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The Numerical Transformation of Slowly Convergent Series by Methods of Comparison*

Part I

by

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Une théorie formelle de la transformation

$$\sum_{s=0}^{\infty} c_s v_s x^s \sim \sum_{s=0}^{m-1} c_s v_s x^s + \sum_{s=0}^{\infty} x^{m+s} \Phi_m^{(s)}(x) \Delta^s v_m \qquad \text{a)}$$

où

$$\Phi_m(x) \sim \sum_{s=0}^{\infty} c_{m+s} x^s \qquad \text{b)}$$

est donnée. Les cas où la fonction $\Phi_o(x)$ satisfait une équation différentielle linéaire en x et où les quantités v_s satisfont une relation récurrente linéaire en s, sont traités en détails. Des exemples numériques sont donnés.

A formal theory of the transformation

a)

where

b)

is given. The cases in which the function $\Phi_o(x)$ satisfies a linear differential equation in x and in which the quantities v_s satisfy a linear recurrence relation in s are treated in detail. Numerical examples are given.

Gegeben ist eine formale Theorie des Transformations :

a)

wo

b)

Die Fälle, wo die Funktion $\Phi_o(x)$ eine lineare Differentialgleichung in x erfüllt und wo die Quantitäten v_s eine lineare Rekurrenzbeziehung in s erfüllen, werden ausführlich abgehandelt. Numerische Beispiele sind beigegeben.

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Автор дает формальную теорию преображения

а)

где б)

Детально изучаются те случаи если функция $\Phi_0(x)$ является решением линейного дифференциального уравнения и если v_s удовлетворяет возвратной линейной зависимости в s. Автор дает численные примеры.

1. Introduction.

1-1. This paper concerns itself with a method, having a wide range of application, for improving the performance of series which are numerically slowly convergent. A number of techniques have been developed for this purpose (particularly powerful among these being the use of the converging factor [1]) but most of these assume that the terms $u_n(x)$ of the series

$$S(x) \sim \sum_{n=0}^{\infty} u_n(x) \qquad (1\text{-}1\text{-}1)$$

satisfy a linear difference equation in n [2], or that the ratio $u_{n+1}(x)/u_n(x)$ may easily be developed as a Laurent series in n [3], or that $S(x)$ satisfies a linear differential equation [2].

The behaviour of most macroscopic physical phenomena may be described by partial differential equations or systems of such equations. The solution to any such equation in which interest is being taken may often be developed as a series expansion, but series of this type do not usually satisfy the requirements of the techniques for improving convergence mentioned above.

For the success of the method to be described however it is merely necessary that the numerical behaviour of the terms in the series to be computed should be similar, in a sense later to be described, to that of those in a series whose sum is known and whose behaviour may be easily be investigated.

Before proceeding to a description of the method it is stated that the term slowly convergent, as it is used generally in Numerical Analysis and as it will be understood here, is an euphemism implying a variety of computational misfortunes, among these.

a) straightforward slow convergence as occurs for example with the series

$${}_2F_1(a, b; c; x) = 1 + \frac{ab}{c\,1\,!}x + \frac{a(a+1)\,b(b+1)}{c(c+1)\,2\,!}x^2 + \dots \qquad (1\text{-}1\text{-}2)$$

when $|x|$ is slightly less than unity;

b) straightforward divergence, as occurs with (1-1-2) when $|x| >> 1$;

c) excessive cancellation, as occurs for example with ascending series of the form

$$J_0(x) = 1 - \frac{1}{(1\,!)^2}\frac{x^2}{4} + \frac{1}{(2\,!)^2}\left(\frac{x^2}{4}\right)^2 - \frac{1}{(3\,!)^2}\left(\frac{x^2}{4}\right)^3 + \dots \tag{1-1-3}$$

and

$$\exp(-x) = 1 - \frac{x}{1\,!} + \frac{x^2}{2\,!} - \frac{x^3}{3\,!} + \dots \tag{1-1-4}$$

with large argument;

d) delayed convergence, as occurs for example with the series

$$I_0(x) = 1 + \frac{1}{(1\,!)^2}\frac{x^2}{4} + \frac{1}{(2\,!)^2}\left(\frac{x^2}{4}\right)^2 + \frac{1}{(3\,!)^2}\left(\frac{x^2}{4}\right)^2 + \dots \tag{1-1-5}$$

and

$$\exp(x) = 1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \dots \tag{1-1-6}$$

with large argument.

e) Series which are asymptotically convergent and yield a limited, and often inadequate, amount of information about the function with which they are associated. An example of such a series is

$$-ze^z\mathrm{E}i(-z) \sim 1 - \frac{1\,!}{z} + \frac{2\,!}{z^2} - \frac{3\,!}{z^3} + \dots \tag{1-1-7}$$

The method to be described meets with success in all these cases.

2. The Euler-Gudermann Transformation [4].

2-1. The transformation to be used is simply the following :

Theorem 1

If

$$\Phi(x) \sim \sum_{s=0}^{h} c_s x^s \tag{2-1-1}$$

then

$$\Theta(x) \sim \sum_{s=0}^{h} c_s v_s x^s \tag{2-1-2}$$

$$\sim \sum_{s=0}^{h} \frac{x^s}{s\,!}\Phi^{(s)}(x)\Delta^s v_0 \tag{2-1-3}$$

If h is infinite and any of the series (2-1-1), (2-1-2) or (2-1-3) diverges then the sign $\sim$ is taken to indicate formal equivalence. If h is finite the result is exact. A discussion of the circumstances under which the series (2-1-3) converges and of the function $\Theta(x)$ which it then defines will be given in a later section.

An operational demonstration of the result (2-1-3) proceeds as follows :

If $E^r v_o \equiv v_r \qquad r = 1, 2, \ldots$

and $\Delta \equiv E - 1$

then

$$\sum_{s=0}^{h} c_s v_s x^s \equiv \sum_{s=0}^{h} c_s x^s E^s v_o$$

$$\equiv \sum_{s=0}^{h} \Phi(xE) v_o$$

$$\equiv \sum_{s=0}^{h} \Phi(x + x\Delta) v_o$$

$$\equiv \sum_{s=0}^{h} \frac{x^s}{s\,!} \Phi^{(s)}(x) \Delta^s v_o \tag{2-1-4}$$

This result includes the generalised Euler transformation

$$\sum_{s=0}^{\infty} (-x)^s v_s \equiv \frac{1}{(1+x)} \sum_{s=0}^{\infty} \left(\frac{-x}{1+x}\right)^s \Delta^s v_o \tag{2-1-5}$$

and the well known transformation

$$\sum_{s=0}^{\infty} \frac{(-x)^s}{s\,!} v_s \equiv e^{-x} \sum_{s=0}^{\infty} \frac{(-x)^s}{s\,!} \Delta^s v_o \tag{2-1-6}$$

2-2. Delayed Application.

It is a matter of numerical experience that it is frequently advantageous to delay the point of application of the Euler transformation. This is also so in the case of the transformation (2-1-3). Accordingly (2-1-3) is generalised to the form

$$\sum_{s=0}^{h} c_s v_s x^s \sim \sum_{s=0}^{m-1} c_s v_s x^s + \sum_{s=0}^{h-m} u_m^{(s)} \Delta^s v_m \qquad h > m \tag{2-2-1}$$

where

$$u_m^{(s)} = \frac{x^{m+s}\,\Phi_m^{(s)}(x)}{s\,!} \tag{2-2-2}$$

and

$$\Phi_m(x) \sim \sum_{s=0}^{h-m} c_{m+s} x^s \tag{2-2-3}$$

(The result (2-1-3) thus corresponds to (2-2-1) with $m = 0$).

If the function $u_0^{(s)}$ $s = 0, 1, \ldots$ and $u_m^{(0)}$ $m = 0, 1, \ldots$ have been computed, a simple recursion system serves to provide the further functions $u_m^{(s)}$ $m,s = 1, 2, \ldots$
For

$$\Phi_m(x) = c_m + x\Phi_{m+1}(x) \tag{2-2-4}$$

accordingly

$$\Phi_m^{(s)}(x) = x\Phi_{m+1}^{(s)}(x) + s\,\Phi_{m+1}^{(s-1)}(x) \tag{2-2-5}$$

or

$$\frac{x^{m+s}\,\Phi_m^{(s)}(x)}{s\,!} = \frac{x^{m+s+1}\,\Phi_{m+1}^{(s)}(x)}{s\,!} + \frac{x^{m+s}\,\Phi_{m+1}^{(s-1)}(x)}{(s-1)\,!} \tag{2-2-6}$$

that is

$$u_m^{(s)} = u_{m+1}^{(s)} + u_{m+1}^{(s-1)} \tag{2-2-7}$$

For the purposes of display the following array (the Θ — array) may be constructed. It gives the various partial sums of (2-2-1) which may be constructed when the quantities v_s $s = 0, 1, \ldots, m$ are given.

$$\begin{array}{cccc}
0 & & & \\
 & \sum_{s=0}^{0} u_0^{(s)}\Delta^s v_0 & & \\
\sum_{s=0}^{0} u_s v_s x^s & & \sum_{s=0}^{1} u_0^{(s)}\Delta^s v_0 & \\
 & \sum_{s=0}^{0} u_s v_s x^s + \sum_{s=0}^{0} u_1^{(s)}\Delta^s v_1 & & \sum_{s=0}^{2} u_0^{(s)}\Delta^s v_0 \\
\sum_{s=0}^{1} u_s v_s x^s & & \sum_{s=0}^{0} u_s v_s x^s + \sum_{s=0}^{1} u_1^{(s)}\Delta^s v_1 & \vdots \\
 & \sum_{s=0}^{1} u_s v_s x^s + \sum_{s=0}^{0} u_2^{(s)}\Delta^s v_2 & \vdots & \\
\sum_{s=0}^{2} u_s v_s x^s & \vdots & & \sum_{s=0}^{\bar{m}-3} u_s v_s x^s + \sum_{s=0}^{2} u_{\bar{m}-2}^{(s)}\Delta^s v_{\bar{m}-2} \\
\vdots & & \sum_{s=0}^{\bar{m}-2} u_s v_s x^s + \sum_{s=0}^{1} u_{\bar{m}-1}^{(s)}\Delta^s v_{\bar{m}-1} & \\
 & \sum_{s=0}^{\bar{m}-1} u_s v_s x^s + \sum_{s=0}^{0} u_{\bar{m}}^{(s)}\Delta^s v_{\bar{m}} & & \\
\sum_{s=0}^{\bar{m}} u_s v_s x^s & & &
\end{array}$$

2-3. General Forms.

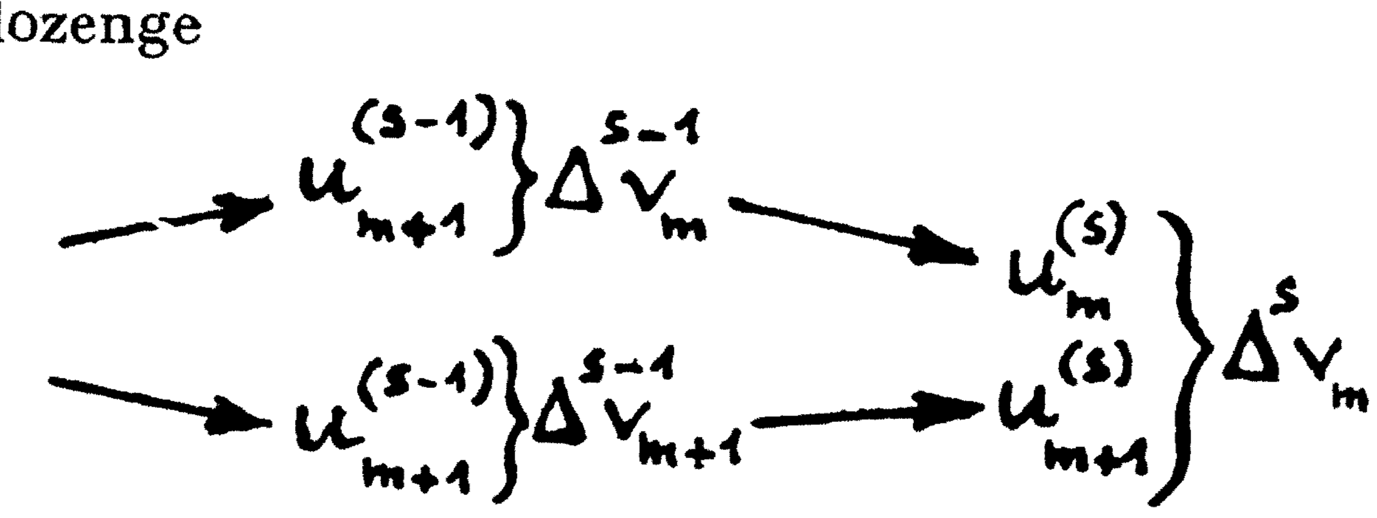
It is a consequence of equation (2-2-7) and the relation

$$\Delta^s v_m = \Delta^{s-1} v_{m+1} - \Delta^{s-1} v_m \qquad (2\text{-}3\text{-}1)$$

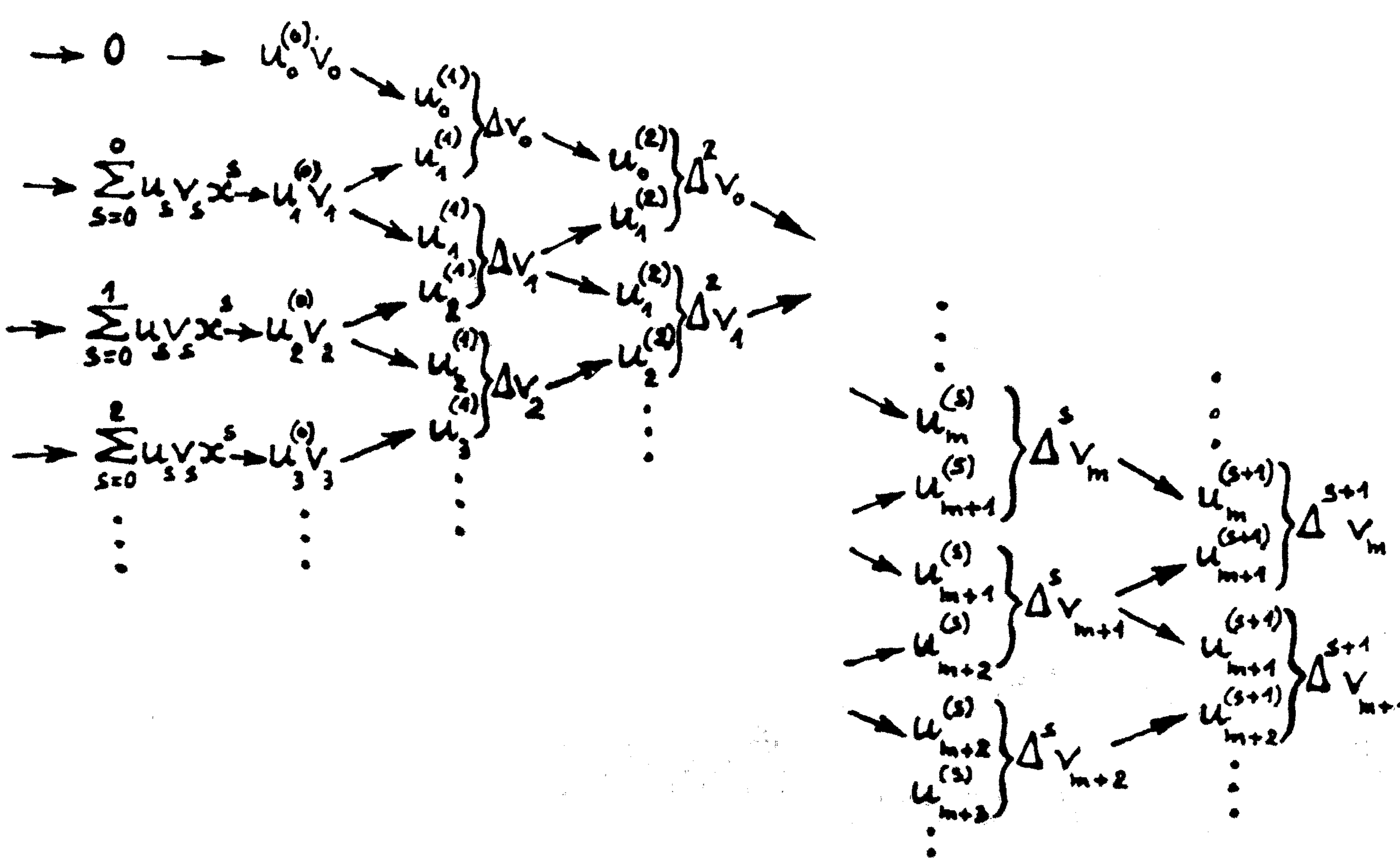
that

$$u_{m+1}^{(s-1)} \Delta^{s-1} v_m + u_m^{(s)} \Delta^s v_m = u_{m+1}^{(s-1)} \Delta^{s-1} v_{m+1} + u_{m+1}^{(s)} \Delta^s v_m \qquad (2\text{-}3\text{-}2)$$

The quantities occurring in equation (2-3-2) may be placed in the following lozenge



when equation (2-3-2) merely asserts that the sum of the quantities occurring along the upper edges of the lozenge is equal to that along the lower edges. Placing a number of such lozenges in contiguity the following diagram is arrived at



Assuming that either $u_m^{(s)}$ $m = 0, 1, \ldots$ or $\Delta^s v_m$ $m = 0, 1, \ldots$ are consistently zero for $s > \bar{s}$ (so that all terms in the Euler-Gudermann series vanish after $s = \bar{s}$) then repeated application of the result (2-3-2) shows that the sum of the terms encountered along any path from left to right in the above diagram is invariant with respect to the path chosen. (This is of course a well known property of all finite difference formulae. In a later section transformations will be developed in which $\Phi_o(x)$ is a factorial series and not a power series. Relationship (2-3-2) still obtains, though its a priori method of derivation is different.)

A number of different forms of the Euler-Gudermann transformation may be therefore be given, among them the Gauss-forward form

$$\Theta(x) \sim \sum_{s=0}^{m-1} c_s v_s x^s + u_m^{(0)} v_m + u_m^{(1)} \Delta v_m + u_m^{(2)} \Delta^2 v_{m-1} + u_{m-1}^{(3)} \Delta^3 v_{m-1} \dots$$
$$\dots u_{m-r+1}^{(2r)} \Delta^{2r} v_{m-r} + u_{m-r}^{(2r+1)} \Delta^{2r+1} v_{m-r} + \dots \qquad (2\text{-}3\text{-}3)$$

the Gauss-backward form

$$\Theta(x) \sim \sum_{s=0}^{m-1} c_s v_s x^s + u_m^{(0)} v_m + u_m^{(1)} \Delta v_{m-1} + u_{m-1}^{(2)} \Delta^2 v_{m-1} + u_{m-1}^{(3)} \Delta^3 v_{m-2}$$
$$+ u_{m-r}^{(2r)} \Delta^{2r} v_{m-r} + u_{m-r}^{(2r+1)} \Delta_{m-r-1}^{2r+1} + \dots \qquad (2\text{-}3\text{-}4)$$

and the mean of these, the Stirling form :

$$\Theta(x) \sim \sum_{s=0}^{m-1} c_s v_s x^s + u_m^{(0)} v_m + u_m^{(1)} \mu \delta v_m + \frac{1}{2}\left(u_m^{(2)} + u_{m-1}^{(2)} \right) \delta^2 v_m$$
$$+ u_{m-1}^{(3)} \mu \delta^3 v_m + \cdots \frac{1}{2}\left(u_{m-r+1}^{(2r)} + u_{m-r}^{(2r)} \right) \delta^{2r} v_m + u_{m-r}^{(r+1)} \mu \delta^{2r+1} v_m + \dots$$
$$. \qquad (2\text{-}3\text{-}5)$$

This latter form is of use when the quantities v_r are one-signed and reach a turning point when $r = s$. The odd order differences change sign along a horizontal line through this point and the mean central differences of odd order therefore tend to be relatively small along this line, in this instance one may expect the numerical convergence of the series (2-3-5) to be rapid. This idea has been applied to the Euler transformation in [5].

2-4. Recursions for the quantities $u_m^{(s)}$

The transformation (2-2-1) relates to a quite general class of functions $\Phi_m(x)$ and requires only that a reasonable number of quantities $\Phi_m^{(r)}(x)$ may easily be derived. Clearly it is a matter of great assistance in the derivation of the quantities $\Phi_m^{(r)}(x)$ if $\Phi_o(x)$ satisfies a linear differential equation of the form

$$\sum_{n=0}^{t} p_n^o(x)\ \Phi_o^{(n)}(x) = f_o(x) \qquad (2\text{-}4\text{-}1)$$

in which the polynomial $p_n^o(x)$ is of degree n^* and $f_o(x)$ is a polynomial of degree k. r-fold differentiation of equation (2-4-1) yields, when $r \leq k$, a system of linear inhomogeneous recursions between the quantities $\Phi_o^{(r)}(x)$, whilst if $r > k$ and $\max(n^* - n) = h$ a system of linear homogeneous recursions between the quantities $\Phi_o^{(r)}(x)$, of order $t+h$, is established. The recursions for the quantities $u_o^{(s)}$ follow from use of the formula

$$\Phi_o^{(s)}(x) = s!\ x^{-s}\ u_o^{(s)} \qquad s = 0, 1, \ldots \qquad (2\text{-}4\text{-}2)$$

The quantities $u_m^{(o)}$ $m = 1, 2, \ldots$ are of course constructed by use of the recursion

$$u_m^{(o)} = u_{m-1}^{(o)} - c_{m-1}x^{m-1} \qquad m = 1, 2, \ldots \qquad (2\text{-}1\text{-}3)$$

Use of (2-2-7) then enables the whole of the u-- array to be constructed.

It is also a consequence of equation (2-4-1) that the quantities $u_m^{(s)}$ $m = 1, 2, \ldots$ should satisfy a system of linear recursions in s, which may be used if desired as a check. For substituting

$$\Phi_o(x) = \sum_{s=0}^{m-1} c_s x^s + x^m \Phi_m(x) \qquad (2\text{-}4\text{-}4)$$

in (2-4-1), there follows

$$\sum_{n=0}^{t} p_n^o(x) \left\{ x^m \Phi_m(x) \right\}^{(n)} = x^m f_m(x) \qquad (2\text{-}4\text{-}5)$$

where the polynomial $f_m(x)$ is also of degree k. By use of Liebnitz' theorem equation (2-4-5) may be written

$$\sum_{n=0}^{t} p_n^m(x)\ \Phi_m^{(n)}(x) = x^t f_m(x) \qquad (2\text{-}4\text{-}6)$$

where

$$p_n^m(x) = \sum_{j=o}^{j=t-n} \binom{n+j}{j} p_{n+j}^o(x) x^{t-j} m(m-1) \ldots (m-j+1) \qquad (2\text{-}4\text{-}7)$$

A system of recursions for the quantities $u_m^{(s)}$ may now be built up as in the case of the quantities $u_o^{(s)}$.

Example :

An example which illustrates the preceding theory and which will be invoked in later sections, is provided by the series

$$\Phi_o(x) = {}_pF_q(\alpha_1,\alpha_2, \ldots, \alpha_p; \varrho_1,\varrho_2, \ldots, \varrho_q; x) \qquad (2\text{-}4\text{-}8)$$

Here

$$u_o^{(s)} = \frac{\Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\ldots\Gamma(\alpha_p+s)\,\Gamma(\rho_1)\,\Gamma(\rho_2)\ldots\Gamma(\rho_q)\,x^s}{\Gamma(\rho_1+s)\,\Gamma(\rho_2+s)\ldots\Gamma(\rho_q+s)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)\,s!} \times \qquad (2.4.9)$$

$${}_pF_q(\alpha_1+s, \alpha_2+s, \ldots, \alpha_p+s; \rho_1+s, \rho_2+s, \ldots, \rho_q+s; x) \qquad s=0,1,\ldots,k$$

where

$$k = \max(q, p-1)$$

the fundamental series satisfies the differential equation

$$\left\{ \delta(\delta+\varrho_1-1) \ldots (\delta+\varrho_q-1) - x(\delta+\alpha_1) \ldots (\delta+\alpha_p) \right\} y = 0 \qquad (2\text{-}4\text{-}10)$$

where $$\delta = x\frac{d}{dx}$$

from which a recursion system, involving the further functions $u_o^{(s)}$, may be constructed.

Furthermore

$$u_m^{(s)}(x) = \frac{\Gamma(\alpha_1+m+s)\,\Gamma(\alpha_2+m+s)\ldots\Gamma(\alpha_p+m+s)\,\Gamma(\rho_1)\,\Gamma(\rho_2)\ldots\Gamma(\rho_q)\;x^s}{\Gamma(\rho_1+m+s)\,\Gamma(\rho_2+m+s)\ldots\Gamma(\rho_q+m+s)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)\,\Gamma(m+1+s)} \times$$

$${}_{p+1}F_{q+1}(\alpha_1+m+s, \alpha_2+m+s, \ldots, \alpha_p+m+s, s+1; \rho_1+m+s, \rho_2+m+s, \ldots, \rho_q+m+s, m+s+1; x)$$

$$m=1,2,\ldots \quad ; \quad s=0,1,\ldots,k+1 \qquad (2.4.11)$$

and the series $\Phi_m(x)$ satisfies the differential equation

$$\left\{ \delta(\delta+\varrho_1-1)(\delta+\varrho_2-1)\ldots(\delta+\varrho_q-1)(\delta+m) - x(\delta+\alpha_1)(\delta+\alpha_2)\ldots (\delta+\alpha_p)(\delta+1) \right\} y = 0 \qquad (2\text{-}4\text{-}12)$$

from which a further recursion system involving the quantities $u_m^{(s)}$ $s = k+2,\ k+3, \ldots$ may be constructed.

The recursion system between the functions $u_m^{(o)}$ $m = 0, 1, \ldots$ is simply

$$u_1^{(0)} = u_0^{(0)} - 1,$$

$$u_{m+2}^{(0)} = u_{m+1}^{(0)} - \frac{\alpha_1(\alpha_1+1)\ldots(\alpha_1+m)\alpha_2(\alpha_2+1)\ldots\alpha_p(\alpha_p+1)\ldots(\alpha_p+m)x^m}{\rho_1(\rho_1+1)\ldots(\rho_1+m)\rho_2(\rho_2+1)\ldots\rho_q(\rho_q+1)\ldots(\rho_q+m)(m+1)} \qquad m = 0, 1, \ldots \quad (2.4.13)$$

Remark :

It will be observed in the previous example that the recursion for $u_m^{(s)}$ $m = 1, 2, \ldots$ is in general slightly more complicated than that for $u_o^{(s)}$. If, however, some α_s in (2-4-8) is unity (this is the case for example when the fundamental series is an Incomplete Beta Function, or an Incomplete Gamma Function with ascending or descending powers of the argument), the recursion for $u_m^{(s)}$ is obtained from that for $u_o^{(s)}$ merely by adding m to all the remaining parameters α_s and ϱ_s.

Summation Checks :

The formation of the quantities $u_m^{(s)}$ may be checked by use of the results

$$\sum_{s=0}^{\infty} u_m^{(s)} \rho^s = x^m \Phi_m \left\{ x(1+\varrho) \right\} \qquad (2\text{-}4\text{-}14)$$

or

$$\sum_{s=0}^{n} (-1)^s u_m^{(s)} = \sum_{s=0}^{\infty} c_{m+n+s+1} (-1)^n \binom{n+s}{s} x^{m+n+s+1} + c_m x^m \qquad (2\text{-}4\text{-}15)$$

or, if the quantities $u_m^{(s)}$ decrease in magnitude with sufficient rapidity

$$\sum_{s=0}^{\infty} (-1)^s u_m^{(s)} = c_m x^m \qquad (2\text{-}4\text{-}16)$$

Two further checks, based on relation (2-2-7) are

$$u_m^{(s)} - u_{m+p}^{(s)} = \sum_{h=1}^{p} u_{m+h}^{(s-1)} \qquad p = 1, 2, \ldots$$

and:

$$\sum_{h=0}^{p-1} u_m^{(s+h)} = u_{m+1}^{(s-1)} + u_{m+1}^{(s+p-1)} + 2\sum_{h=0}^{p-2} u_{m+1}^{(s+h)} \qquad p = 1, 2, \ldots$$

2-5. Recursions for the quantities $\Delta^s v_m$

If the quantities v_m satisfy a linear recursion of the form

$$\sum_{r=0}^{k} \delta_r(n) v_{n+r} = 0 \qquad n = 0, 1, \ldots \qquad (2\text{-}5\text{-}1)$$

where $\delta_r(n)$ $r = 0, 1, \ldots, k$ is a polynomial in n of degree $\bar{r}$, a number of further recursion systems may be derived.

Using the result that

$$\Delta^s p(n) q(n) = \sum_{t=o}^{t=s} \binom{s}{t} \Delta^t p(n) \Delta^{s-t} q(n+t) \qquad (2\text{-}5\text{-}2)$$

it follows, by applying the operator Δ^s to equation (2-5-1), that

$$\sum_{r=0}^{k} \sum_{t=0}^{t=s} \binom{s}{t} \Delta^t v_{n+r} \Delta^{s-t} \delta_r(n+t) = 0 \qquad (2\text{-}5\text{-}3)$$

which involves, and may be used to check the formation of, the block of differences $\Delta^s v_{n+r}, \Delta^{s-1} v_{n+r}, \ldots, \Delta^{s-\bar{r}} v_{n+r}$, $r = 0, 1, \ldots, k$.

A recursion formula involving a vertical line of differences, which is perhaps easier to apply, may be derived as follows. Suppose that max(F $= \bar{\bar{\text{F}}}$, so that $\Delta^{s-\bar{\bar{r}}} v_{n+r}$ is the lowest order difference in the double sum (2-5-3) without a zero coefficient. Then, by use of the result

$$\Delta^{s-\bar{\bar{r}}+t'} v_{n+r} = \sum_{j=0}^{j=t'} (-1)^{t'-j} \binom{t'}{j} \Delta^{s-\bar{\bar{r}}} v_{n+r+j} \qquad (2.5.4)$$

(2.5.3) becomes:

$$\sum_{r=0}^{k} \sum_{t'=0}^{t'=\bar{\bar{r}}} \binom{s}{s-\bar{\bar{r}}+t'} \Delta^{s-\bar{\bar{r}}+t'} v_{n+r} \, \Delta^{\bar{\bar{r}}-t} \delta_r(n+s-\bar{\bar{r}}+t')$$
$$= \sum_{r=0}^{k} \sum_{t'=0}^{t'=\bar{\bar{r}}} \binom{s}{s-\bar{\bar{r}}+t'} \Delta^{\bar{\bar{r}}-t'} \delta_r(n+s-\bar{\bar{r}}+t') \sum_{j=0}^{j=t'} (-1)^{t'-j} \binom{t'}{j} \Delta^{s-\bar{\bar{r}}} v_{n+r+j}$$
$$= 0 \qquad (2.5.5)$$

a recursion involving the line of differences $\Delta^{s-r} v_{n+r}$ $\bar{\bar{r}} = 0, 1, \ldots$

A further recursion which relates to a diagonal line of differences, $\Delta^l v_n$ $l = s - \bar{o}, s - \bar{o} + 1, \ldots$

may also be derived from equation (2-5-3). Using the relation

$$\Delta^t v_{n+r} = \sum_{i=0}^{r} \binom{r}{i} \Delta^{t+i} v_n \tag{2-5-6}$$

the recursion (2-5-3) becomes

$$\sum_{r=0}^{k} \sum_{t=0}^{t=s} \binom{s}{t} \Delta^{s-t}\delta_r(n+t) \sum_{i=0}^{i=r} \binom{r}{i} \Delta^{t+i} v_n = 0 \tag{2-5-7}$$

which again may be used to check the formation of the quantities $\Delta^s v_{n+r}$, or to provide a method for prolonging the computation of the quantities $\Delta^s v_m$ alternative to differencing the quantities v_r.

In the following sections a number of examples will be given which indicate how the Euler-Gudermann transformation copes with the different types of slow convergence described in the introduction.

3. The Case in which the Fundamental Series is a Hypergeometric Series.

3-1. The series now being considered are of the form

$$\Theta(x) \sim v_o + \frac{ab}{1!c} x v_1 + \frac{a(a+1)b(b+1)}{2!c(c+1)} x^2 v_2 + \ldots \tag{3-1-1}$$

where the fundamental series

$$\Phi_o(x) = 1 + \frac{ab}{1!c} x + \frac{a(a+1)b(b+1)}{2!c(c+1)} x^2 + \ldots \tag{3-1-2}$$

satisfies the differential equation

$$x(1-x)y'' + \{ c - (a+b+1)x \} y' - aby = 0 \tag{3-1-3}$$

Thus

$$u_0^{(0)} = {}_2F_1(a,b;c;x), \quad u_0^{(1)} = \frac{ab}{c} x\, {}_2F_1(a+1,b+1;c+1;x) \tag{3-1-4}$$

and furthermore

$$u_0^{(s+2)} = \frac{\{(a+s)(b+s)x u_0^{(s)} - [c+s-(a+b+2s+1)x](s+1)u_0^{(s+1)}\}}{(1-x)(s+2)(s+1)} \tag{3.1.5}$$

$s=0,1,\ldots$

The recursion between the quantities $u_m^{(0)}$ $m=0,1,\ldots$ is :

$$u_1^{(0)} = u_0^{(0)} - 1, \quad u_{m+2}^{(0)} = u_{m+1}^{(0)} - \frac{a(a+1)\ldots(a+m)b(b+1)\ldots(b+m)x^{m+1}}{(m+1)!c(c+1)\ldots(c+m)} \tag{3.1.6}$$

$m=0,1,\ldots$

$$\phi_m(x) = \frac{\Gamma(a+m)\Gamma(b+m)\Gamma(c)}{\Gamma(m+1)\Gamma(c+m)\Gamma(b)}\, {}_3F_2(a+m, b+m, 1; c+m, 1+m, x) \qquad (3.1.7)$$

satisfies:

$$x(1-x)y''' - \{(a+b+2m+4)x - c - 2m - 2\}xy''$$
$$-\left[\{(a+m+2)(b+m+2)-2\}x - (m+c)(m+1)\right]y' - (a+m)(b+m)y = 0 \qquad (3.1.8)$$

upon which may be based the recursion:

$$u_m^{(s+3)} = (x-1)^{-1}\left[\frac{s(1-2x) - \{(a+b+2m+4)x - 2m - 2\}x\,u_m^{(s+2)}}{(s+3)}\right.$$
$$-\frac{\{(a+m+2)(b+m+2)-2\}x - (m+c)(m+1) + s\{2(a+b+2m+4)x - c - 2m - s - 3\}x\,u_m^{(s+1)}}{(s+3)(s+2)}$$
$$\left.-\frac{(a+m)(b+m) + s\{(a+m+2)(b+m+2) - 2 + (s-1)(a+b+2m+4)\}x^2 u_m^{(s)}}{(s+3)(s+2)(s+1)}\right] \qquad (3.1.9)$$

Note that when

$$\Phi_o(x) = {}_2F_1(1,b;c;x) \qquad (3\text{-}1\text{-}10)$$

the simpler recursion system

$$u_m^{(1)} = (1-x)^{-1}\left[(c+m-1)x^m \frac{\Gamma(a+m)\Gamma(c)}{\Gamma(c+m)\Gamma(a)} + \{(a+m)x + 1 - c - m\}u_m^{(0)}\right] \qquad (3.1.11)$$

$$u_m^{(s+2)} = \frac{(b+m+s)x\,u_m^{(s)} - \{c+m+s-(b+m+2s+2)x\}u_m^{(s+1)}}{(1-x)(s+2)} \qquad (3.1.12)$$

obtains (note the Remark to § 2.4)

3-2. The Generalised Euler Transformation.

Since

$$(1 + x)^{-1} = {}_2F_1(1,1;1; -x) \qquad (3\text{-}2\text{-}1)$$

the generalised Euler transformation is subsumed within those being considered in this section. In the event, however, the recursion systems relating to this transformation are far simpler.

The general term $u_m^{(s)}$ may be written in the simple closed form

$$u_m^{(s)} = (-x)^{m+s}(1 + x)^{-s-1} \qquad (3\text{-}2\text{-}2)$$

which leads to the trivial recursions

$$u_m^{(s+1)} = \frac{-x}{1+x} u_m^{(s)} \qquad m, s = 0, 1, \ldots \qquad (3\text{-}2\text{-}3)$$

and

$$u_{m+1}^{(s)} = -x\, u_m^{(s)} \qquad m, s = 0, 1, \ldots \qquad (3\text{-}2\text{-}4)$$

3-3. The Logarithmic Series.

Particular interest also attaches to the series (3-1-1) when $a = b = 1$, $c = 2$ and x is replaced by $-x$. Equations (3-1-4) then become

$$u_0^{(0)} = x^{-1}\log(1+x), \quad u_0^{(1)} = (1+x)^{-1} - 2x^{-1}\log(1+x) \quad (3\text{-}3\text{-}1)$$

and the recursion (3-1-5) becomes

$$u_0^{(s+2)} = \frac{-\left\{ s+2+(2s+3)x \right\} u_0^{(s+1)} + (s+1)\, x\, u_0^{(s)}}{(1+x)(s+2)} \qquad s = 0, 1, \ldots \qquad (3\text{-}3\text{-}2)$$

Furthermore

$$u_{m+1}^{(0)} = u_m^{(0)} - \frac{(-x)^m}{(m+1)} \qquad m = 0, 1, \ldots \qquad (3\text{-}3\text{-}3)$$

and the recursion system for the $u_m^{(s)}$ becomes

$$u_m^{(s+2)} = -\frac{\left\{ m+s+2+(2s+m+3)x \right\} u_m^{(s+1)} + (m+s+1)\, x\, u_m^{(s)}}{(1+x)(s+2)} \qquad (3\text{-}3\text{-}4)$$

3-4. Mestel's Integral.

An opportunity for contrasting the numerical performances of the two transformation just described, is provided by the evaluation of the integral

$$4(\lambda\pi^{1/2})^{-1}G(\lambda) = \int_0^\infty \frac{t^2 dt}{(1+e^{t^2}/\lambda)}$$
$$\sim \sum_{s=0}^{\infty} (s+1)^{-3/2}(-\lambda)^s \qquad (3\text{-}4\text{-}1)$$

which occurs in the theory of the conductivity of dense stars [6].

Two obvious substitutions which can be made to transform the series (3-4-1) are

$$v_n = (n+1)^{-3/2} \qquad (3\text{-}4\text{-}2)$$

corresponding to the generalised Euler transformation, and

$$v_n = (n+1)^{-1/2} \qquad (3\text{-}4\text{-}3)$$

corresponding to the fundamental series of (3-3-1). The Θ-arrays for these two transformations when $\lambda = 10$ are displayed in Tables I and II. Since a more accurate value of $(5\Gamma(3/2))^{-1}G(10)$ is 0.3285 it will be seen that the latter transformation provides, in this case, the better result. (For the sake of completeness it is remarked that Mestel was primarily interested, not in the integral $G(\lambda)$, but in the ratio $F(\lambda)/G(\lambda)$ where

$$F(\lambda) = \int_0^\infty t^2 \log(1+\lambda e^{-t^2})dt \sim \frac{\lambda\pi^{1/2}}{4} \sum_{s=0}^{\infty} (s+1)^{-5/2}(-\lambda)^s \qquad (3\text{-}4\text{-}4)$$

$F(\lambda)$ may also be evaluated by recourse to the two transformations described. For this integral the fundamental series

$$\Phi_o(x) = 1 - \frac{x}{1.2} + \frac{x^2}{2.3} - \dots + \frac{(-1)^r}{r(r+1)} x^r + \dots \qquad (3\text{-}4\text{-}5)$$

$$= 2 - x^{-1}\log(1+x) - \log(1+x)$$

may also be used. It is not subsumed within those hitherto considered in this section; the functions $u_o^{(s)}$ are given by

$$u_o^{(1)} = 1 - \mathrm{Log}(1+x) - u_o^{(0)} \qquad (3.4.6)$$

$$u_o^{(2)} = -\frac{1}{2}\left\{\frac{x}{1+x} + 2u_o^{(1)}\right\} \qquad (3.4.7)$$

$$u_o^{(s+3)} = -\left[\frac{\{(2x+1)s+4x+3\}u_o^{(s+2)} + x(s+1)u_o^{(s+1)}}{(s+3)(1+x)}\right] \qquad (3.4.8)$$

	s	0	1	2	3	4	5	6	7	8	9	10
0	0.0											
		+0.091										
1	1.0		+0.144									
		+0.679		+0.181								
2	-2.536		+0.545		+0.208							
		-0.786		+0.475		0.228						
3	+16.709		-0.229		+0.433		+0.244					
		+5.346		+0.011		+0.406		+0.257				
4	-108.291		+2.407		+0.132		+0.387		+0.268			
		-26.979		+1.344		+0.200		+0.374		+0.276		
5	+786.137		-9.292		+0.879		+0.241		.264		+0.284	
		+167.579		-3.766		+0.650		+0.267		.357		+0.290
6	-6018.800		+51.492		-1642		+0.527		.284		+0.351	
		-1109.372		+19.594		-0.707		+0.456		+0.296		
7	+47976.923		-299.393		+8.651		-0.252		+0.413			
		+7800.402		-100.774		+4.296		-0.014				
8	-393964.815		+1885.412		-39.252		+2.366					
		-57264.478		+573.964		-16.932						
9	+3309738.889		-12518.524		+206.269							
		+434941.016		-3490.680								
10	-28313037.713		+86787.762									
		-3394744.773										
11	+245788184.630											

Table I

	s	0	1	2								
m												
0	0.0											
		+0.240										
1	1.0		+0283									
		+0.462		+0.301								
2	-2.536		+0.383		+0.310							
		-0.088		+0.357		+0.316						
3	+16.709		+0.193		+0.345		+ 0.319					
		+2.163		+0.270		+0.339		+0.321				
4	-108.291		+0.818		+0.299		.335		+0.323			
		-9.498		+0.510		+0.312		.333		+ 0.324		
5	+786.137		-1.883		+0.409		.319		.332		+0.325	
		+59.825		-0.380		+0.368		.322		.331		+0.326
6	-6018.800		+11.866		+0.051		.350		.324		+0.330	
		-391.028		+3.582		+0.205		.341		+ 0.326		
7	+47976.923		-66.265		+1.468		.268		+0.336			
		+2732.848		-16.419		+0.790		+0.296				
8	-393964.815		+414.302		-4.969		+0.536					
		-19954.672		+94.219		1.627						
9	+3309738.889		-2723.126		+27.388							
		+150914.204		-561.922								
10	-28313037.713		+18735.504									
		-1273724.402										
11	+245788184.630											

Table II

4. The Case in which the Fundamental Series is a Bessel Type Series.

4-1. The series now being considered are of the form

$$\Theta(x) \sim v_o + \frac{x}{1!c} v_1 + \frac{x^2}{2!c(c+1)} v_2 + \dots \qquad (4\text{-}1\text{-}1)$$

where the fundamental series

$$\Phi_o(x) = 1 + \frac{x}{1!c} + \frac{x^2}{2!c(c+1)} + \dots \qquad (4\text{-}1\text{-}2)$$

$$= {}_0F_1(c; x)$$

satisfies the differential equation

$$xy'' - cy' - y = 0 \qquad (4\text{-}1\text{-}3)$$

Thus

$$u_o^{(0)} = {}_0F_1(c; x) \qquad u_o^{(1)} = \frac{x}{c}\, {}_0F_1(c + 1; x) \qquad (4\text{-}1\text{-}4)$$

and thereafter

$$u_o^{(s+2)} = \frac{x}{(s+2)(s+1)} u_o^{(s)} - \frac{(s-c)}{(s+2)} u_o^{(s+1)} \qquad (4\text{-}1\text{-}5)$$

The recursion between the quantities $u_m^{(0)}$ $m = 0, 1, \dots$ is

$$u_1^{(0)} = u_o^{(0)} - 1, \quad u_{m+2}^{(0)} = u_{m+1}^{(0)} - \frac{x^{m+1}}{(m+1)!\, c\, (c+1) \dots (c+m)} \quad m = 0, 1, \dots \qquad (4\text{-}1\text{-}6)$$

In this case

$$\Phi_m(x) = \frac{\Gamma(c)}{\Gamma(m+1)\, \Gamma(c+m)}\, {}_1F_2(1; c + m, 1 + m; x) \qquad (4\text{-}1\text{-}7)$$

and satisfies the differential equation

$$x^2y''' + (c+2m+2)xy'' + \left\{ (m+c)(m+1) - x \right\} y' - y = 0 \qquad (4\text{-}1\text{-}8)$$

which leads to the recursion system

$$u_m^{(s+3)} = \frac{x}{(s+3)(s+2)} u_m^{(s)} - \frac{(c+2m+s+1)s + (m+c)(m+1) - x}{(s+3)(s+2)} u_m^{(s+1)}$$

$$- \frac{c+2m+2s+2}{(s+3)} u_m^{(s+2)} \qquad s = 0, 1, \dots \qquad (4.1.9)$$

4-2. Example :

A simple example of the application of the transformations of this section is provided by the transformation of the series

$$I_o(z) = \sum_{s=0}^{\infty} \frac{(z/2)^{2s}}{(s\,!)^2} \qquad (4\text{-}2\text{-}1)$$

using as the fundamental series the expansion

$$z^{-1}\sinh(z) = 1 + \frac{1}{\frac{3}{2}\,.\,1\,!}\left(\frac{z^2}{4}\right) + \frac{1}{\frac{3}{2}\,.\,\frac{5}{2}\,.\,2\,!}\left(\frac{z^2}{4}\right)^2 + \cdots \qquad (4\text{-}2\text{-}2)$$

This corresponds to the substitution

$$v_m = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(m+1\right)}{\Gamma\left(m+\frac{3}{2}\right)} \qquad m = 0,\ 1, \ldots \qquad (4\text{-}2\text{-}3)$$

The recursion system between the quantities v_n is

$$v_o = 1$$

$$v_r = \frac{2r+1}{2r}\,.\,v_{r-1} \qquad r = 1,\ 2, \ldots \qquad (4\text{-}2\text{-}4)$$

whilst that between the functions $u_m^{(s)}$, is

$$u_0^{(0)} = x^{-1}\sinh(x)\ , \quad x = 2Z^{1/2} \qquad (4.2.5)$$

$$u_0^{(1)} = \left\{x\cosh(x) - \sinh(x)\right\}/(2x) \qquad (4.2.6)$$

$$u_0^{(s+2)} = \left\{\frac{x\,u_0^{(s)}}{(s+1)} - \left(s-\frac{3}{2}\right)u_0^{(s+1)}\right\}/(s+2) \qquad s=0,1,\ldots \qquad (4.2.7)$$

$$u_1^{(0)} = u_0^{(0)} - 1\ ,\quad u_{m+2}^{(0)} = u_{m+1}^{(0)} - \frac{x^{m+1}}{(m+1)!\,\frac{3}{2}\,\frac{5}{2}\cdots\left(m+1+\frac{1}{2}\right)} \qquad m=0,1,\ldots \qquad (4.2.8)$$

The Θ-array for this example, when $z = 16$, is displayed in Table III. It will be recalled that $I_o(8) \doteq 428$.

$m \diagdown s$		0	1	2	3
0	0				
		186			
1	1		512		
		279		358	
2	17		454		425
		344		406	
3	81		436		
		388			
4	195				

Table III

5. The Case in which the Fundamental Series is a Confluent Hypergeometric Series.

5-1. The series now being considered are of the form

$$\Theta(x) = v_o + \frac{ax}{1!c} v_1 + \frac{a(a+1)x^2}{2!\,c\,(c+1)} v_2 + \ldots \qquad (5\text{-}1\text{-}1)$$

where the fundamental series

$$\Phi_o(x) = 1 + \frac{a}{1!c} x + \frac{a(a+1)x^2}{2!\,c\,(c+1)} + \ldots \qquad (5\text{-}1\text{-}2)$$

satisfies the differential equation

$$xy'' + (c - x)y' - ay = 0 \qquad (5\text{-}1\text{-}3)$$

Thus

$$u_o^{(0)} = {}_1F_1(a;c;x)\ , \qquad u_o^{(1)} = \frac{ax}{c}\, {}_1F_1(a+1;c+1;x) \qquad (5\text{-}1\text{-}4)$$

and furthermore

$$u_o^{(s+2)} = \left\{ (a+s)x\, u_o^{(s)} - (c+s-x)(s+1)\, u_o^{(s+1)} \right\} / (s+1)(s+2) \qquad (5\text{-}1\text{-}5)$$
$$s = 0, 1, \ldots$$

The recursion between the quantities $u_m^{(0)}$ $m = 0, 1, \ldots$ is

$$u_1^{(0)} = u_0^{(0)} - 1, \quad u_{m+2}^{(0)} = u_{m+1}^{(0)} - \frac{a(a+1)\dots(a+m)x^{m+1}}{(m+1)!\,c(c+1)\dots(c+m)} \qquad m=0,1,\dots \qquad (5.1.6)$$

In this case:

$$\Phi_m(x) = \frac{\Gamma(c)\Gamma(a+m)}{\Gamma(m+1)\Gamma(c+m)\Gamma(a)}\, {}_2F_2(a+m,1;c+m;1+m;x) \qquad (5.1.7)$$

and satisfies:

$$x^3y''' + (c+2m+2-x)xy'' + \{(m+1)(m+c)-(a+m+2)x\}y' - (a+m)y = 0 \qquad (5.1.8)$$

giving rise to the recursion

$$u_m^{(s+3)} = \frac{s(a+m+s+1)+a+m}{(s+3)(s+2)(s+1)}\, x u_m^{(s)} - \frac{s(c+2m+s+1-2x)+(m+1)(m+c)-(a+m+2)x}{(s+3)(s+2)}\, u_m^{(s+1)}$$

$$- \frac{c+2m+2s+2-x}{(s+3)}\, u_m^{(s+2)} \qquad (5.1.9)$$

Note that when

$$\Phi_o(x) = {}_1F_1(1;c;x) \qquad (5\text{-}1\text{-}10)$$

the simpler recursion system

$$u_m^{(1)} = (x+1-c-m)\,u_m^{(0)} + \frac{(c+m-1)x^m\Gamma(c)}{\Gamma(c+m)} \qquad (5.1.11)$$

$$u_m^{(s+2)} = \left\{x u_m^{(s)} - (c+m+s-x)\,u_m^{(s+1)}\right\}/(s+2) \qquad (5.1.12)$$

obtains.

5-2. The Exponential Series.

Since

$$\exp(x) = {}_1F_1(1;1;x) \qquad (5\text{-}2\text{-}1)$$

the transformation based upon the use of the exponential series as fundamental series is subsumed within those considered in this section. In the event, however, the recursion systems relating to this transformation are far simpler.

In this special case

$$u_o^{(0)} = \exp(x) \qquad (5\text{-}2\text{-}2)$$

$$u_o^{(s)} = \frac{x}{s}\, u_o^{(s-1)} \qquad s = 1, 2, \dots \qquad (5\text{-}2\text{-}3)$$

$$u_{m+1}^{(0)} = u_m^{(0)} - \frac{x^m}{m!} \qquad m = 0, 1, \dots \qquad (5\text{-}2\text{-}4)$$

Furthermore

$$\Phi_m(x) = (m!)^{-1}{}_1F_1(1;m+1;x) \qquad (5\text{-}2\text{-}5)$$

and thus the recursion system obtaining between the functions $u_m^{(s)}$ is

$$u_m^{(s+2)} = \{x\, u_m^{(s)} - (s+m-x+1)\, u_m^{(s+1)}\} / (s+2) \quad s = 0, 1, \ldots \qquad (5\text{-}2\text{-}6)$$

in agreement with (5-1-11).

Kummer's Transformation

If

$$v_r = \frac{\Gamma(a+r)\Gamma(c)}{\Gamma(c+r)\Gamma(a)} \qquad (5\text{-}2\text{-}7)$$

then, as is easily verified

$$\Delta^s v_o = (-1)^s \frac{\Gamma(c-a+s)\Gamma(c)}{\Gamma(c+s)\,\Gamma(c-a)} \qquad (5\text{-}2\text{-}8)$$

Thus Kummer's transformation

$${}_1F_1(a;c;x) = e^x{}_1F_1(c-a;c;-x) \qquad (5\text{-}2\text{-}9)$$

is a special case of the Euler-Gudermann transformation in which the fundamental series is the exponential series.

As a simple example of the above transformation there follows

$$\begin{aligned}(x')^{-1}\int_0^{x'} e^{t^2}dt &= 1 + \frac{x'^2}{3.1!} + \frac{x'^4}{5.2!} + \frac{x'^6}{7.3!} + \ldots \\ &= {}_1F_1\left(\tfrac{1}{2}; \tfrac{3}{2}; x'^2\right) \\ &= e^{x'^2}{}_1F_1\left(1; \tfrac{3}{2}; -x'^2\right) \\ &= e^{x'^2}\left\{1 - \frac{(2x'^2)}{1.3} + \frac{(2x'^2)^2}{1.3.5} - \ldots\right\} \qquad (5.2.10)\end{aligned}$$

which will be of use later.

5-3. Wilson's Integral.

A practical example of the manner in which the transformations being discussed in this section may be employed is provided by the computation of the integral

$$F_3(\alpha,\beta) = \int_0^{\pi/2} \sec^3\Theta e^{-\alpha^2\sec^2\theta} \cos(\beta\sec\Theta)d\Theta \qquad (5\text{-}3\text{-}1)$$

which has been investigated by Goodwin and Olver. Subsequently integrals of the more general form

$$F_m(\alpha,\beta) = \int_0^{\pi/2} \sec^m\Theta e^{-\alpha^2 \sec^2\theta} \cos(\beta\sec\Theta)d\Theta \qquad (5\text{-}3\text{-}2)$$

were studied and computed for extensive values of α and β by Wilson [7]. Integrals of this form quite clearly satisfy partial differential equations of the form

$$\frac{\partial F_m}{\partial(\alpha^2)} - \frac{\partial^2 F_m}{\partial\beta^2} = 0 \qquad (5\text{-}3\text{-}3)$$

and occur in problems concerned with surface waves.

Expanding the term $\cos(\beta\sec\Theta)$ in powers of β, and noting the result

$$-\frac{1}{2} e^{-\alpha^2/2} K_0(\alpha^2/2) = \int_0^{\pi/2} \sec\Theta e^{-\alpha^2 \sec^2\theta} d\Theta \qquad (5\text{-}3\text{-}4)$$

it follows that

$$F_3(\alpha,\beta) = \sum_{n=0}^{\infty} \frac{\beta^{2n}}{(2n)!} A_n(\alpha^2) \qquad (5\text{-}3\text{-}5)$$

where

$$A_n(\alpha^2) = -\frac{1}{2}\left(\frac{d}{d(\alpha^2)}\right)^{n+1} \left\{ e^{-\alpha^2/2} K_0\left(\frac{\alpha^2}{2}\right)\right\} \qquad (5\text{-}3\text{-}6)$$

For large values of n

$$A_n(\alpha^2) \sim \text{const}\, (-1)^n\, n!\, \alpha^{-2n} \qquad (5\text{-}3\text{-}7)$$

and hence it follows that the series (5-3-5) is ultimately convergent. However, for large values of β^2/α^2 the initial terms increase rapidly in magnitude, and summation is accompanied by considerable cancellation of figures. This is illustrated in Table IV which gives certain of the terms in the series (5-3-5) when $\beta = 1.7$, $\alpha = 0.25$.

n		n		n	
0	1.0758186	11	—73590.7192816	43	—0.0000098
1	—23.9185633	12	+73957.0426119	44	+0.0000026
2	+181.1074373	13	—68381.6903476	45	—0.0000007
					—0.1239364

Table IV

As a first attempt to transform this series the substitution

$$\frac{\beta^{2n}}{(2n)!} A_n(\alpha^2) = \frac{x^n}{n!} v_n \tag{5-3-8}$$

is made, where

$$\frac{\beta^2}{\alpha^2} = 4x \tag{5-3-9}$$

The fundamental series is thus exp(— x), and the recursion systems prevailing between the functions $u_m^{(s)}$ have already been given.

The quantities v_n are given by

$$v_0 = \frac{1}{4} e^{-\frac{\alpha^2}{2}} \left\{ K_0\left(\frac{\alpha^2}{2}\right) + K_1\left(\frac{\alpha^2}{2}\right) \right\} \tag{5.3.10}$$

$$v_1 = -\frac{1}{2} e^{-\frac{\alpha^2}{2}} \left\{ \alpha^2 K_0\left(\frac{\alpha^2}{2}\right) + (1+\alpha^2) K_1\left(\frac{\alpha^2}{2}\right) \right\} \tag{5.3.11}$$

and, by manipulating the differential equation satisfied by $K_o(x)$

$$v_n = \frac{2(n+\alpha^2)v_{n-1}}{2n-1} - \frac{2\alpha^2 v_{n-2}}{2n-3} \tag{5-3-12}$$

The Θ-array for this choice of fundamental series is displayed in Table V.

A somewhat more sensitive choice of fundamental series, yielding better results, is obtained by writing

$$A_n(\alpha^2) = (-\alpha^2)^n n! v_n \tag{5-3-13}$$

The fundamental series is then

$$\begin{aligned} u_0^{(0)} = \Phi_0(y^2) &= 1 - \frac{y^2}{2}\left\{1 - \frac{y^2}{2}\cdot\frac{1}{1.3} + \frac{y^4}{4}\cdot\frac{1}{1.3.5} - \dots\right\} \\ &= 1 - y\left\{e^{-\frac{y^2}{4}} \int_0^{\frac{y}{2}} e^{t^2} dt\right\} \\ &= {}_1F_1\left(1, \frac{1}{2}; -\frac{y^2}{4}\right) \end{aligned} \tag{5.3.14}$$

(see (5-2-10)), where

$$y = \beta/\alpha \tag{5-3-15}$$

Numerical values of (5-3-14) may be extracted from [8].

The subsequent functions $u_m^{(s)}$ are obtained from

$$u_0^{(1)} = -\frac{y}{2}\left\{\left(e^{-\frac{y^2}{4}}\int_0^{\frac{y}{2}} e^{t^2}dt\right)\left(1-\frac{y^2}{2}\right)+\frac{y}{2}\right\} \tag{5.3.16}$$

$$u_0^{(s+2)} = -\left\{\frac{y^2}{4}u_0^{(s)}+\left(s+\frac{1}{2}+\frac{y^2}{4}\right)u_0^{(s+1)}\right\}\Big/(s+1)_{\quad s=0,1,\ldots} \tag{5.3.17}$$

More generally

$$\phi_m(y^2) = \frac{m!}{(2m)!}\,{}_1F_1\left(1, m+\frac{1}{2}; -\frac{y^2}{4}\right) \tag{5.3.18}$$

giving rise to the recursion:

$$u_m^{(s+2)} = -\left\{\frac{y^2}{4}u_m^{(s)}+\left(m+s+\frac{1}{2}+\frac{y^2}{4}\right)u_m^{(s+1)}\right\}\Big/(s+2)_{\quad s=0,1,\ldots} \tag{5.3.19}$$

The quantities v_n are now given by:

$$v_0 = \frac{1}{4}e^{-\frac{\alpha^2}{2}}\left\{K_0\left(\frac{\alpha^2}{2}\right)+K_1\left(\frac{\alpha^2}{2}\right)\right\} \tag{5.3.20}$$

$$v_1 = -\frac{1}{4}e^{-\frac{\alpha^2}{2}}\left\{\alpha^2 K_0\left(\frac{\alpha^2}{2}\right)+(1+\alpha^2)K_1\left(\frac{\alpha^2}{2}\right)\right\} \tag{5.3.21}$$

$$v_m = \left(\frac{m+\alpha^2}{m}\right)v_{m-1} - \frac{\alpha^2\left(m-\frac{1}{2}\right)}{m(m-1)}v_{m-2} \qquad m=2,3,\ldots \tag{5.3.22}$$

The Θ-array for this choice of fundamental series is displayed in Table VI.

(The transformations illustrated here are useful but their effect is not particulary striking. It is not difficult to choose a more favorable example, but that given was drawn to the author's attention by Dr F. W. J. Olver as having arisen as a source of difficulty in a practical computation and has therefore the merit of authenticity in contrast with a carefully contrived, totally convincing but, in the light of subsequent application, misleading display. The entries in Table IV were computed by Dr Olver and subsequently recomputed by the author.)

m	s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0.0															
		0.000														
1	+1.0758	-0.0001														
		-0.9932	-0.0003													
2	-22.8427	-0.3519	-0.0009													
		+5.7802	-0.2342	-0.0022												
3	+158.2647	+0.7741	-0.1882	-0.0049												
		-23.6788	+0.1607	.1650	-0.0097											
4	-674.4261	-2.5713	-0.0105	.1515	-0.0171											
		+67.2710	-0.7028	.0740	.1430	-0.0273										
5	+2068.3845	+4.9990	-0.3057	.1010	.1373	-0.0399										
		-151.5158	+0.8091	.1895	.1133	.1334	-0.0541									
6	-4966.3127	-8.9829	+0.1096	.1495	.1190	.1306	-0.0686									
		+280.6251	-1.4081	.5496	.1343	.1218	.1285	-0.0822								
7	+9803.6684	+12.9938	-0.3860	.1015	.1282	.1230	.1270	-0.0940								
		- 444.1808	+1.4309	.1882	.1162	.1257	.1236	.1260	-0.1035							
8	-164443608	-17.1468	+0.1402	.1416	.1212	.1247	.1238	-0.1253								
		+612.8662	-1.8094	.0692	.1291	.1229	.1242	-0.1239								
9	+23989.7695	+19.5644	-0.3665	.1111	.1255	.1236	-0.1241									
		-751.3996	+1.5324	.1670	.1207	.1244	-0.1238									
10	-30976.5512	-20.6813	+0.0810	.1327	.1231	-0.1241										
		+828.2586	-1.6142	.0924	.1259	-0.1237										
11	+35885.3647	+19.4526	-02843	.1183	-0.1244											
		-830.6170	+1.1132	.1456	-0.1228											
12	-37705.3546	-17.2679	-0.0069	-0.1273												
		+763.4631	-1.0778	-0.1099												
13	+36251.6880	+13.7783	-0.2039													
		-648.6284	+0.5629													
14	-32130.0023	-10.6238														
		+511.7122														
15	+26414.8771															

Table V

m	s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0.															
		-0.0546														
1	+1.0758		-0.0571													
		-0.0112		-0.0589												
2	-22.8427		-0.0314		-0.0600											
		-0.4107		-0.04604		-0.0602										
3	+158.2647		-0.2899		.0580		-0.0593									
		+0.4893		-0.2300		.0685		-0.0575								
4	-674.4261		+0.1194		.1948		.0780		-0.0553							
		-1.3384		-0.0102		.1721		.0867		-0.0537						
5	+2068.3845		-0.4791		.0662		.1568		.0946		-0.0540					
		+1.9745		-0.2510		.09321		.1461		.1017		-0.0570				
6	-4966.3127		+0.3522		.1747		.1072		.1388		.1078		-0.0631			
		-3.2920		+0.0118		.1456		.1148		.1336		.1127		-0.0716		
7	+9803.6684		-0.7015		.0798		.1335		.1189		.1301		.1165		-0.0815	
		+4.1055		-0.2589		.1084		.1283		.1212		.1277		.1193		-0.0915
8	-16444.3608		+0.5117		.1604		.1182		.1259		.1225		.1262		-0.1212	
		-5.1773		+0.0004		.1347		.1218		.1248		.1232		-0.1252		
9	+23989.7695		-0.7625		.0954		.1273		.1231		.1243		-0.1235			
		+5.3371		-0.2303		.1167		.1250		.1236		-0.1241				
10	-30976.5512		+0.4657		.1449		.1220		.1243		-0.1238					
		-5.5.101		-0.0392		.1285		.1234		-0.1241						
11	+35885.3647		-0.6273		.1094		.1250		-0.1238							
		+4.7612		-0.1871		.1211		-0 .1242								
12	-37705.3516		+0.2755		.1335		-0.1233									
		-4.2251		-0.0798		-0.1256										
13	+36251.6880		-0.4201		-0.1180											
		+3.0809		-0.1530												
14	-32130.0023		+0.8215													
		-2.4664														
15	+26414.8771															

Table VI

6. The Case in which the Fundamental Series is a Certain Asymptotic Series.

6-1. The series now being considered are of the form

$$\Theta(x) = v_o + \frac{abx}{1\,!} v_1 + \frac{a(a+1)b(b+1)}{2\,!} x^2 v_2 + \dots \qquad (6\text{-}1\text{-}1)$$

where the fundamental series

$$\Phi_o(x) = 1 + \frac{abx}{1\,!} x + \frac{a(a+1)b(b+1)}{2\,!} x^2 + \dots \qquad (6\text{-}1\text{-}2)$$

satisfies the differential equation

$$x^2 y'' + \left\{ (a+b+1)x - 1 \right\} y' + aby = 0 \qquad (6\text{-}1\text{-}3)$$

Thus

$$u_o^{(0)} = {}_2F_0(a,b;x)\ , \quad u_o^{(1)} = abx\,{}_2F_0(a+1,b+1;x) \qquad (6\text{-}1\text{-}4)$$

and furthermore

$$u_o^{(s+2)} = -x^{-1}\left[\frac{(a+b+2s+1)x-1}{(s+2)} u_o^{(s+1)} + \frac{s(a+b+s)+ab}{(s+2)(s+1)} x u_o^{(s)}\right] \qquad (6.1.6)$$

$s = 0,1,\dots$

The recursion between the quantities $u_m^{(0)}$ $m=0,1,\dots$ is:

$$u_1^{(0)} = u_o^{(0)} - 1$$

$$u_{m+2}^{(0)} = u_{m+1}^{(0)} - \frac{a(a+1)\dots(a+m)b(b+1)\dots(b+m)}{(m+1)!} x^{m+1} \qquad (6.1.7)$$

$m = 0,1,\dots$

In this case:

$$\Phi_m(x) = \frac{\Gamma(a+m)\Gamma(b+m)}{\Gamma(m+1)\Gamma(a)\Gamma(b)}\, {}_3F_1(a+m, b+m, 1; m+1; x) \qquad (6.1.8)$$

and satisfies the equation

$$x^3 y''' + \left\{(a+b+2m+4)x - 1\right\} x y'' + \left[\left\{(a+m+2)(b+m+2) - 2\right\} x - m - 1\right] y' + (a+m)(b+m) y = 0 \qquad (6.1.9)$$

from which may be derived the recursion system:

$$u_m^{(s+3)} = -x^{-1}\left[\frac{(a+b+2m+3s+4)x-1}{(s+3)}u_m^{(s+2)} + s\,\frac{\{(2a+2b+4m+3s+5)x-1\}+\{(a+m+2)(b+m+2)-2\}x-m-1}{(s+3)(s+2)}u_m^{(s+1)} + \frac{(a+m)(b+m)+s\{(a+m+2)(b+m+2)-2+(s-1)\{a+b+2m+s+2\}\}x}{(s+3)(s+2)(s+1)}u_m^{(s)}\right]$$

$$s=0,1,\cdots \qquad (6.1.10)$$

Note that when

$$\Phi_o(x) = {}_2F_1(1,b;x) \qquad (6\text{-}1\text{-}11)$$

the simpler recursion system

$$u_m^{(1)} = -x^{-1}\left[\{(b+m)x-1\}u_m^{(0)} + \frac{x^m\Gamma(b+m)}{\Gamma(b)}\right] \qquad (6.1.12)$$

$$u_m^{(s+2)} = -\frac{1}{x(s+2)}\left[\{(b+m+2s+2)x-1\}u_m^{(s+1)} + (s+b+m)x\,u_m^{(s)}\right] \qquad (6.1.13)$$

obtains.

6-2. The Integral of Goodwin and Staton [9].

The preceding theory may be illustrated by the evaluation of the integral

$$\int_0^\infty \frac{e^{-t^2}}{z'+t}\,dt \sim -2\sum_{s=0}^{\infty}\Gamma\left(\frac{s+1}{2}\right)(-z')^{-s-1} \qquad (6\text{-}2\text{-}1)$$

(This may be shown [10] to be equal to

$$e^{-z'^2}\left\{\pi^{1/2}\int_0^{z'} e^{t^2}\,dt - \frac{1}{2}\,\mathrm{E}i\,(z'^2)\right\}\Bigg)$$

The fundamental series is taken to be

$${}_2F_1(1,1;x) = -ze^z\mathrm{E}i(-z) \qquad (6\text{-}2\text{-}2)$$

where

$$z = -x^{-1} \qquad (6\text{-}2\text{-}3)$$

This corresponds to the substitutions

$$v_m = \frac{2^{m-1}\,\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m+1)}, \qquad x = (-2z')^{-1} \qquad (6\text{-}2\text{-}4)$$

The recursion system between these quantities is

$$v_o = \frac{1}{2}\pi^{1/2}$$

$$v_1 = 1.0$$

$$v_{m+2} = \frac{2}{m+2} v_m \qquad m = 0,\ 1, \dots \qquad (6\text{-}2\text{-}5)$$

The recursions among the functions $u_m^{(s)}$ are

$$u_0^{(0)} = -z e^z Ei(-z) \qquad (6.2.6)$$

$$u_0^{(1)} = \{1 + (z+1)e^z Ei(-z)\} z \qquad (6.2.7)$$

$$u_m^{(s+2)} = -\frac{\{(2s+m+3)x-1\} u_m^{(s+1)} + (s+m+1) x u_m^{(s)}}{(s+2)x} \qquad (6.2.8)$$

$$u_1^{(0)} = u_0^{(0)} - 1,\ u_{m+1}^{(0)} = u_m^{(0)} - m!(-x)^m \qquad m = 0,1,\dots \qquad (6.2.9)$$

The Θ-array for this example, when $x = -0.5$, is displayed in Table VII. (The results along the line $m = 0$ have already been produced by van Wijngaarden [11].)

6-3. Wilson's Integral.

A second example is provided by the transformation of the asymptotic series for the integral (5-3-1). The series is derived by expanding the term $e^{-\alpha^2 \sec^2\theta}$ in ascending powers of α^2 and noting the result

$$Y_o(\beta) = \left(\frac{2}{\pi}\right) \int_0^{\pi/2} \sec\Theta \cos(\beta \sec\Theta) d\Theta \qquad (6\text{-}3\text{-}1)$$

there follows

$$F_3(\alpha,\beta) \sim \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{n!} Y_o^{(2n+2)}(\beta) \qquad (6\text{-}3\text{-}2)$$

Since

$$\beta\, Y_o^{(n+3)}(\beta) + (n+2)\, Y_o^{(n+2)}(\beta) + \beta\, Y_o^{(n+1)}(\beta) + (n+1)\, Y_o^{(n)}(\beta) = 0 \qquad (6\text{-}3\text{-}3)$$

it follows that for large β

$$Y_o^{(n)}(\beta) \sim \text{const}(-1)^n (n-1)!\, \beta^{-n} \qquad (6\text{-}3\text{-}4)$$

and further that the series (6-3-2) diverges.

The transformation of the asymptotic series is based on relation (6-3-4); the substitution

$$Y_o^{(2r+2)}(\beta) = \frac{2}{\pi}(2r)!\,\beta^{-2r-2}v_r \qquad r = 0, 1, \ldots \qquad (6\text{-}3\text{-}5)$$

is made and the quantities v_r computed by means of (6-3-3).

The corresponding fundamental series is

$$u_0^{(0)} = \Phi_0(x) = 1 + \frac{3}{2}x + \frac{3}{2}\cdot\frac{5}{2}\cdot x^2 + \ldots$$

$$\sim 2h^2\left\{2he^{-h^2}\int_0^h e^{t^2}dt - 1\right\} \qquad (6.3.6)$$

where: $$x = h^{-2} = 4\frac{\alpha^2}{\beta^2} \qquad (6.3.7)$$

The functions $u_m^{(s)}$ $m,s = 1, 2, \ldots$
are computed by means of the recursions

$$u_0^{(1)} = \left\{\left(1 - \frac{3}{2}x\right)u_0^{(0)} - 1\right\}x^{-1} \qquad (6.3.8)$$

$$u_m^{(s+2)} = -\frac{\left\{\left(2s+m+\frac{7}{2}\right)x - 1\right\}u_m^{(s+1)} + x\left(m+s+\frac{3}{2}\right)u_m^{(s)}}{(s+2)x} \qquad m,s = 0,1,\ldots \qquad (6.3.9)$$

$$u_1^{(0)} = u_0^{(0)} - 1 \qquad u_{m+1}^{(0)} = u_m^{(0)} - \frac{3}{2}\cdot\frac{5}{2}\cdots\left(m+\frac{1}{2}\right)x^m \qquad m = 1,2,\ldots \qquad (6.3.10)$$

The initial values for (6.3.3) are:

$$Y_0^{(0)}(\beta) = Y_0(\beta), \quad Y_1^{(1)}(\beta) = -Y_1(\beta), \quad Y_0^{(2)}(\beta) = \beta^{-1}\{Y_1(\beta) - \beta Y_0(\beta)\}$$

The Θ-array for this example when $\alpha = 0.2$, $\beta = 1.0$ are displayed in Table VIII.

		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0.0															
		+0.6404383														
1	0.8862269		+0.6213276													
		+0.6088842		+0.6079933												
2	0.3862693		+0.5964407		+0.6049793											
		+0.5835518		+0.6018115		+0.6047448										
3	0.8293404		+0.6084249		+0.6051834		+0.6049303									
		+0.6444452		+0.6084249		.6051834		+0.6050643								
4	+0.3293404		+0.6077698		+0.6063102		.6052655		+0.6051214							
		+0.5387812		+0.6029866		.6054051		.6052144		+0.6051381						
5	+0.9940106		+0.5932328		+0.6036616		.6051209		.6051673		+0.6051397					
		+0.7200525		+0.6051355		.6045357		.6050757		.6051430		+0.6051376				
6	-0.0059894		+0.6345494		+0.6065609		.6049753		.6050921		.6051335		+0.6051356			
		+0.3961593		+0.6102710		.6060488		.6051309		.6051133		.6051313		+0.6051344		
7	+1.6556861		+0.5394673		+0.6046526		.6055293		.6051645		.6051264		.6051317		+0.6051338	
		+1.0059838		+0.5875913		.6040527		.6052541		.6051594		.6051324		.6051325		+0.6051336
8	-1.3443139		+0.7498456		+0.6021624		.6044411		.6051455		.6051482		.6051344		+0.6051331	
		-0.2052296		+0.6528972		.6057069		.6048133		.6051159		.6051400		+0.6051347		
9	+4.4715503		+0.2811311		+0.6184210		.6060648		.6050226		.6051154		+0.6051355			
		+2.3271720		+0.4827223		.6073846		.6057472		.6051137		+0.6051219				
10	-7.5284497		+1.3503486		+0.5628711		.6045450		.6054376		+0.6051434					
		-3.2295279		+0.9153136		.5928261		.6042370		+0.6052521						
11	+18.6429392		-1.1632462		+0.7272527		.6042370		+0.6045160							
		+9.5257795		-0.1875615		.6492002		+0.6057169								
12	-41.3570608		+4.9404207		+0.2629630		+0.6186654									
		-21.0120322		+2.6670625		+0.4650734										
13	+102.5555781		-10.3745889		+1.5594184											
		+55.0290920		-4.8779756												
14	-257.4144219		+29.3120941													
		-141.4102016														
15	+678.2127310															

Table VII

m	s	0	1	2	3	4	5	6	7	8	9
0	0.0										
		-1.97043									
1	-1.36576		-1.63352								
		1.723 05		-1.46136							
2	1.55944		1.74258		-1.91133						
		1.74776		1.79001		-2.27267					
3	1.64862		1.74746		1.77289		-2.01739				
		1.74851		1.74840		-1.71122		-1.35586			
4	1.69893		1.74784		1.74846		-1.67727		-0.89120		
		1.74875		1.74835		1.74882		1.70991		-1.05088	
5	1.73533		1.74830		1.74894		1.75006		-1.78709		-1.79073
		1.74878		1.74832		1.74903		1.75096		-1.85212	
6	1.76744		1.74851		1.74854		1.74872		-1.75004		
		1.74875		1.74842		1.74869		-1.74833			
7	1.80088		1.74860		1.74847		-1.74870				
		1.74871		1.74850		-1.74856					
8	1.84103		1.74864		-1.74849						
		1.74866		-1.74856							
9	1.89568		-1.74865								
		-1.74861									
10	-1.97876										

Table VIII

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