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NOTE ON THE SOLUTION OF A CERTAIN BOUNDARY-VALUE PROBLEM*

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Abstract.

A simple algorithm is described for inverting the operator D_xD_y (D_x and D_y here and subsequently denote partial differentiation with respect to x and y respectively) which occurs in the iterative solution of the equation

$$D_xD_yf(x,y) = g(x,y,f,D_xf,D_x^2f,D_xD_yf,D_y^2f)$$

when boundary values of $f(x,y)$ are given along the sides of the rectangle in the xy-plane whose corners are at the points (a,b); $(a+(n+1)k,b)$; $(a+(n+1)k,b+(n+1)h)$; $(a,b+(n+1)h)$.

When carrying out a series of numerical experiments in the iterative solution of partial differential equations the author was concerned with the problem of obtaining the function $f^{(m+1)}(x,y)$ from the equation

$$(1)\quad D_xD_yf^{(m+1)}(x,y) = g(x,y,f^{(m)},D_xf^{(m)},D_yf^{(m)},D_x^2f^{(m)},D_xD_yf^{(m)},D_y^2f^{(m)})$$

where boundary values of $f(x,y)$ are given along the sides of the rectangle in the xy-plane whose corners are at the points (a,b); $(a+(n+1)k,b)$; $(a+(n+1)k,b+(n+1)h)$; $(a,b+(n+1)h)$.

Equation (1) may be solved numerically by replacing the left hand member by the substitution [1]

$$(2)\quad D_xD_yf^{(m+1)}(x,y) = \{f^{(m+1)}(x+k,y+h)-f^{(m+1)}(x+k,y-h) + f^{(m+1)}(x-k,y-h)-f^{(m+1)}(x-k,y+h)\}/(4hk)+\Delta$$

where

$$(3)\quad \Delta = O\ \text{(terms of the second and higher orders in } h \text{ and } k)$$

Δ is ignored in equation (2), and the solution of (1) is thus reduced to the problem of determining the n^2 quantities

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$$f^{(m+1)}\big(a+jk, b+(n+1-i)h\big), \qquad i,j = 1(1)n\,.$$

If these quantities are ordered by the relation

(4) $$v_r = f^{(m+1)}\big(a+jk, b+(n+1-i)h\big) \qquad r = 1(1)n^2$$

(5) $$i = [(r-1)/n]+1, \qquad j = r-n(i-1)$$

there results the matrix equation

(6) $$\boldsymbol{A}\boldsymbol{v} = \boldsymbol{c}\,.$$

The right hand side of this equation is the vector of order n^2, whose r^{th} element c_r is given by

(7) $$c_r = 4hkg\big(a+jk, b+(n+1-i)h, \ldots\big)+\Delta'$$

in conjunction with (5), where Δ' is a correction term derived from the boundary conditions, which affects the $4n-2$ elements with suffices $r=s$, $r=n(n-1)+s$, $s=1(1)n$; $r=ns+1$, $s=2(1)n-1$. The dots in equation (7) are taken to imply the values of $f\big(a+jk, b+(n+1-i)h\big)$ and its various derivatives; in an algorithmic dramatisation of this equation, to be given later, even these dots will be omitted.

The matrix $\boldsymbol{A}$ is a compound matrix having the form

(8) $$\boldsymbol{A} = \left(\begin{array}{ccc:cccc} 0 & \boldsymbol{D} & 0 & & & & \\ -\boldsymbol{D} & 0 & \boldsymbol{D} & & & 0 & \\ 0 & -\boldsymbol{D} & 0 & & & & \\ \hdashline & & & 0 & \boldsymbol{D} & 0 & 0 \\ & 0 & & -\boldsymbol{D} & 0 & \boldsymbol{D} & 0 \\ & & & 0 & -\boldsymbol{D} & 0 & \boldsymbol{D} \\ & & & 0 & 0 & -\boldsymbol{D} & 0 \end{array}\right)$$

where

(9) $$\boldsymbol{D} = \left(\begin{array}{cccc:cccc} 0 & -1 & 0 & & & & & \\ 1 & 0 & -1 & & & 0 & & \\ 0 & 1 & 0 & & & & & \\ \hdashline & & & & 0 & -1 & 0 & 0 \\ & 0 & & & 1 & 0 & -1 & 0 \\ & & & & 0 & 1 & 0 & -1 \\ & & & & 0 & 0 & 1 & 0 \end{array}\right).$$

There are n rows and columns in $\boldsymbol{D}$, n rows and columns of submatrices in $\boldsymbol{A}$, and hence $\boldsymbol{A}$ is a square matrix of order n^2.

The main purpose of this note is to point out that the matrix $\boldsymbol{A}$ has an easily constructed inverse, which enables the system of equations (6) to be solved analytically. Indeed as is easily verified

$$\boldsymbol{A}^{-1} = \begin{pmatrix} 0 & -\boldsymbol{D}^{-1} & 0 & -\boldsymbol{D}^{-1} & 0 & -\boldsymbol{D}^{-1} & \cdots \\ \boldsymbol{D}^{-1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -\boldsymbol{D}^{-1} & 0 & -\boldsymbol{D}^{-1} & \cdots \\ \boldsymbol{D}^{-1} & 0 & \boldsymbol{D}^{-1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\boldsymbol{D}^{-1} & \cdots \\ \boldsymbol{D}^{-1} & 0 & \boldsymbol{D}^{-1} & 0 & \boldsymbol{D}^{-1} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{10}$$

where

$$\boldsymbol{D}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1 & \cdots \\ -1 & 0 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ -1 & 0 & -1 & 0 & -1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{11}$$

The solution to the problem originally proposed may now be formulated as follows*:

Stage I: set up equation (7)

for $i := 1(1)n$ **do for** $j := 1(1)n$ **do**
$c_{(i-1)n+j} := 4hkg(a+jk, b+(n+1-i)h)$; } First term on right-hand side of equation (7).

for $s := 1(1)n$ **do begin**
$c_s := c_s - g(a+(s+1)k, b+(n+1)h) + g(a+(s-1)k, b+(n+1)h)$;
$c_{n(n-1)+s} := c_{n(n-1)+s} + g(a+(s+1)k, b) - g(a+(s-1)k, b)$;
$c_{n(s-1)+1} := c_{n(s-1)+1} - g(a, b+(n-s)h) + g(a, b+(n-s+2)h)$;
$c_{n(s-1)} := c_{n(s-1)} + g(a+(n+1)k, b+(n-s)h) - g(a+(n+1)k, b+(n-s+2)h)$ **end**; } Correction for sides of rectangle

$c_1 := c_1 - g(a, b+(n+1)h)$;
$c_n := c_n + g(a+(n+1)k, b+(n+1)h)$;
$c_{n(n-1)} := c_{n(n-1)} + g(a, b)$; $c_{n^2} := c_{n^2} - g(a+(n+1)k, b)$; } Recorrection at corners

* The algorithm described here is written in a language closely related to normal mathematics. Anyone interested should find it very easy to translate it e. g. into ALGOL 60.

Stage II: the matrix multiplication $A^{-1}c$

(Note: the general element of A^{-1} has row suffix $(i'-1)n+i$ and column suffix $(j'-1)n+j$)

for $p := +1, -1$ **do for** $q := +1, -1$ **do**
for $i' := (3-p)/2\ (2)\ n$ **do for** $i := (3-q)/2\ (2)\ n$ **do**
begin sum $:= 0$;
for $j' := i'+p(2p)\{n+1+p(n-1)\}/2$ **do for**
$j := i+q(2q)\{n+1+q(n-1)\}/2$ **do**
sum $:=$ sum $- pqc_{(j'-1)n+j}$; $f_{a+ik,\, b+(n+1-i')h} :=$ sum **end**;

It is important to note that when n is odd, the matrix A in (6) is singular, and the above procedure breaks down.

Further if the equation

$$D_xD_yf = g(x,y,f,D_xf,D_yf,D_x^2f,D_xD_yf,D_y^2f)$$

is hyberbolic over any part of the domain in which the solution is required, then some compatibility condition must prevail among the boundary conditions. However this is a matter concerning the posing of the problem; this note is concerned only with the formal inversion of a finite difference operator.

Techniques for accelerating the convergence of the sequence of approximate solutions to (1) $f^{(m)}(x,y)$, $m=0,1,\ldots$ are discussed in [2].

REFERENCES

1. Buckingham, R. A., *Numerical Methods*, Pitman, London 1957, ch. 15, p. 503.
2. Wynn, P., *Acceleration Techniques for Iterated Vector and Matrix Problems*, Maths. of Comp., to appear.

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