2)}(z)\right\}'}{H_1^{(2)}(z)} - \frac{\left\{H_1^{(1)}(z)\right\}'}{H_1^{(1)}(z)} = -\frac{4i}{\pi z\left\{J_1^2(z) + Y_1^2(z)\right\}} .$$

Therefore we find by integration of this relation

$$\int_0^\infty \left\{g(u) - 1\right\} du = \lim_{N\to\infty}\left[\left\{-\frac{1}{2i}\ln\frac{H_1^{(2)}(u)}{H_1^{(1)}(u)} - u\right\}\Bigg|_0^N\right] = -\frac{\pi}{4} .$$

It is possible to prove other properties of g(u).