KONINKL. NEDERL. AKADEMIE VAN WETENSCHAPPEN – AMSTERDAM
Reprinted from Proceedings, Series A, 65, No. 1 and Indag. Math., 24, No. 1, 1962

MATHEMATICS

THE NUMERICAL EFFICIENCY OF CERTAIN CONTINUED FRACTION EXPANSIONS [1]). IA

BY

P. WYNN

(Communicated by Prof. A. VAN WIJNGAARDEN at the meeting of September 30, 1961)

The author has recently conducted investigations into the numerical efficiency of certain continued fraction expansions for the purpose of computing a prescribed function of a real variable. He proposes in the near future to extend the investigation into the complex domain. The purpose of this note is to place on record the results of the investigations so far.

It is in order first briefly to recapitulate methods for deriving continued fractions which are likely to be of use for computational purposes. These are as follows:

1. Gauss' Method ([1] Ch. XVIII)

It is assumed that functions $A_n, B_n, \ldots, P_n$ satisfy p linear recursions of the form

$$-a_nA_n+b_nA_{n+1}+c_nB_n=0 \tag{1}$$

$$-A_{n+1}+d_nB_n+e_nC_n=0 \tag{2}$$

$$-B_n+f_nC_n+g_nD_n=0 \tag{3}$$

$$\cdots\cdots\cdots\cdots$$

$$-O_n+x_nP_n+y_nA_{n+1}=0 \tag{4}$$

$$-P_n+z_nA_{n+1}+A_{n+2}=0 \tag{5}$$

where $a_n, b_n, \ldots, z_n$ are functions of n. Equation (1) gives

$$\frac{A_{n+1}}{A_n}=\cfrac{a_n}{b_n+c_n\cfrac{B_n}{A_{n+1}}} \tag{6}$$

equation (2) gives

$$\frac{B_n}{A_{n+1}}=\cfrac{1}{d_n+e_n\cfrac{C_n}{B_n}} \tag{7}$$

and finally one obtains

$$\frac{A_{n+1}}{A_n}=\frac{a_n}{b_n+}\,\frac{c_n}{d_n+}\cdots\cfrac{y_n}{z_n+\cfrac{A_{n+2}}{A_{n+1}}}. \tag{8}$$

---

1) Communication MR 37 of the Computation Department of the Mathematical Centre at Amsterdam.

Successive use of equation (8) with $n=0, 1, \ldots$ gives

$$(9) \quad \frac{A_1}{A_0} = \frac{a_0}{b_0+} \frac{c_0}{d_0+} \cdots \frac{y_0}{z_0+} \frac{a_1}{b_1+} \frac{c_1}{d_1+} \cdots \frac{a_n}{b_n+} \frac{c_n}{d_n+} \cdots \frac{y_n}{z_n+} \frac{a_{n+1}}{b_{n+1}+} \cdots$$

2. Application of the $q-d$ Algorithm

If the continued fraction

$$(10) \qquad F(x) = \frac{c_0}{1-} \frac{q_1^{(0)}x}{1-} \frac{e_1^{(0)}x}{1-} \cdots \frac{q_r^{(0)}x}{1-} \frac{e_r^{(0)}x}{1-} \cdots$$

equivalent to the power series

$$(11) \qquad F(x) = c_0 + c_1x + c_2x^2 + \ldots$$

is characterised by imposing the condition that the x power series expansion of the rth convergent of (10) should agree with the power series (11) at least as far as the term in x^{2r-1} then it may be shown [2] that the coefficients in (10) may be derived by constructing quantities $q_r^{(m)}$, $e_r^{(m)}$ by means of the relations

$$(12) \qquad q_r^{(m)} + e_r^{(m)} = q_r^{(m+1)} + e_{r-1}^{(m+1)}, \quad q_r^{(m)} e_{r-1}^{(m)} = q_{r-1}^{(m+1)} e_{r-1}^{(m+1)}$$

from the initial conditions

$$(13) \qquad e_0^{(m)} = 0 \qquad q_1^{(m)} = c_{m+1}/c_m.$$

Continued fractions equivalent to (10) are

$$(14) \qquad z^{-1}F(z^{-1}) = \frac{c_0}{z-} \frac{q_1^{(0)}}{1-} \frac{e_1^{(0)}}{z-} \cdots \frac{q_r^{(0)}}{1-} \frac{e_r^{(0)}}{z-} \cdots$$

and the even part of (14)

$$(15) \qquad z^{-1}F(z^{-1}) = \frac{c_0}{z-q_1^{(0)}-} \frac{e_1^{(0)}q_1^{(0)}}{z-q_2^{(0)}-e_1^{(0)}-} \cdots \frac{e_r^{(0)}q_r^{(0)}}{z-q_{r+1}^{(0)}-e_r^{(0)}-} \cdots$$

This method is formally equivalent to a number of others. A discussion of these is contained in reference [3].

3. Euler's Method [4] [5]

This method yields a continued fraction expansion for the quotient y/y' (the dash denotes differentiation with respect to x) where y satisfies a homogeneous linear differential equation of the second order

$$(16) \qquad y = Q_0 y' + P_1 y''.$$

Successive differentiation of equation (16) yields the system of equations

$$(17) \qquad \begin{cases} y^{(\nu)} = Q_\nu y^{(\nu+1)} + P_{\nu+1} y^{(\nu+2)} \\ Q_\nu = \dfrac{Q_{\nu-1} + P_\nu'}{1 - Q'_{\nu-1}}, \quad P_{\nu+1} = \dfrac{P_\nu}{1 - Q'_{\nu-1}}. \end{cases}$$

Substitution in the equations (17) then gives the continued fraction

$$\frac{y}{y'} = Q_0 + \frac{P_1}{Q_1+}\,\frac{P_2}{Q_2+}\cdots\frac{P_\nu}{Q_\nu+}\cdots \tag{18}$$

(The class of continued fractions produced by this method is thus seen to be a subclass of those produced by the method of 1).)

4. Expansion of a Laplace Transform

Two equivalent methods which, although having a limited field of application, may in certain cases be more convenient for expanding in a continued fraction the Laplace integral

$$F(z) = \int_0^\infty e^{-zt}\phi(a+t)\,dt \tag{19}$$

than application of the $q-d$ algorithm (or some equivalent method) to the series

$$F(z) \sim \sum_{s=0}^{\infty} \phi^{(s)}(a) z^{-s-1} \tag{20}$$

have been given by Stieltjes ([6], [1] Ch. XI, p. 202) and Rutishauser [8].

The first, due to Stieltjes, is as follows:
If

$$\left\{\begin{aligned}\phi(a)\phi(a+x+y) = \phi(a+x)\phi(a+y) + \{\phi(a)\}^2\beta_0\phi_1(a+x)\phi_1(a+y) + \\ + \{\phi(a)\}^2\beta_0\beta_1\phi_2(a+x)\phi_2(a+y) + \dots\end{aligned}\right. \tag{21}$$

where

$$\phi_r(a+x) = \sum_{s=r}^{\infty} \frac{\phi_r^{(s)}(a)\,x^s}{s\,!} \qquad r=1,\,2,\,\dots \tag{22}$$

and

$$\phi_r^{(r)}(a) = 1 \qquad r=1,\,2,\,\dots \tag{23}$$

then the coefficients in the continued fraction

$$F(z) = \frac{\phi(a)}{z-\alpha_0-}\,\frac{\beta_0}{z-\alpha_1-}\cdots\frac{\beta_{r-1}}{z-\alpha_r-}\cdots \tag{24}$$

may be determined from (21) and the further result

$$\phi(a)\,\alpha_0 = \phi'(a),\;\; \phi(a)\,\alpha_r = \phi_r^{(r+1)}(a) - \phi_{r-1}^{(r)}(a) \qquad r=1,\,2,\,\dots \tag{25}$$

The second, due to Rutishauser, states that if the sequences of functions $E_r(t)$, $Q_r(t)$ are constructed by means of the non-linear difference-differential relations

$$Q_{r+1}(t) - Q_r(t) = E_r'(t)/E_r(t) \tag{26}$$

$$E_r(t) - E_{r-1}(t) = Q_r'(t) \tag{27}$$

from the initial members

(28) $$E_0(t)=0, \quad Q_1(t)=\phi'(t)/\phi(t)$$

then the coefficients in (24) are given by

(29) $$\alpha_{r-1}=Q_r(a), \quad \beta_{r-1}=E_r(a).$$

5. Use of reciprocal differences

A continued fraction expansion is derived from a number of argument-function value pairs. The numerical efficiency of continued fractions of this type has not been examined.

Numerical efficiency tables

The numerical efficiency of a particular continued fraction expansion will be displayed in the following manner. The least value of n is computed for which the relation

(30) $$\left|\frac{C-C_n}{C}\right| \leqslant \epsilon$$

obtains, where

(31) $$C_n = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n}$$

is the nth convergent of the continued fraction

(32) $$C = b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n+}\cdots$$

and ϵ is a preassigned relative error. The coefficients a_s, b_s in (32) are functions of a variable x or z, as is C. ϵ is made to take the values $\epsilon={}^1/_2\,10^{-h}$ $h=1, 2, \ldots$ and the determination of n is conducted for a number of values of the argument. Each set of results is displayed in a table as follows

TABLE I

h	1	2	$\ldots$
x, z			
x_1, z_1	$n_{1,1}$	$n_{1,2}$	$\ldots$
x_2, z_2	$n_{2,1}$	$n_{2,2}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	
x_p, z_p	$n_{p,1}$	$n_{p,2}$	$\ldots$

The functions dealt with group themselves under seven headings: the results of this note will be correspondingly presented.

a) *Continued Fractions Associated with Bessel Functions*

The function

(33) $${}_0F_1(c;x) = 1 + \frac{x}{1!\,c} + \frac{x^2}{2!\,c(c+1)} + \cdots\frac{x^r}{r!\,c(c+1)\ldots(c+r-1)} + \cdots$$

satisfies the recursion

(34) $$c(c+1)\,{}_0F_1(c;x) - c(c+1)\,{}_0F_1(c+1;x) - x\,{}_0F_1(c+2;x) = 0$$

and the differential equation

$$xy'' - cy' - y = 0 \tag{35}$$

from which, cf. 1) and noting that $\{{}_0F_1(c;x)\}' = c^{-1}\,{}_0F_1(c+1;x)$, follows the continued fraction expansion

$$\frac{{}_0F_1(c+1;x)}{{}_0F_1(c;x)} = \frac{c}{c+}\,\frac{x}{c+1+}\cdots\frac{x}{c+r+}\cdots \tag{36}$$

Noting the relation

$$J_n(x) = \left(\frac{x}{2}\right)^n \{\Gamma(n+1)\}^{-1}\,{}_0F_1\left(n+1;\,-\left(\frac{x}{2}\right)^2\right) \tag{37}$$

there follows immediately [8]

$$\frac{J_n(x)}{J_{n-1}(x)} = \frac{x/2}{n-}\,\frac{(x/2)^2}{n+1-}\cdots\frac{(x/2)^2}{n+r-1}\cdots \tag{38}$$

and from

$$I_n(x) = \left(\frac{x}{2}\right)^n \{\Gamma(n+1)\}^{-1}\,{}_0F_1\left(n+1;\left(\frac{x}{2}\right)^2\right) \tag{39}$$

there follows

$$\frac{I_n(x)}{I_{n-1}(x)} = \frac{x/2}{n+}\,\frac{(x/2)^2}{n+1+}\cdots\frac{(x/2)^2}{n+r-1+}\cdots \tag{40}$$

The numerical efficiency of expansion (38), within $n=1$, is indicated in Table II, whilst that of expansion (40), again with $n=1$, is indicated in Table III.

TABLE II

x \ h	1	2	3	4	5	6	7	8	9	10	11
0.25	1	2	2	2	3	3	3	4	4	4	5
0.50	1	2	2	3	3	4	4	5	5	5	6
1.00	2	3	3	4	4	5	5	6	6	7	7
2.00	3	4	5	6	6	7	7	8	8	9	10
4.00	6	7	8	9	10	10	11	12	12	13	13
8.00	10	12	13	14	15	16	16	17	18	19	20

TABLE III

x \ h	1	2	3	4	5	6	7	8	9	10	11
0.25	1	2	2	2	3	3	3	4	4		
0.50	1	2	2	3	3	4	4	5	5		
1.00	2	2	3	4	4	5	5	6	6	6	
2.00	2	3	4	5	5	6	7	7	8	8	
4.00	4	5	6	7	7	8	9	10	10	11	12
8.00	5	7	8	9	10	11	12	13	14	15	16

Writing $n = {}^1/_2$ in (38) there follows the well known expansion

$$\tan(x) = \frac{x}{1-}\,\frac{x^2}{3-}\,\frac{x^2}{7-}\cdots \tag{41}$$

the numerical efficiency of which is illustrated in Table IV.

TABLE IV

x \ h	1	2	3	4	5	6	7	8	9	10
0.25	1	2	2	3	3	3	4	4	4	5
0.50	2	2	3	3	4	4	4	5	5	6
1.00	2	3	4	4	5	5	6	6	7	7
2.00	4	5	5	6	7	7	8	8	9	9
4.00	6	7	8	9	10	10	11	12	12	13
8.00	11	13	14	15	16	16	17	18	19	20

b) *Continued Fractions Associated with the Hypergeometric Function*

The hypergeometric function

$$ {}_2F_1(a,b;c;x) = 1 + \frac{abx}{1!\,c} + \frac{a(a+1)b(b+1)x^2}{2!\,c(c+1)} + \cdots \tag{42}$$

satisfies the recursions

$$\left\{\begin{aligned} & c(c+1)\,{}_2F_1(a,b;c;x) - c(c+1)\,{}_2F_1(a,b+1;c+1;x) + \\ & \qquad + a(c-b)\,x\,{}_2F_1(a+1,b+1;c+2;x) = 0 \end{aligned}\right. \tag{43}$$

and

$$\left\{\begin{aligned} & (c+1)(c+2)\,{}_2F_1(a,b+1;c+1;x) - \\ & \quad -(c+1)(c+2)\,{}_2F_1(a+1,b+1;c+2;x) + \\ & \qquad +(b+1)(c-a+1)\,x\,{}_2F_1(a+1,b+2;c+3;x) = 0. \end{aligned}\right. \tag{44}$$

These yield (cf. 1)) the continued fraction

$$\left\{\begin{aligned} & \frac{{}_2F_1(a,b+1;c+1;x)}{{}_2F_1(a,b;c;x)} = \frac{c}{c-}\,\frac{a(c-b)x}{c+1-}\,\frac{(b+1)(c-a+1)x}{c+2-}\cdots \\ & \qquad \cdots\frac{(a+r-1)(c-b+r-1)x}{c+2r-1-}\,\frac{(b+r)(c-a+r)x}{c+2r-}\cdots \end{aligned}\right. \tag{45}$$

Writing $a = {}^1/_2$, $b = -{}^1/_2$, $c = {}^1/_2$ and replacing x by x^2, (45) becomes

$$\frac{\arcsin(x)}{\sqrt{1-x^2}} = \frac{x}{1-}\,\frac{1.2x^2}{3-}\,\frac{1.2x^2}{5-}\,\frac{3.4x^2}{7-}\,\frac{3.4x^2}{9-}\cdots\frac{r(r-1)\,x^2}{2r-1-}\,\frac{r(r-1)\,x^2}{2r+1-}\cdots \tag{46}$$

the numerical efficiency of which is illustrated in Table V:

TABLE V

x \ h	1	2	3	4	5	6	7	8	9	10	11	12
0.2	1	2	2	3	3	4	4	5	5	6	6	6
0.4	2	2	3	4	5	5	6	7	8	8	9	9
0.6	2	3	4	5	6	7	8	9	10	12	13	
0.8	3	5	6	7	10	11	13	15	16	18	19	

When $b=0$ (45) gives the continued fraction expansion

$$(47)\quad \begin{cases} {}_2F_1(a, 1; c; x) = 1 + \frac{a}{c}x + \frac{a(a+1)}{c(c+1)}x^2 + \dots \\ \qquad = \frac{1}{1-}\,\frac{ax}{c-}\,\frac{1(c-a)x}{c+1-}\,\frac{(a+1)cx}{c+2-}\dots \\ \qquad \dots\frac{r(c-a+r-1)x}{c+2r-1-}\,\frac{(a+r)(c+r-1)x}{c+2r-}\dots \end{cases}$$

of the incomplete Beta Function.

When $a=1$, $c=2$, (47) gives

$$(48)\qquad \log(1+x) = \frac{x}{1+}\,\frac{1^2x}{2+}\,\frac{1^2x}{3+}\,\frac{2^2x}{4+}\,\frac{2^2x}{5+}\dots\frac{r^2x}{2r+}\,\frac{r^2x}{2r+1+}\dots$$

The numerical efficiency of this continued fraction is indicated in Table VI:

TABLE VI

x \ h	1	2	3	4	5	6	7	8	9	10	11	12
0.25	2	2	3	4	5	6	6	7	8	9	10	11
0.50	2	3	4	5	6	7	8	9	10	11	12	13
1.00	2	4	5	6	8	9	10	12	13	14	16	17
2.00	3	5	7	8	10	12	14	15	17	19	21	22
4.00	4	6	9	11	14	16	18	21	23	26	28	29
8.00	6	9	12	16	19	22	25	29	32	35	38	40

Writing $a={}^1/_2$, $c={}^3/_2$ and x^2 for x in (47) there follows

$$(49)\qquad \operatorname{arc\,tan}(x) = \frac{x}{1+}\,\frac{1x^2}{3+}\,\frac{4x^2}{5+}\dots\frac{(r-1)^2x^2}{2r-1+}\dots$$

the numerical efficiency of which is illustrated in Table VII:

TABLE VII

x \ h	1	2	3	4	5	6	7	8	9	10	11
0.25	1	2	2	3	4	4	5	5	6	6	7
0.50	2	2	3	4	5	6	6	7	8	9	10
1.00	2	4	5	6	8	9	10	12	13	14	16
2.00	4	7	8	11	14	16	19	21	23	26	28
4.00	8	12	17	22	26	31	36	40	45	50	54
8.00	15	24	34	43	52	61	71	80	89	99	105

When $a={}^1/_2$, $c={}^3/_2$, and x is replaced by $1/z$, (47) gives

$$(50)\qquad \log\left(\frac{z+1}{z-1}\right) = \frac{2}{z-}\,\frac{1}{3z-}\,\frac{4}{5z-}\dots\frac{(r-1)^2}{(2r-1)z-}\dots$$

valid for z not lying upon the line -1 to $+1$ through zero. The numerical efficiency of expansion (50) is illustrated in Table VIII:

TABLE VIII

z \ h	1	2	3	4	5	6	7	8	9	10	11	12
8.0	1	2	2	2	3	3	4	4	4	5	5	8
4.0	1	2	2	3	4	4	5	5	6	6	6	7
2.0	2	3	4	4	5	6	7	8	9	10	11	11

The hypergeometric function also satisfies the differential equation

$$x(1-x)y''+\{c-(a+b+1)x\}y'-aby=0 \tag{51}$$

which thereby yields the continued fraction

$$\frac{ab}{c-(a+b+1)x-}\;\frac{(a+1)(b+1)x(1-x)}{c+1-(a+b+3)x-}\cdots\frac{(a+r)(b+r)x(1-x)}{c+r-(a+b+2r+1)x-}\cdots \tag{52}$$

This continued fraction converges [9] to

$$\begin{cases} \dfrac{d}{dx}\log\{{}_2F_1(a,b;c;x)\} = \dfrac{ab}{c}\dfrac{{}_2F_1(a+1,b+1;c+1;x)}{{}_2F_1(a,b;c;x)} \\ \qquad\qquad\qquad\qquad \text{when } \operatorname{Re}(x)<{}^1\!/_2 \text{ or } x={}^1\!/_2 \end{cases} \tag{53}$$

and to

$$\begin{cases} \dfrac{d}{dx}\log\{{}_2F_1(a,b;a+b-c+1;1-x)\} = \\ \dfrac{-ab}{(a+b-c+1)}\dfrac{{}_2F_1(a+1,b+1;a+b-c+2;1-x)}{{}_2F_1(a,b;a+b-c+1;1-x)} \text{ when } \operatorname{Re}(x)>{}^1\!/_2. \end{cases} \tag{54}$$

It diverges, except at the point $x={}^1\!/_2$ upon the line $\operatorname{Re}(x)={}^1\!/_2$.

Writing $a=b=0$, $c=1$ and $-x$ for x, it follows that

$$\frac{x}{1+x-}\;\frac{1^2x(1+x)}{2+3x-}\cdots\frac{r^2x(1+x)}{r+1+(2r+1)x-}\cdots \tag{55}$$

converges to $\log(1+x)$ for $\operatorname{Re}(x)>-{}^1\!/_2$ or $x=-{}^1\!/_2$. The numerical efficiency of this continued fraction is indicated in Table IX

TABLE IX

x \ h	1	2	3	4	5	6	7	8	9	10	11	12
0.25	2	3	4	6	7	8	10	11	12	14	15	16
0.50	2	4	6	8	10	12	14	16	18	20	22	28
1.00	3	6	9	12	15	18	21	24	27	30	33	35
2.00	5	9	14	19	24	29	35	40	45	51		
4.00	7	15	24	33	42	52	61	71	81	99		
8.00	12	27	43	60	77	95	113	132	150	170		

c) *Continued Fractions Associated with the Confluent Hypergeometric Function*

The confluent hypergeometric function

$$(56)\quad {}_1F_1(a;c;x) = 1 + \frac{ax}{1!\,c} + \frac{a(a+1)x^2}{2!\,c(c+1)} + \ldots + \frac{a(a+1)\ldots(a+r-1)\,x^r}{r!\,c(c+1)\ldots(c+r-1)} + \ldots$$

satisfies the recursions

$$(57)\quad c(c+1)\,{}_1F_1(a;c;x) - c(c+1)\,{}_1F_1(a;c;x) + x(c-a)\,{}_1F_1(a+1;c+2;x) = 0$$

and

$$(58)\quad \left\{ \begin{array}{l} (c+1)(c+2)\,{}_1F_1(a+1;c+1;x) - (c+1)(c+2)\,{}_1F_1(a+1;c+2;x) \\ \qquad\qquad - (a+1)\,x\,{}_1F_1(a+2;c+3;x) = 0 \end{array} \right.$$

which may be used (cf. 1)) to derive the continued fraction

$$(59)\quad \left\{ \begin{array}{l} \dfrac{{}_1F_1(a+1;c+1;x)}{{}_1F_1(a;c;x)} = \dfrac{c}{c-}\,\dfrac{(c-a)\,x}{c+1+}\,\dfrac{(a+1)\,x}{c+2-}\ldots \\ \qquad\qquad\qquad\qquad \dfrac{(c-a+r-1)\,x}{c+2r-1+}\,\dfrac{(a+r)\,x}{c+2r-}\ldots \end{array} \right.$$

When $a=0$ (59) gives the expansion

$$(60)\quad {}_1F_1(1;c+1;x) = \frac{1}{1-}\,\frac{x}{c+1+}\,\frac{1.x}{c+2-}\ldots\frac{(c+r)\,x}{c+2r+1-}\,\frac{(r+1)\,x}{c+2r+2-}\ldots$$

When $c=0$ (60) gives in turn

$$(61)\quad \exp\,(x) = 1 + \frac{x}{1-}\,\frac{x}{2+}\,\frac{x}{3-}\ldots\frac{rx}{2r+}\,\frac{rx}{2r+1+}\ldots$$

The numerical efficiency of expansion (61) is indicated in Table X

TABLE X

x \ h	1	2	3	4	5	6	7	8	9	10	11	12
0.25	1	2	3	4	4	5	5	6	7	7	8	8
0.50	2	3	4	4	5	6	7	7	8	9	9	10
1.00	3	4	5	6	7	8	8	9	10	11	11	13
2.00	5	6	7	8	9	10	11	12	13	14	15	15
4.00	7	9	11	12	13	14	15	16	17	19	19	
8.00	13	15	17	19	20	21	23	24	25	26	27	29

The numerical efficiency of the continued fraction [10],

$$(62)\quad \exp\,(x) = 1 + \frac{2x}{2-x+}\,\frac{x^2}{6+}\,\frac{x^2}{10+}\,\frac{x^2}{14+}\ldots$$

which is the odd part of (61) may be obtained by replacing each entry s by $[(s+1)/2]$. The expansions (63) and (64) below may be contracted in a similar way.

When $c={}^1/_2$ and x is replaced by x^2 in (60) there follows

$$(63)\qquad x^{-1}e^{x^2}\int_0^x e^{-t^2}\,dt = \frac{1}{1-}\,\frac{2x^2}{3+}\,\frac{4x^2}{5-}\cdots\frac{(4r-2)\,x^2}{4r-1+}\,\frac{4rx^2}{4r+1-}\cdots$$

The numerical efficiency of this continued fraction is indicated in Table XI:

TABLE XI

x \ h	1	2	3	4	5	6	7	8	9	10	11	12
0.25	1	2	3	3	4	4	5	5	5	6	6	7
0.50	2	3	3	4	5	5	6	7	7	8	8	
1.00	3	5	6	6	7	8	9	10	11	11	12	12
2.00	8	10	11	12	14	15	16	17	18	19	20	21
4.00	27	29	30	31	33	35	36	38				

Writing $-x^2$ for x^2 in (63) there follows

$$(64)\qquad x^{-1}e^{-x^2}\int_0^x e^{t^2}\,dt = \frac{1}{1+}\,\frac{2x^2}{3-}\,\frac{4x^2}{5+}\cdots\frac{(4r-2)\,x^2}{4r-1-}\,\frac{4rx^2}{4r+1+}\cdots$$

The numerical efficiency of this continued fraction is indicated in Table XII:

TABLE XII

x \ h	1	2	3	4	5	6	7	8	9	10	11
0.25	1	2	3	3	4	4	5	5	5		
0.50	2	3	3	4	5	5	6	7	7	8	8
1.00	3	4	5	6	7	8	9	10	10	11	
2.00	6	8	9	11	12	13	14	16	17		
4.00	12	16	18	20	22	24	26	28	30		
8.00	26	32	36	40	44	48	50	54	56	59	

The confluent hypergeometric function also satisfies the differential equation

$$(65)\qquad xy'' + (c-x)\,y' - ay = 0$$

and since

$$\frac{d}{dx}\,{}_1F_1(a;\,c;\,x) = \frac{a}{c}\,{}_1F_1(a+1;\,c+1;\,x)$$

the continued fraction of the form (18) is

$$(66)\qquad \frac{{}_1F_1(a+1;\,c+1;\,x)}{{}_1F_1(a;\,c;\,x)} = \frac{c}{c-x+}\,\frac{(a+1)\,x}{c+1-x+}\,\frac{(a+2)\,x}{c+2-x+}\cdots$$

A special case of (66) is the expansion

$$(67)\qquad \left\{\begin{aligned} &\operatorname{erf}(x) = \int_0^x e^{-t^2}\,dt = \\ &\qquad xe^{-x^2}\left\{\frac{1}{1-2x^2+}\,\frac{4x^2}{3-2x^2+}\,\frac{8x^2}{5-2x^2+}\cdots\frac{4(r-1)\,x^2}{2r-1-2x^2+}\cdots\right\}\end{aligned}\right.$$

the numerical efficiency of which is indicated in Table XIII:

TABLE XIII

x h	1	2	3	4	6	6	7	8	9	10	11	12
0.25	2	2	3	4	4	5	5	6	6	7	7	8
0.50	3	4	4	5	6	7	7	8	9	9	10	10
1.00	5	7	8	9	10	11	12	13	14	15	15	
2.00	16	18	19	21	22	24	25	26	27	28	30	

Replacing x^2 by $-x^2$ in (67), there follows

$$x^{-1} e^{-x^2} \int_0^x e^{t^2}\, dt = \frac{1}{1+2x^2-} \frac{4x^2}{3+2x^2-} \cdots \frac{4(r-1)x^2}{2r-1+2x^2-} \cdots \tag{68}$$

The numerical efficiency of this continued fraction is illustrated in Table XIV from which it may be seen that the continued fraction (68) is unique among this collection of expansions in that there are two points ($x=0$ and $x=\infty$) near which it is rapidly convergent.

TABLE XIV

x h	1	2	3	4	5	6	7	8	9	10	11
0.25	2	2	3	4	4	5	5	6	6		
0.50	2	3	4	5	6	6	7	8	8	9	10
1.00	4	5	7	8	9	10	11	12	13	13	
2.00	4	8	11	13	15	17	18	20	21	22	
4.00	2	3	4	6	10	18	25	29	32		
8.00	1	2	2	3	4	5	6	7	8	8	

THE NUMERICAL EFFICIENCY OF CERTAIN CONTINUED FRACTION EXPANSIONS. IB

BY

P. WYNN

(Communicated by Prof. A. VAN WIJNGAARDEN at the meeting of September 30, 1961)

d) *Continued Fractions Associated with Certain Asymptotic Series*

The function associated with the asymptotic series

(69) $$ {}_2F_0(a,b;x) \sim 1 + \frac{abx}{1!} + \frac{a(a+1)b(b+1)x^2}{2!} + \dots $$

satisfies the recursions

(70) $$ {}_2F_0(a,b;x) - {}_2F_0(a,b+1;x) + ax\,{}_2F_0(a+1,b+1;x) = 0 $$

and

(71) $$ {}_2F_0(a,b+1;x) - {}_2F_0(a+1,b+1;x) + (b+1)x\,{}_2F_0(a+1,b+2;x) = 0 $$

which may be used to derive the continued fraction

(72) $$ \frac{{}_2F_0(a,b+1;x)}{{}_2F_0(a,b;x)} = \frac{1}{1-}\,\frac{ax}{1-}\,\frac{(b+1)x}{1-}\,\frac{(a+1)x}{1-}\,\frac{(b+2)x}{1-}\dots $$

Writing $x = -(z)^{-1}$, (72) becomes

(73) $$ \left\{ \begin{aligned} &\frac{{}_2F_0(a,b+1;-(z)^{-1})}{{}_2F_0(a,b;-(z)^{-1})} = \frac{1}{1+}\,\frac{a}{z+}\,\frac{(b+1)}{1+}\,\frac{(a+1)}{z+}\,\frac{(b+2)}{1+}\dots \\ &\qquad\qquad\qquad\qquad \dots\frac{(a+r-1)\,(b+r)}{z+\quad 1+}\dots \end{aligned} \right. $$

which, when $a=1$, $b=0$, in turn becomes

(74) $$ -ze^z\,Ei(-z) = \frac{1}{1+}\,\frac{1}{z+}\,\frac{1}{1+}\,\frac{2}{z+}\,\frac{2}{1+}\dots\frac{r}{z+}\,\frac{r}{1+}\dots $$

The numerical efficiency of expansion (74) is indicated in Table XV:

TABLE XV

z \ h	1	2	3	4	5	6	7	8	9	10	11	12
8.00	2	3	4	6	7	9	11	13	16	18	21	22
4.00	2	4	6	8	11	14	17	21	25	29	34	36
2.00	3	6	9	13	18	23	28	35	42	50	58	65
1.00	4	9	14	22	30	40	50	62	76	90	106	120
0.50	6	14	24	38	54	72	92	116	142	170	198	214
0.25	10	24	44	68	98	132	172	218	270	323	363	381

Writing $a={}^1/_2$ in (73), and effecting an equivalence transformation, there follows

$$e^{z^2}\int_z^\infty e^{-t^2}\,dt = \frac{1}{2z+}\,\frac{2}{2z+}\,\frac{4}{2z+}\cdots\frac{2(r-1)}{2z+}\cdots \tag{75}$$

the numerical efficiency of which is illustrated in Table XVI:

TABLE XVI

z \ h	1	2	3	4	5	6	7	8	9	10	11
0.25	32	80	149	238	350						
0.50	10	23	41	64	93	127	166	211	261	316	358
1.00	4	8	13	19	27	36	47	58	72	86	98
2.00	2	3	5	7	10	13	16	19	23	27	31
4.00	1	2	3	4	5	6	7	9	10		
8.00	1	2	2	3	3	4	5				

Expansion (73) also yields

$$e^{-z^2}\int_0^z e^{t^2}\,dt \sim \frac{1}{2z-}\,\frac{2}{2z-}\,\frac{4}{2z-}\cdots\frac{2(r-1)}{2z-}\cdots \tag{76}$$

which, although easily shown to be divergent, has as Table XVII indicates, a certain asymptotic property, and may be used to compute Dawson's integral for large argument.

TABLE VXII

z \ h	1	2	3	4	5	6	7	8	9	10	11
8.00	1	2	2	3	3	4	5	5	6	39	92
4.00	1	2	3	4	6	9	10	43	318	318	
2.00	3	3	193	225							
1.00	11	44									
0.50	78	123									
0.25	19										

The even part of (73) is

$$\frac{{}_2F_0(a, b+1; -(z)^{-1})}{{}_2F_0(a, b; -(z)^{-1})} = \frac{z}{z+a-}\,\frac{a(b+1)}{z+(a+b+2)-}\,\frac{(a+1)(b+2)}{z+(a+b+4)-}\cdots \tag{77}$$

and that of (74) is

$$-ze^z\,Ei(-z) = \frac{1}{z+1-}\,\frac{1^2}{z+3-}\cdots\frac{r^2}{z+2r+1-}\cdots \tag{78}$$

The numerical efficiency of expansion (78) may be deduced from Table XV by replacing each entry s by $[(s+1)/2]$.

A further continued fraction expansion for the ratio of two functions of the form (69) may be derived by noting that the function ${}_2F_0(a, b; x)$ satisfies the differential equation

$$x^2y'' + \{(a+b+1)x-1\}y' + aby = 0. \tag{79}$$

There then follows (cf. 3)

$$(80)\qquad \frac{{}_2F_0'(a,b;x)}{{}_2F_0(a,b;x)} = \frac{ab}{1-(a+b+1)x-}\;\frac{(a+1)(b+1)x^2}{1-(a+b+3)x-}\cdots$$

Writing $x=-(z)^{-1}$, and noting that

$$(81)\qquad {}_2F_0'(a,b;x) = ab\,{}_2F_0(a+1,b+1;x)$$

(80) becomes

$$(82)\qquad \begin{cases} \dfrac{{}_2F_0(a+1,b+1;-(z)^{-1})}{{}_2F_0(a,b;-(z)^{-1})} = \dfrac{z}{z+(a+b+1)-}\;\dfrac{(a+1)(b+1)}{z+(a+b+3)-}\cdots \\ \qquad\qquad\qquad\qquad\qquad\qquad \dfrac{(a+r-1)\,(b+r-1)}{z+(a+b+2r-1)-}\cdots \end{cases}$$

The continued fractions (72) and (80) are not essentially distinct, for the odd part of (72) is

$$(83)\qquad \frac{{}_2F_0(a,b+1;x)}{{}_2F_0(a,b;x)} = 1+\frac{ax}{1-(a+b+1)x-}\;\frac{(a+1)(b+1)x^2}{1-(a+b+3)x+}\cdots$$

and (80) and (83) may be derived from each other by use of the relationship

$$(84)\qquad \frac{{}_2F_0(a,b+1;x)}{{}_2F_0(a,b;x)} = 1+ax\,\frac{{}_2F_0(a+1,b+1;x)}{{}_2F_0(a,b;x)}.$$

e) *A Continued Fraction Associated with a Special Asymptotic Series*

The numerical efficiency of the continued fraction

$$(85)\qquad f_k(z) = \int_0^\infty (\operatorname{sech} t)^k e^{-zt}\,dt = \frac{1}{z+}\;\frac{1.k}{z+}\;\frac{2(k+1)}{z+}\cdots\frac{(r-1)\,(k+r)}{z+}\cdots$$

has also been examined. The expansion (85) may be derived by use of a number of methods (those of 4), and by application of the method 2) to the series

$$(86)\qquad F_k(z) \sim \sum_{n=0}^{\infty} E_n^{(k)} z^{-n-1}.$$

The numerical efficiency of the expansion (85) is indicated in Tables XVIII and XIX which refer to the cases $k=1$ and $k=2$ respectively.

TABLE XVIII

z \ h	1	2	3	4	5	6	7	8	9	10	11
32.0	1	1	2	2	2	3	3	4	4	5	6
16.0	1	1	2	3	3	4	5	6	7	8	9
8.0	1	2	3	4	5	7	10	13	17	23	31
4.0	2	3	5	10	17	31	55	97	151		
2.0	3	10	30	96	242						
1.0	14	108	252								

TABLE XIX

z \ h	1	2	3	4	5	6	7	8	9	10	11
32.0	1	1	2	2	3	3	4	4	5	6	6
16.0	1	2	2	3	4	4	5	7	8	9	11
8.0	1	2	3	5	7	9	12	17	22	30	39

A method for transforming the continued fraction (85), thereby increasing its rate of convergence, is given in [11]. The expansion (85) belongs to the class of (52), indeed it is evident from comparison of (85) and (52) that

$$F_k(z) = \frac{1}{z+k}\, {}_2F_1\left(1, k; \frac{z+k}{2} + 1; {}^1/_2\right). \tag{87}$$

f) *Continued Fraction Expansions Associated with Three Slowly Convergent Series*

Three expansions, which may be derived by the method of 4) are [12]

$$\text{(88)}\quad \begin{cases} \sum\limits_{p=0}^{\infty} (z+p)^{-2} = \int\limits_0^\infty \dfrac{u\, e^{(-2z+1)u}}{\sinh u}\, du \\[2ex] \qquad = \dfrac{2}{2z-1+}\, \dfrac{1^4}{3(2z-1)+}\, \dfrac{2^4}{5(2z-1)+} \cdots \dfrac{(r-1)^4}{(2r-1)\,(2z-1)+} \cdots \end{cases}$$

$$\text{(89)}\quad \begin{cases} \sum\limits_{p=0}^{\infty} (-1)^p (z+p)^{-2} = \int\limits_0^\infty \dfrac{u\, e^{(-2z+1)u}}{\cosh u}\, du \\[2ex] \qquad = \dfrac{(2z)^{-1}}{z-1+}\, \dfrac{1^2}{z+}\, \dfrac{1^2}{z-1+}\, \dfrac{2^2}{z+}\, \dfrac{2^2}{z-1+} \cdots \dfrac{r^2}{z+}\, \dfrac{r^2}{z-1+} \cdots \end{cases}$$

and [13]

$$\text{(90)}\quad \begin{cases} 4z^3 \sum\limits_{p=0}^{\infty} (z+p)^{-3} = 2z + 2 + \dfrac{1}{z+}\, \dfrac{1^2.2}{6z+}\, \dfrac{1.2^2}{z+}\, \dfrac{2^2.3}{10z+} \cdots \\[2ex] \qquad \cdots \dfrac{r^2(r+1)}{(4r+2)\,z+}\, \dfrac{r(r+1)^2}{z+} \cdots \end{cases}$$

(note the error in Wall [1], p. 373, eqn. (94.14)).

The numerical efficiency of these expansions is illustrated in Tables XX, XXI and XXII:

TABLE XX

z \ h	1	2	3	4	5	6	7	8	9	10	11	12
32.0	1	1	1	2	2	2	2	3	3	3	4	4
16.0	1	1	1	2	2	2	3	3	4	4	4	5
8.0	1	1	2	2	3	3	4	4	5	5	6	7
4.0	1	2	2	3	4	4	5	7	8	10	11	13
2.0	1	2	4	5	8	12	16					
1.0	3	10	16									

TABLE XXI

z \ h	1	2	3	4	5	6	7	8	9	10	11
32.0	1	1	2	2	2	3	3	3	4	4	5
16.0	1	1	2	2	3	3	4	4	5	6	6
8.0	1	2	2	3	4	5	6	7	8	10	12
4.0	2	2	4	6	8	12	17	24	32	46	64

TABLE XXII

z \ h	1	2	3	4	5	6	7	8	9	10	11	12
32.0	0	0	0	1	1	1	2	2	2	3	3	3
16.0	0	0	1	1	1	2	2	3	3	3	4	4
8.0	0	1	1	1	2	2	3	4	4	5	6	6
4.0	0	1	1	2	3	4	5	6	7	9	10	12
2.0	1	2	3	4	6	9	12	17	23	32	40	
1.0	1	4	8	15	27	50	77					

g) Continued Fractions Associated with Theta Functions

The series

(91) $$\phi_0(q,x) = 1 + qx + q^4x^2 + q^9x^3 + \dots$$

may be transformed by the method of 2) into the continued fraction

(92) $$1 + \frac{qx}{1-}\,\frac{q^3x}{1+}\,\frac{q^3(1-q^2)x}{1-}\,\frac{q^7x}{1+}\,\frac{q^5(1-q^4)x}{1-}\dots$$

For certain reasons, a full discussion of which is given in [14], it is in practice more suitable to use series of the type (91) for the computation of theta functions. Nevertheless results which indicate the numerical efficiency of (92) will be given, since at some future time it may happen that continued fractions whose coefficients approximate to those of (92), occur in practice. Accordingly the numerical efficiency of expansion (92) for the case is shown in Tables XXIII to XXVI:

TABLE XXIII

x \ h	1	2	3	4	5	6	7	8	9	10	11
0.25	0	1	1	2	2	2	3	3	3	3	3
0.50	1	1	1	2	2	3	3	3	3	3	4
1.00	1	1	2	2	3	3	3	3	3	4	4
2.00	1	1	2	3	3	3	3	3	4	4	4
4.00	1	2	2	3	3	3	3	4	4	4	5
8.00	1	2	3	3	3	3	4	4	5	5	5

TABLE XXIV

x h	1	2	3	4	5	6	7	8	9	10	11
0.25	1	1	2	2	3	3	3	4	4	4	5
0.50	1	2	2	3	3	3	4	4	5	5	5
1.00	1	2	3	3	3	4	4	5	5	5	5
2.00	2	3	3	3	4	4	5	5	5	5	6
4.00	2	3	3	4	5	5	5	5	6	6	7
8.00	3	3	4	5	5	5	5	6	7	7	7

TABLE XXV

x h	1	2	3	4	5	6	7	8	9	10	11
0.25	1	2	2	3	3	4	4	5	5	5	6
0.50	1	2	3	3	4	5	5	5	6	6	7
1.00	2	3	3	4	5	5	6	6	7	7	7
2.00	3	3	5	5	5	6	7	7	7	8	8
4.00	3	5	5	6	7	7	7	8	9	9	9
8.00	5	5	7	7	7	8	9	9	9	10	10

TABLE XXVI

x h	1	2	3	4	5	6	7	8	9	10	11
0.25	1	2	3	4	4	5	6	6	7	7	8
0.50	2	3	4	5	5	6	7	7	8	9	9
1.00	3	4	5	6	7	8	9	9	10	11	11
2.00	5	6	7	8	9	10	11	11	12	13	13
4.00	7	9	9	11	11	12	13	13	14	15	15
8.00	10	11	12	13	13	15	15	16	17	17	17

Comparison of Two Types of Continued Fraction Expansion

It is clear (by comparison of Tables IV and V, VII and IX) that in most cases where differing continued fractions for the same function may be derived by application of the methods of 1) and 3), the numerical efficiency of that relating to 3) is inferior, though expansion (68) provides an interesting exception.

Comparison with Power Series

In certain cases the numerical convergence of a continued fraction expansion is strikingly better than that of an equivalent power series; indeed a continued fraction may be used to define a certain function in the whole of an arbitrary bounded cut plane, whereas the equivalent power series defines the function only within a circle in the complex plane.

An explanation of this is as follows. If the function

$$f(x) \sim \sum_{s=0}^{\infty} c_s x^s \tag{93}$$

has a pole at the point $x = \varrho e^{i\theta}$ and no other poles within the circle $|x| = \varrho$, then the numerical rate of convergence of the power series (91) for x within the circle $|x|=\varrho$ is largely determined by the ratio $|x|/\varrho$. The successive convergents of an equivalent continued fraction (determined, for example, as in 2)) are rational functions of x. The presence of the denominators in these rational expressions tends to remove the effect of the successive poles (simple or multiple) of $f(x)$. (For a more precise formulation of this idea in the context of the more general rational expressions of the Padé Table, reference should be made to a theorem of MONTESSUS DE BALLOIRE ([15], [1] Ch. XX, p. 412).) Where this reasoning is relevant, as in the case of $f(x)=\tan(x)$, the improvement in convergence is emphatic, where it is not, as in the case of $f(x)=\exp(x)$, the improvement is correspondingly not so marked.

COMPARISON WITH TSCHEBYSCHEFF POLYNOMIAL APPROXIMATIONS

It is of interest to compare the numerical performance of Tschebyscheff polynomial approximations with that of continued fraction approximations of the type discussed in this paper. A general comparison is, of course, not possible, since Tschebyscheff polynomial approximations are designed to provide uniform accuracy upon a fixed contour, whilst continued fraction approximations of the type previously discussed, are in general valid over some domain in the complex plane but do not, as Tables II to XXVI indicate, provide uniformly good numerical approximation. Nevertheless some sort of comparison can be made by contrasting the numerical performance of the two methods upon the fixed contour upon which the Tschebyscheff polynomials approximation is valid, and using as a standard of the numerical performance of the continued fraction of which the rational approximation is a convergent, the slowest convergence over the range considered. Table XXVII indicates the degree n of the Tschebyscheff series approximation $\sum_{s=0}^{n} a_{n,s}x^s$ to the function $f(x)$ which is sufficient to ensure that the error

$$(92) \qquad \left|f(x)-\sum_{s=0}^{n} a_{n,s}x^s\right| \leqslant {}^1/_2\, 10^{-h}$$

where h increases by steps of unity as indicated in the table headings, and the function being approximated is

a) $\log(1+x)$ when $0 \leqslant x \leqslant 1$
b) $\exp(x)$ when $0 \leqslant x \leqslant 1$
c) $-ze^z Ei(-z)$, $z=x^{-1}$ and $0 \leqslant x \leqslant 1$

TABLE XXVII

h	0	1	2	3	4	5	6	7	8
a	1	2	3	4	6	7	8	9	10
b	1	2	3	4	4	5	6	6	7
c	1	2	2	4	5	7	9	11	14

The entries in this table may be deduced from figures given in references [16] and [17].

Now it will be recalled that the nth convergent of the continued fraction

$$b_0 + \frac{a_1 x}{b_1 +} \frac{a_2 x}{b_2 +} \cdots \frac{a_r x}{b_r +} \cdots \tag{93}$$

is the quotient of two polynomials of the nth degree in x, whilst the $(2n+1)$th convergent is the quotient of a polynomial of the $(n+1)$th degree divided by a polynomial of the nth degree in x, and that the nth convergent of the continued fraction

$$\frac{c_m}{z-\alpha_0^{(m)} -} \frac{\beta_0^{(m)}}{z-\alpha_1^{(m)} -} \cdots \frac{\beta_{r-1}^{(m)}}{z-\alpha_r^{(m)} -} \cdots \tag{94}$$

is the quotient of a polynomial of the $(n-1)$th degree in divided by a polynomial of the nth degree; it may thus be seen by comparing Tables VI, X and XV with Table XXVII, that where the numerical performance of the two methods may legitimately be contrasted and the function $f(x)$ is relatively well behaved, the Tschebyscheff polynomial series are better than the rational approximation.

If the function $f(x)$ has a pole for some value of the argument which lies in the range over which an approximation is required, a Tschebyscheff series approximation cannot, of course, be derived, and if $f(x)$ has also a zero or sequence of zeros when the argument lies in this range, then the difficulty may not be overcome by trying to obtain an approximation to the function $\{f(x)\}^{-1}$. The difficulty may be overcome by a suitable artifice, such as writing

$$f(x) = \frac{r(x)}{s(x)} \tag{95}$$

where $r(x)$ and $s(x)$ are analytic in the range considered, and obtaining separate polynomial approximations for the functions $r(x)$ and $s(x)$. Such a state of affairs is encountered when considering the function $J_1(x)/J_0(x)$ when $-10 \leqslant x \leqslant 10$. The most obvious choices for $r(x)$ and $s(x)$ in this case are

$$xr\left(\frac{x^2}{100}\right) = J_1(x) \qquad s\left(\frac{x^2}{100}\right) = J_0(x). \tag{96}$$

Table XXVIII indicates, in the same manner as Table XXVII, the numerical efficiency of the Tschebyscheff polynomial series for $J_1(x)$ and $J_0(x)$ when $-10 \leqslant x \leqslant 10$.

TABLE XXVIII

h	0	1	2	3	4	5	6	7
$J_0(x)$	3	5	7	8	9	9	10	12
$J_1(x)$	0	4	5	6	7	8	9	10

(The entries in this table were also deduced from figures given in reference [16].)

Comparison of Tables II and XXVIII shows that even in this case greater economy of computational effort is achieved by use of Tschebyscheff series. MAEHLY [18], however, shows that for (41) over the range $|x| \leqslant \pi/4$ the numerical performance of the Tschebyscheff series is inferior.

It is perhaps more meaningful to compare Tschebyscheff type approximations with rational approximations which are best in a certain sense, and for this reference should be made to the wrok of MAEHLY [18][19] and others.

CONCLUSION

The successive convergents, defined by equation (32), of the continued fractions occurring in this paper have been evaluated by effecting the twin recursions

$$\left\{ \begin{array}{l} A_n = b_n A_{n-1} + a_n A_{n-2} \\ B_n = b_n B_{n-1} + a_n B_{n-2} \end{array} \right. \tag{97}$$

from the initial values

$$A_{-1} = 1, \quad A_0 = b_0, \quad B_{-1} = 0, \quad B_0 = 1,$$

and thereafter carrying out the division

$$C_n = A_n / B_n.$$

The computations have been carried out using fixed length (forty binary figures, equivalent to slightly more than twelve decimal figures) floating point arithmetic. The various functions have either been extracted from published tables (refs. [20] to [29])or computed independently: this accounts for the varying format of the tables.

If the coefficients of the continued fraction (32) are positive and real and the growth of error in equations (97) is not too rapid, the inequality (30) may be replaced by

$$\left| \frac{C_n - C_{n+1}}{C_{n+1}} \right| \leqslant \epsilon \tag{98}$$

without serious consequences. (In practice this leads to the computation of a convergent of somewhat higher order than is necessary.)

If, however, no restriction may be placed upon the coefficients, use of the inequality (98) to estimate the relative error made by accepting the convergent C_{n+1} as the value of the function being computed, may lead to monumental blunders.

One disadvantage attendant upon the use of the recursions (97) is that the numerical values of A_n and B_n may rapidly exceed the capacity of the arithmetic facilities being used. This may be avoided by effecting the equivalence transformation

$$\left\{ \begin{array}{l} C = b_0 + \dfrac{a_1}{b_1 +} \dfrac{a_2}{b_2 +} \cdots \dfrac{a_r}{b_r +} \cdots \\ \quad = b_0 + \dfrac{\gamma_1 a_1}{\gamma_1 b_1 +} \dfrac{\gamma_1 \gamma_2 a_2}{\gamma_2 b_2 +} \dfrac{\gamma_2 \gamma_3 a_3}{\gamma_3 b_3 +} \cdots \dfrac{\gamma_{r-1} \gamma_r a_r}{\gamma_r b_r +} \cdots \end{array} \right. \tag{99}$$

and choosing the $\gamma_r \; r=1, 2, \ldots$ such that

$$\max |\gamma_{r-1}\, \gamma_r\, a_r| \;,\; |\gamma_r\, b_r| = 1$$

but this hardly improves the propagation of rounding errors in (97).

There is a sense in which the tables given in this paper represent the theoretical optimum performance which may be attained by use of the continued fractions treated. One fact which may certainly not be deduced by perusing the tables is the manner in which error is introduced in the evalution of the successive convergents by the growth of unwanted complementary solutions in the difference equations in (97). Indeed it would appear that in the case of certain continued fractions, if the argument is given to a certain precision, and the computations are carried out with a certain length arithmetic, than there is no point in prolonging the computation beyond a certain limit, beyond which the estimation of C by C_n becomes steadily worse, though C_n may appear to converge steadily to a spurious result. (This is another reason why the inequality (98) must be used with care.) One could construct tables, similar in forms to those in this paper, having as arguments x or z, and the relative errors in x or z, and giving the order n of the convergent belong which there is no point in prolonging the computation; but these tables would, of course, depend upon the length of the arithmetic used.

Most of the continued fractions considered above are of the form

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_r}{b_r+}\cdots \tag{100}$$

where

$$a_r = \sum_{s=0}^{s=p} \alpha_s\, r^s \qquad b_r = \sum_{s=0}^{s=q} \beta_s\, r^s. \tag{101}$$

Provisional work which the author has carried out on the $q-d$ algorithm indicates that for certain classes of power series (namely those in which the coefficients satisfy linear recursions of a certain form) the coefficients in the equivalent continued fraction are expressible as Laurent series in their suffices, that is to say that the numerical behaviour of these continued fractions is dominated by those of the form (100). It is therefore possible that the continued fractions which are described in this note are archetypal for certain of those which will find numerical application in the future.

Acknowledgements

The manuscript of this paper was first drafted whilst the author was member of the Institute of Applied Mathematics of the University of Mainz (he is grateful to the Deutsche Forschungsgemeinschaft for providing him with a grant) and was revised at the Mathematical Centre in Amsterdam. Tables II–XXVI were computed upon the X1 in Amsterdam.

REFERENCES

1. WALL, H., Analytic Theory of Contained Fractions, Van Nostrand 1948.
2. RUTISHAUSER, H., Der Quotient-Differenz Algorithmus, Birkhauser Verlag.
3. WYNN, P., The Rational Approximation of a Function which is Formally Defined by a Power Series Expansion, Math. of Comp. 14, 1960, p. 147.
4. PERRON, O., Die Lehre von den Kettenbrüchen, vol. I, B. E. Teubner, Stuttgart, 1954.
5. INCE, E. L., Ordinary Differential Equations, Longmans Green and Co Ltd, 178 (1927).
6. STIELTJES, T. J., Sur la Réduction en Fraction Continue d'une Série Précédent Suivant les Puissances Descendant d'une Variable, Ann. Fac. Sci. Toulouse 3, (1889).
7. RUTISHAUSER, H., Ein Kontinuierliches Analogon zum Quotienten Differenzen Algorithmus, Arch. Math. 5, 132–137 (1954)
8. WATSON, G. N., A Treatise on the Theory of Bessel Functions, Cambridge, 153 (1922).
9. INCE, E. L., On the Continued Fractions Connected with the Hypergeometric Equation, Proc. London Math. Soc., 18, 236 (1918).
10. MACON, N., A Continued Fraction for e^x MTAC 9, 1941 (1955).
11. RUTISHAUSER, H., Beschleunigung der Konvergenz einer gewissen Klasse von Kettenbrüchen, ZAMM 38, 187 (1958).
12. ROGERS, L. J., On the Representation of Certain Asymptotic Series as Convergent Continued Fractions, Proc. London Math. Soc. (2) 4, 72 (1907).
13. STIELTJES, T. J., Sur quelques Intégrales Définies et leur Développement en Fractions Continues, Quart. Jrn of Math. 24, 370 (1890).
14. WYNN, P., Continued Fraction Expansions of the Jacobian Theta Functions and the Jacobian Elliptic Functions, to appear;
15. MONTESSUS DE BALLOIRE, R. DE, Sur les Fractions Continues Algébraiques, Bull. Soc. Math. de France, 30, 28 (1902).
16. CLENSHAW, C. W., Polynomial Approximations to Elementary Functions, MTAC, VIII, 47, 143 (July).
17. ————, Private Communication.
18. MAEHLY, H., Rational Approximation for Transcendental Functions, Proceedings of International Conference on Information Processing, Unesco, Paris 1959.
19. ————, Methods for Fitting Rational Approximations, Jrnl. Ass. Comp. Mach. 7, 150 (1960).
20. Tables of the Bessel Functions of the First Kind and Orders, Cambridge, Mass. 1947.
21. Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments, New York 1943.
22. Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments, New York, 1943.
23. British Association for the Advancement of Science, Mathematical Tables, Vol. I, Cambridge 1946.
24. Table of Natural Logarithms, Vol. III and Vol. IV, New York 1941.
25. Table of Arctan x, New York 1942.
26. Table of Arcsin x, New York 1945.
27. Tables of the Exponential Function e^x, New York 1947.
28. Tables of Probability Functions Vol. I, New York 1941.
29. BARKLEY ROSSER J.,, Theory and Application of

$$\int_0^z e^{-x^2} \text{ and } \int_0^z e^{-p^2y^2}dy \int_0^y e^{-x^2}dx.$$