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A comparison technique for the numerical transformation of slowly convergent series based on the use of rational functions*

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Introduction

In [*1*] an expository account was given of a technique for the transformation of slowly convergent series by methods of comparison. The method used proceeds as follows: the series to be transformed is $\sum_{s=0}^{\infty} c_s v_s x^s$. It is assumed that the function

$$\varphi_0(x) \sim \sum_{s=0}^{\infty} c_s x^s \tag{1}$$

and its derivatives may easily be computed. The transformation is then

$$\sum_{s=0}^{\infty} c_s v_s x^s \sim \sum_{s=0}^{\infty} \frac{x^s}{s!} \varphi_0^{(s)}(x) \Delta^s v_0 \tag{2}$$

and in its delayed form

$$\sum_{s=0}^{\infty} c_s v_s x^s \sim \sum_{s=0}^{m-1} c_s v_s x^s + \sum_{s=0}^{\infty} \frac{x^{m+s}}{s!} \varphi_m^{(s)}(x) \Delta^s v_m \tag{3}$$

where

$$\varphi_m(x) \sim \sum_{s=0}^{\infty} c_{m+s} x^s. \tag{4}$$

If the quantities v_{m+s} $s=0, 1, \ldots$ vary slowly with s then the successive differences of v_m decrease rapidly in magnitude as, in favorable cases, do the successive terms on the right hand side of equation (3).

An Algorithm

In this paper we shall develop a transformation based upon the expression of

$$\left\{\sum_{s=0}^{\infty} c_s v_s x^s\right\} \left\{\sum_{s=0}^{\infty} c_s x^s\right\}^{-1}$$

as a continued fraction. An algorithm for doing this is already to hand (see for example [*2*], p. 31). We are concerned to express the quotient

$$\{\varphi_0(x)\} \left\{\frac{\alpha_{1,0}+\alpha_{1,1} x+\alpha_{1,2} x^2+\alpha_{1,3} x^3+\cdots}{\alpha_{0,0}+\alpha_{0,1} x+\alpha_{0,2} x^2+\alpha_{0,3} x^3+\cdots}\right\} \tag{5}$$

where

$$\alpha_{0,s} = c_s \qquad \alpha_{1,s} = c_s v_s \qquad s = 0, 1, \ldots \tag{6}$$

as a continued fraction. (5) may be written

$$\{\varphi_0(x)\}\left\{\frac{1}{\dfrac{\alpha_{0,0}}{\alpha_{1,0}} + \dfrac{\alpha_{0,0}+\alpha_{0,1}\,x+\alpha_{0,2}\,x^2+\cdots}{\alpha_{1,0}+\alpha_{1,1}\,x+\alpha_{1,2}\,x^2+\cdots} - \dfrac{\alpha_{0,0}}{\alpha_{1,0}}}\right\} \tag{7}$$

or

$$\{\varphi_0(x)\}\left\{\frac{\alpha_{1,0}}{\alpha_{0,0} + x\,\dfrac{\alpha_{2,0}+\alpha_{2,1}\,x+\alpha_{2,2}\,x^2+\cdots}{\alpha_{1,0}+\alpha_{1,1}\,x+\alpha_{1,2}\,x^2+\cdots}}\right\} \tag{8}$$

where

$$\alpha_{2,s} = \alpha_{1,0}\,\alpha_{0,s+1} - \alpha_{0,0}\,\alpha_{1,s+1}. \tag{9}$$

This process may clearly be repeated and we obtain for (5) the continued fraction

$$\{\varphi_0(x)\}\left\{\frac{\alpha_{1,0}}{\alpha_{0,0}+}\;\frac{\alpha_{2,0}\,x}{\alpha_{1,0}+}\;\frac{\alpha_{3,0}\,x}{\alpha_{2,0}+}\cdots\right\} \tag{10}$$

and in general (delaying the comparison until the m^{th} term) for the quotient

$$\sum_{s=0}^{m-1}\alpha_{1,s}\,x^s + \left\{\varphi_0(x) - \sum_{s=0}^{m-1}\alpha_{0,s}\,x^s\right\}\left\{\frac{\alpha_{1,m}+\alpha_{1,m+1}\,x+\alpha_{1,m+2}\,x^2+\cdots}{\alpha_{0,m}+\alpha_{0,m+1}\,x+\alpha_{0,m+2}\,x^2+\cdots}\right\} \tag{11}$$

the continued fraction

$$\sum_{s=0}^{m-1}\alpha_{1,s}\,x^s + \left\{\varphi_0(x) - \sum_{s=0}^{m-1}\alpha_{0,s}\,x^s\right\}\left\{\frac{\alpha_{1,0}^{(m)}}{\alpha_{0,0}^{(m)}+}\;\frac{\alpha_{2,0}^{(m)}\,x}{\alpha_{1,0}^{(m)}+}\;\frac{\alpha_{3,0}^{(m)}\,x}{\alpha_{2,0}^{(m)}+}\right\} \tag{12}$$

The coefficients in (12) are computed as the leading column of the following array

Table 1

$$\begin{array}{ccccc}
\alpha_{0,0}^{(m)} & \alpha_{0,1}^{(m)} & \alpha_{0,2}^{(m)} & \cdots\,\alpha_{0,s}^{(m)} & \cdots \\
\alpha_{1,0}^{(m)} & \alpha_{1,1}^{(m)} & \alpha_{1,2}^{(m)} & \cdots\,\alpha_{1,s}^{(m)} & \cdots \\
\alpha_{2,0}^{(m)} & \alpha_{2,1}^{(m)} & \alpha_{2,2}^{(m)} & \cdots\,\alpha_{2,s}^{(m)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
\alpha_{r,0}^{(m)} & \alpha_{r,1}^{(m)} & \alpha_{r,2}^{(m)} & \cdots\,\alpha_{r,s}^{(m)} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

which is formed by use of the relationship

$$\alpha_{r,s}^{(m)} = \alpha_{r-1,0}^{(m)}\,\alpha_{r-2,s+1}^{(m)} - \alpha_{r-2,0}^{(m)}\,\alpha_{r-1,s+1}^{(m)} \tag{13}$$

from the initial values

$$\alpha_{0,s}^{(m)} = \alpha_{0,m+s} \qquad \alpha_{1,s}^{(m)} = \alpha_{1,m+s} \qquad s = 0, 1, \ldots. \tag{14}$$

The successive convergents $C_r^{(m)}$ $r = 0, 1, \ldots$ of (12) may be computed by means of the twin recursions

$$\begin{aligned} A_r^{(m)} &= \alpha_{r-1,0}^{(m)}\,A_{r-1}^{(m)} + \alpha_{r,0}^{(m)}\,x\,A_{r-2}^{(m)} \\ B_r^{(m)} &= \alpha_{r-1,0}^{(m)}\,B_{r-1}^{(m)} + \alpha_{r,0}^{(m)}\,x\,B_{r-2}^{(m)} \end{aligned} \tag{15}$$

from the initial conditions

$$A_0^{(m)} = \sum_{s=0}^{m-1} c_s v_s x^s, \qquad B_0^{(m)} = 1, \tag{16}$$

$$A_1^{(m)} = \alpha_{0,0}^{(m)} A_0^{(m)} + \alpha_{1,0}^{(m)} \left\{\varphi_0(x) - \sum_{s=0}^{m-1} c_s x^s\right\}, \qquad B_1^{(m)} = \alpha_{0,0}^{(m)} \tag{17}$$

when

$$C_r^{(m)} = A_r^{(m)} / B_r^{(m)}. \tag{18}$$

A Programme

Given $m1+1$ pairs of coefficients c_s, cv_s $(\equiv c_s v_s)$ $s=0, 1, \ldots, m1$ we can, if we are so disposed, construct a triangular array of transformed results of the form (18) for which $m+r \leqq m1+1$. Taking into account remarks 1—4 below, a programme for carrying this out proceeds as follows:

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begin real a0, a1, a2, b0, b1, b2, t; integer m, i, j; array f, g[0:m1];
for m:=0 step 1 until m1+1 do
begin a0:=a1:=0; b0:=1·0; for j:=m − 1 step − 1 until 0 do
begin a0:=a0×x+cv[j]; a1:=a1×x+c[j] end;
C[m,0]:=a0; a1:=c[m]×a0+(phi0x − a1)×cv[m]; b1:=c[m]; C[m,1]:=a1/b1;
for j:=m step 1 until m1 do
begin f[j]:=c[j]; g[j]:=cv[j] end;
for i:=2 step 1 until m1 − m+1 do
begin if 2×[i ÷ 2]=i then
begin t:=f[m]; for j:=m step 1 until m1 − i+1 do f[j]:=g[m]×f[j+1] − t×g[j+1];
a2:=g[m]×a1+x×f[m]×a0; b2:=g[m]×b1+x×f[m]×b0 end else
begin t:=g[m]; for j:=m step 1 until m1 − i+1 do g[j]:=f[m]×g[j+1] − t×f[j+1];
a2:=f[m]×a1+x×g[m]×a0; b2:=f[m]×b1+x×g[m]×b0 end;
C[m,i]:=a2/b2; a0:=a1; a1:=a2; b0:=b1; b1:=b2 end i end m end;
```

1. In this programme the numbers occurring in the even and odd order rows of the array in Table 1 are retained in the vectors f and g respectively.

2. It is assumed that the transformed results (18) are to be stored as the triangular array $C_{m,i}$ $m=0(1)m1+1$, $i=0(1)m1-m+1$; this clearly implies a prodigal and unnecessary use of storage space and one can quite clearly print out the transformed results as they are produced.

3. It is further assumed that the coefficients c_s, cv_s $s=0, 1, \ldots, m1$ have already been computed and that these arrays together with $C_{m,i}$ have already been declared as global arrays.

4. Lastly it is assumed that the values of the global variables x and $phi\,0\,x$ $(\equiv \varphi_0(x))$ have also been computed.

Numerical Results

We shall give three examples of the application of the above programme. The first concerns the transformation of the series

$$\int_0^\infty \frac{t^2\,dt}{(1+e^{t^2}/\lambda)} \sim \sum_{s=0}^{\infty} (s+1)^{-\frac{3}{2}} (-\lambda)^s \tag{19}$$

which occurs in work by MESTEL on the conductivity of dense stars [3]. This series development is compared with that of

$$\lambda^{-1} \log(1+\lambda) \sim \sum_{s=0}^{\infty} (s+1)^{-1} (-\lambda)^s \quad (20)$$

when $\lambda = 10.0$. The results are displayed in Table 2 and confirm the value 0.3285 computed by an alternative method.

The second concerns the transformation of the series

$$I_0(z) = \sum_{s=0}^{\infty} \frac{(z/2)^{2s}}{(s!)^2} \quad (21)$$

by comparing it with

$$z^{-1} \sinh(z) = \sum_{s=0}^{\infty} \frac{z^{2s}}{(2s+1)!} \quad (22)$$

when $z = 8.0$ (Table 3). It will be seen that some improvement in the convergence of the series (21) to the value $I_0(8.0) \doteqdot 427.56$ is effected. In the event this example is a little idle, since the addition of three further terms of the original series brings about the same measure of improvement. However the point made here is that the method described works for series with delayed convergence of the type illustrated by (21), and in a less trivial case the transformation would be of some practical use.

The last example concerns the transformation of the asymptotic development of the integral of GOODWIN and STATON [4]

$$z \int_0^{\infty} \frac{e^{-t^2}}{z+t} dt \sim 2 \sum_{s=0}^{\infty} (-z)^{-s} \Gamma\left(\frac{s+1}{2}\right) \quad (23)$$

by comparing it with the series for the exponential integral

$$-z e^z Ei(-z) \sim \sum_{s=0}^{\infty} (-z)^{-s} \Gamma(s+1) \quad (24)$$

when $z = 2.0$ (Table 4) when the required value is 0.60513.

Table 2

m \ s	0	1	2	3	4	5	6	7	8	9	10	11
0	0	+.2398	−.5163	+.3023	+.3961	+.3205	+.3397	+.3261	+.3311	+.3278	+.3292	+.3284
1	+.1000(1)	+.1625	+.3407	.3581	.2435	.3363	.3153	.3307	.3258	.3292	+.3280	
2	−.2536(1)	−.8769(−1)	−.5115(3)	.2657	+.5469	.3158	.3577	.3256	.3339	+.3278		
3	+.1671(2)	+.2163(1)	−.7689(2)	+.5212	−.4612	.3585	.2419	.3342	+.3154			
4	−.1083(3)	−.9498(1)	+.2524(3)	−.4181	+.3811(1)	.2387	+.6412	+.3147				
5	+.7861(3)	+.5982(2)	−.1208(4)	+.3724(1)	−.1717(2)	+.6518	−.9634					
6	−.6018(4)	−.3910(3)	+.6921(4)	−.1699(2)	+.9685(2)	−.1002(1)						
7	+.4798(5)	+.2733(4)	−.4405(5)	+.9664(2)	−.5711(3)							
8	−.3940(6)	−.4995(5)	+.3010(6)	−.5724(3)								
9	+.3310(7)	+.1509(6)	−.2165(7)									
10	−.2831(8)	−.1174(7)										
11	+.2458(9)											

Table 3

m \ s	0	1	2	3	4	5	6	7	8
0	0	+186.31	−42.99	+350.10	+611.93	+417.84	+432.62	+427.08	+427.69
1	+1.00	278.96	+1390.82	437.08	405.72	425.18	428.28	+427.54	
2	17.00	344.46	455.93	441.48	476.90	426.96	+427.70		
3	81.00	388.37	426.79	431.96	429.20	+427.44			
4	194.75	412.56	426.02	428.49	+427.75				
5	308.56	422.96	427.01	+427.72					
6	381.37	426.43	+427.43						
7	413.74	+427.34							
8	+424.30								

Table 4

m \ s	0	1	2	3	4	5	6	7	8	9	10	11
0	0	+.64044	+.60181	+.55822	+.59865	+.60521	+.60708	+.60542	+.60515	+.60584	+.60510	+.60511
1	+.88623	.60888	.57328	.60170	.60485	.59465	.60562	.60535	.60569	.60514	+.60498	
2	.36823	.58355	.70025	.60429	.60914	.60532	.60552	.60534	.60516	+.60566		
3	.82934	.64445	.26793	.60874	.59543	.60567	.60382	.60523	+.60488			
4	.32934	.53878	.46814(2)	.59797	.62592	.60388	.60800	+.60485				
5	+.99401	.72005	+.18038(1)	.61832	.56079	.60749	+.59923					
6	−.59894(−2)	.39616	−.58496	.58053	.70226	+.60075						
7	+.16557(1)	+.10060(1)	+.22678(1)	.65276	+.38455							
8	−.13443(1)	−.20523	−.21372(1)	+.50882								
9	+.44716(1)	+.23272(1)	+.56504(1)									
10	−.75284(1)	−.32295(1)										
11	+.18643(2)											

A Comparison between Two Methods

An array of transformed results, similar to the C-array, may also be developed from the transformation (3), and indeed such arrays, corresponding to the three examples treated here, are given in [1]. Comparison of the numerical results reveals that, in terms of the accuracy of the transformed estimates, the method described in this paper is slightly inferior in the third example, and slightly superior in the first two.

In general the functions $\frac{x^{m+s}}{s!}\varphi_m^{(s)}(x)$ of (3) are computed by recourse to a differential equation satisfied by $\varphi_m(x)$, the complexity of which varies from case to case; so it is impossible to make any general comparison between the amounts of computation involved in the two methods.

One disadvantage of the method proposed here is that it is a purely formal process and at the moment no convergence theory may be given, whereas this is possible in certain cases for the transformation (3).

One advantage of the present method, which should at least make some appeal to a human mathematician, is that once the coefficients in the original series and those in the series $\sum_{s=0}^{\infty} c_s x^s$ have been computed, and the value of $\varphi_0(x)$ is known, the method may immediately be applied; there is no need to set up recursion systems between further auxilary functions as in the use of (3).

Concluding Remark

It is possible to set up determinantal formulae in terms of the coefficients c_s and $c_s v_s$ for the quantities $\alpha_{r,0}^{(m)}$. Adopting a slightly different formulation it is shown in [5] that if

$$\left(\sum_{s=0}^{\infty} c_{m+s}\, x^s\right)\left(\sum_{s=0}^{\infty} d_{m+s}\, x^s\right)^{-1} = \frac{c_m}{d_m} + \frac{x}{r_1^{(m)}+}\;\frac{x}{r_2^{(m)}-r_0^{(m)}+}\;\frac{x}{r_3^{(m)}-r_1^{(m)}+}\cdots\frac{x}{r_s^{(m)}-r_{s-2}^{(m)}+}\cdots \tag{25}$$

then

$$r_{2n}^{(m)} = \frac{\begin{vmatrix} 0 & 0 & 0 & \dots & 0 & c_m & d_m \\ 0 & 0 & 0 & \dots & d_m & c_{m+1} & d_{m+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{m+n-2} & c_{m+n-1} & d_{m+n-1} \\ 0 & c_m & d_m & \dots & d_{m+n-1} & c_{m+n} & d_{m+n} \\ c_m & c_{m+1} & d_{m+1} & \dots & d_{m+n} & c_{m+n+1} & d_{m+n+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ c_{m+n} & c_{m+n+1} & d_{m+n+1} & \dots & d_{m+2n-1} & c_{m+2n} & d_{m+2n} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & \dots & 0 & c_m & d_m \\ 0 & 0 & 0 & \dots & d_m & c_{m+1} & d_{m+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{m+n-2} & c_{m+n-1} & d_{m+n-1} \\ 0 & c_m & d_m & \dots & d_{m+n-1} & c_{m+n} & d_{m+n} \\ d_m & c_{m+1} & d_{m+1} & \dots & d_{m+n} & c_{m+n+1} & d_{m+n+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ d_{m+n} & c_{m+n+1} & d_{m+n+1} & \dots & d_{m+2n-1} & c_{m+2n} & d_{m+2n} \end{vmatrix}} \tag{26}$$

and

$$r_{2n+1}^{(m)} = \frac{\begin{vmatrix} 0 & 0 & 0 & \dots & 0 & c_m & d_m \\ 0 & 0 & 0 & \dots & d_m & c_{m+1} & d_{m+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & c_m & \dots & d_{m+n-1} & c_{m+n} & d_{m+n} \\ 0 & d_m & c_{m+1} & \dots & d_{m+n} & c_{m+n+1} & d_{m+n+1} \\ d_m & d_{m+1} & c_{m+2} & \dots & d_{m+n+1} & c_{m+n+2} & d_{m+n+2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ d_{m+n} & d_{m+n+1} & c_{m+n+2} & \dots & d_{m+2n} & c_{m+2n+1} & d_{m+2n+1} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & \dots & 0 & c_m & d_m \\ 0 & 0 & 0 & \dots & d_m & c_{m+1} & d_{m+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & c_m & \dots & d_{m+n-1} & c_{m+n} & d_{m+n} \\ c_m & d_m & c_{m+1} & \dots & d_{m+n} & c_{m+n+1} & d_{m+n+1} \\ c_{m+1} & d_{m+1} & c_{m+2} & \dots & d_{m+n+1} & c_{m+n+2} & d_{m+n+2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ c_{m+n+1} & d_{m+n+1} & c_{m+n+2} & \dots & d_{m+2n} & c_{m+2n+1} & d_{m+2n+1} \end{vmatrix}} \qquad (27)$$

Now it would be desirable to set up recursion systems between the quantities $r_s^{(m)}$ $m, s=0, 1, \dots$ involving consecutive values of m and s (in analogy with the $q-d$ algorithm). If these recursion systems are to be proved by the use of compound determinants, as is the case with the $q-d$ and ε-algorithms (see [6] and [7]) then the determinants involved in expressions (26) and (27) with $m=\overline{m}$ and $m=\overline{m}+1$ must have a common minor, but at a glance this is seen to be untrue, so that it seems unlikely that such a recursion system exists. This is not however very serious, for in all truth the recursion (13) is quite simple.

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