



**A SUFFICIENT CONDITION FOR THE INSTABILITY
OF THE ϵ -ALGORITHM***

BY

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In a previous treatment [1] of the q - d algorithm a sufficient condition for the instability in the construction of certain auxiliary quantities by use of this algorithm was given. The criterion was established by contrasting a special distribution of error in the initial conditions with that in the derived quantities. The special error distribution arose only as a possibility (but this in itself of course shows that a certain degree of instability can occur) and was chosen to allow certain differential forms of the q - d algorithm to be used in the analysis. At the time at which the note referred to was written, corresponding differential forms of the ϵ -algorithm had not been discovered, but this has latterly been remedied and accordingly a similar treatment of the ϵ -algorithm is possible.

If quantities $\epsilon_s^{(m)}$ $m, s = 0, 1, \dots$ are produced by application of the ϵ -algorithm relationships [2]

$$\epsilon_{s+1}^{(m)} = \epsilon_{s-1}^{(m+1)} + \frac{1}{\epsilon_s^{(m+1)} - \epsilon_s^{(m)}} \quad (1)$$

to the initial conditions

$$\epsilon_{-1}^{(m)} = 0, \epsilon_0^{(m)} = \sum_{s=0}^{m-1} \phi^{(s)}(a)z^{-s-1}, m = 1, 2, \dots; \epsilon_0^{(0)} = 0 \quad (2)$$

then it may be shown, by an extension of methods similar to those employed in [3], that

$$\begin{aligned} \{\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)}\} \{\phi(a) - z\epsilon_{2s}^{(m)} + \frac{\partial}{\partial a} \epsilon_{2s}^{(m)}\} &= z \\ \{\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)}\} \{z\epsilon_{2s+1}^{(m)} + \frac{\partial}{\partial a} \epsilon_{2s+1}^{(m)}\} &= z \end{aligned} \quad (3)$$

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Now, in any application of the ϵ -algorithm to the initial condition

$$\epsilon_{-1}^{(m)} = 0, \epsilon_0^{(m)} = S_m \quad m = 1, 2, \dots, \epsilon_0^{(0)} = 0 \quad (4)$$

the quantities S_m $m = 0, 1, \dots$ may be construed as the partial sums of the series

$$S_m = \sum_{s=0}^{m-1} \phi^{(s)}(a) \quad m = 1, 2, \dots \quad (5)$$

corresponding to $z = 1$ in (2).

A certain error distribution may be produced in the initial conditions (4) by letting an increase to $a + \Delta a$; in this case the error in $\epsilon_0^{(m')}$ is, from (5),

$$\{\epsilon_0^{(m'+1)} - \epsilon_0^{(1)}\} \Delta a \quad (6)$$

The corresponding errors in the quantities $\epsilon_{2s}^{(m)}$ are, from (3) given by

$$\left\{ \epsilon_{2s}^{(m)} - \epsilon_0^{(1)} + \frac{1}{\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)}} \right\} \Delta a \quad (7)$$

and in $\epsilon_{2s+1}^{(m)}$, by

$$\left\{ \frac{1}{\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)}} - \epsilon_{2s+1}^{(m)} \right\} \Delta a. \quad (8)$$

The ratios of the relative errors in $\epsilon_{2s}^{(m)}$ and $\epsilon_{2s+1}^{(m)}$ to those in $\epsilon_0^{(m')}$ which indicate the condition of the ϵ -process, are given by

$$\left\{ \frac{(\epsilon_{2s}^{(m)} - \epsilon_0^{(1)})(\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)}) + 1}{(\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)})\epsilon_{2s}^{(m)}} \right\} \frac{\epsilon_0^{(m')}}{(\epsilon_0^{(m'+1)} - \epsilon_0^{(1)})} \quad (9)$$

$$m - 2s \leq m' \leq m; s = 0, 1, \dots; m' = 1, 2, \dots$$

and

$$\left\{ \frac{1 - \epsilon_{2s+1}^{(m)}(\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)})}{\epsilon_{2s+1}^{(m)}(\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)})} \right\} \frac{\epsilon_0^{(m')}}{\epsilon_0^{(m'+1)} - \epsilon_0^{(1)}} \quad (10)$$

$$m - 2s - 1 \leq m' \leq m; s = 0, 1, \dots; m' = 1, 2, \dots$$

These results may be extended by the following considerations

If quantities $\epsilon_s^{(m)}$ are produced by applying relationships (1) to the initial conditions (4), and quantities $\tilde{\epsilon}_s^{(m)}$ are produced by applying relationships (1) to the initial conditions

$$\tilde{\epsilon}_{-1}^{(m)} = 0, \tilde{\epsilon}_0^{(m)} = b + S_m \quad m = 1, 2, \dots; \tilde{\epsilon}_0^{(0)} = b \quad (11)$$

then

$$\tilde{\epsilon}_{2s}^{(m)} = b + \epsilon_{2s}^{(m)}, \tilde{\epsilon}_{2s+1}^{(m)} = \epsilon_{2s+1}^{(m)} \quad m, s = 0, 1, \dots \quad (12)$$

That is to say that any one of the initial values $S_{m''}$ $m'' = 1, 2, \dots$ in (4) can be made to play the rôle of $\epsilon_0^{(1)}$ in (9) and (10), merely by subtracting $\epsilon_0^{(m''-1)}$ from all the initial values. The ratios (9) and (10) then become

$$\frac{\{(\epsilon_{2s}^{(m)} - \epsilon_0^{(m'')})(\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)}) + 1\}(\epsilon_0^{(m')} - \epsilon_0^{(m''-1)})}{(\epsilon_{2s+1}^{(m)} - \epsilon_{2s-1}^{(m)})(\epsilon_{2s}^{(m)} - \epsilon_0^{(m''-1)})(\epsilon_0^{(m'+1)} - \epsilon_0^{(m'')})} \quad (13)$$

$$m - 2s \leq m' \leq m; s = 0, 1, \dots; m'' = 1, 2, \dots; m' = m'', m'' + 1, \dots$$

and

$$\frac{\{1 - \epsilon_{2s+1}^{(m)}(\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)})\}(\epsilon_0^{(m')} - \epsilon_0^{(m''-1)})}{\epsilon_{2s+1}^{(m)}(\epsilon_{2s+2}^{(m)} - \epsilon_{2s}^{(m)})(\epsilon_0^{(m'+1)} - \epsilon_0^{(m'')})} \quad (14)$$

$$m - 2s - 1 \leq m' \leq m; s = 0, 1, \dots; m'' = 1, 2, \dots; \\ m' = m'', m'' + 1, \dots$$

The main result of this note may thus be stated as:

If either of the expressions (13) and (14) has a modulus greater than unity the ϵ -process is unstable.

It would appear from expressions (13) and (14) that if the differences $\epsilon_0^{(m'+1)} - \epsilon_0^{(m'')} m'' = 1, 2, \dots; m' = m'', m'' + 1, \dots$ are consistently small then the ϵ -process is likely to be unstable, but this corresponds to the case in which the ϵ -algorithm is applied to the transformation of a slowly convergent series of terms which have approximately constant modulus and argument, and in this case it is a matter of numerical experience that the ϵ -process is unstable.

REFERENCES

- [1] P. WYNN, A Sufficient Condition for the Instability of the q - d Algorithm, Num. Math. 1, (1959), 203-207.
- [2] P. WYNN, On a Device for Computing the $e_m(S_n)$ Transformation, MTAC, X, (1956), 91-96.
- [3] P. WYNN, Upon a Second Confluent Form of the ϵ -Algorithm, to appear.

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