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Continued Fractions Whose Coefficients Obey a Non-Commutative Law of Multiplication

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Continued Fractions whose Coefficients Obey a Non-Commutative Law of Multiplication

P. WYNN

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Contents

Introduction

In this paper* we shall establish a formal theory of continued fractions of the form

$$b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \cdots \frac{a_n}{b_n+} \cdots \tag{1}$$

in which the coefficients a_s $(s=1, 2, \ldots)$, b_s $(s=0, 1, \ldots)$ obey a non-commutative law of multiplication.

The theory has already found application in the acceleration of slowly convergent iterative processes in numerical analysis and has therefore some relevance to this subject, but in any case it is of considerable interest as a self-contained intellectual discipline. Moreover, non-linear multi-dimensional iterative processes are of frequent occurrence in all branches of applied mathematics, and the present inquiry may well herald a break-through to a systematic theory of such processes.

The indicated domain of inquiry is, it would appear, completely unstudied, and therefore all the results to be given are original, but many of them are quite transparent adaptations of existing results in the conventional theory of continued fractions, and references (either to an original source or to an appropriate textbook) are consistently inserted for these existing results.

* Communication MR 48 of the Computation Department of the Mathematical Centre, Amsterdam.

Since this is a somewhat long paper, it is perhaps in order to outline the scheme of its development. In the first section both the continued fractions and the nature of their coefficients are preliminarily defined. In the second, various fundamental formulae — the three-term recurrence relation between the numerators and denominators of successive convergents, the Euler-Minding sum formulae, and a number of others — are established. In the third section the notions of pre and post orthogonal polynomials are introduced, and with their help continued fractions of types which may be said to correspond to (*korrespondierende*, PERRON [5], vol. II, ch. III) and to be associated with (*assoziierte*, PERRON [5], vol. II, ch. III) power series, are established. In the fourth section the non-commutative version of the $q-d$ algorithm (which may be regarded as a device for transforming the coefficients of a power series into those of the continued fractions mentioned above) is derived. Various transformations of such continued fractions are then described. Section five deals with the non-commutative versions of certain non-linear sequence to sequence transformations which have recently been discovered. It culminates in a fundamental theorem relating to the ε-algorithm. In Section six continued fractions, which relate respectively to functions which satisfy systems of three term recurrence relationships and to functions which satisfy homogeneous linear differential equations of the second order, are derived. The next section contains a short account of the use of continued fractions to interpolate in sequences of functions which obey a non-commutative multiplication law. In a final section a further restriction relating to the scalar elements of the argument field is introduced, and analogues of a number of non-linear difference-differential relationships are discussed. The conclusion contains a brief comment upon some outstanding difficulties.

There are two appendices. The first deals with determinants whose elements obey a non-commutative law of multiplication. One might have expected that such determinants would have played a leading role in the theory of this paper, but for reasons which are discussed in the appendix they have only limited application. In the second, a few details are given of an application which a certain part of the theory has found.

1. A First Definition

If the coefficients in the expression (1) are scalars, then the concept of an infinite continued fraction is immediately tangible. It is the limit, if such exists, of the sequence of convergents

$$C_n = b_0 + \frac{a_1}{b_1 +} \, \frac{a_2}{b_2 +} \cdots \frac{a_n}{b_n} \qquad (n = 0, 1, \ldots)$$

as n tends to infinity. The convergent, C_n, may immediately be computed by means of the following rules: divide a_n by b_n, and add the quotient to b_{n-1}; divide a_{n-1} by the result, and add the quotient to b_{n-2}, and so on. More concisely,

$$C_n = D_n$$

where

$$D_0 = b_n,$$

and

$$D_{r+1} = b_{n-r-1} + \frac{a_{n-r}}{D_r} \qquad (r = 0, 1, \ldots, n-1). \tag{2}$$

In this paper we shall assume that the coefficients in (1) belong to the set N, whose elements we may denote by $A, B, C, \ldots$. The following assumptions are made:

1. To every pair A, B there corresponds an element C such that

$$A + B = B + A = C.$$

2. $$(A + B) + C = A + (B + C) = (A + C) + B.$$

3. To every pair A, B there corresponds an element D such that

$$AB = D;$$

in general

$$AB \neq BA \tag{3}$$

but

$$(AB)C = A(BC).$$

4. There exists a subset S (whose elements are referred to as scalars) of N such that for every member T of S

$$TA = AT,$$

and in particular there exist two members, I the unit element and O the zero element, of S such that

$$IA = AI = A$$

and

$$OA = AO = O,$$

$$A + O = A.$$

5. To every element $E \neq 0$ of N there corresponds an inverse E^{-1} such that

$$EE^{-1} = E^{-1}E = I. \tag{4}$$

Bearing equations (2) and (3) in mind, we see that when a_s $(s = 1, 2, \ldots)$, b_s $(s = 0, 1, \ldots) \in N$, then the convergent C_n derived from the expression (2) may be given 2^n meanings according as to whether at each stage pre or post-multiplication by the inverse of b_s is effected. This paper will concern itself with two cases, that in which premultiplication is consistently used and that in which the contrary is true. Accordingly

$$\text{pre}\left[b_0 + \frac{a_1}{a_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n}\right] \tag{5}$$

is defined by

$$D_0 = b_n, \qquad D_{r+1} = b_{n-r-1} + D_r^{-1} a_{n-r} \qquad (r = 0, 1, \ldots, n-1),$$

$$\text{pre}\left[b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n}\right] = D_n \qquad (n = 0, 1, \ldots), \tag{6}$$

and correspondingly we have

$$D_0 = b_n, \qquad D_{r+1} = b_{n-r-1} + a_{n-r} D_r^{-1} \qquad (r = 0, 1, \ldots, n-1),$$

$$\text{post}\left[b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n}\right] = D_n \qquad (n = 0, 1, \ldots). \tag{7}$$

Notation 1. At this point we have provisionally introduced the notational operators pre[...] and post[...] rather than define separate sets of symbols for the pre and post systems of continued fractions respectively. We shall shortly see, by application of a simple rule, that it is possible to dispense with this device.

Now in the development of the convergence theory of conventional continued fractions, conditions were first derived which were sufficient to ensure that none of the convergents C_n became indeterminate, that is, that the continued fraction took on some meaning, and then further conditions were derived to ensure that the sequence C_n $(n=0, 1, \ldots)$ possessed a limit.

In this paper no convergence theory is given. The first point of the previous paragraph is taken care of by assuming that throughout the manipulations point 5 above always holds. The second is ignored completely. The concepts of a normalisable space and subsequently of distance are not introduced. There is no suggestion of proceeding to a limit. The theorems derived are formal algebraic identities involving a finite number of rational operations. That a convergence theory of the continued fractions treated in this paper exists may be inferred from the results of numerical experiments (an example of which is contained in Appendix II). But we shall first establish a formalism, and proceed perhaps at some later date to establish a convergence theory; and in this respect we are of course simulating the historical development of conventional continued fractions, this time precisely diagnosing the limitations of our achievements.

We conclude this section by remarking that the definitions (6) and (7) have placed in our hands the fundamental result

Theorem 1.1. *If* $a_1, a_2, \ldots \in S$, *then*

$$\text{pre}\left[b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_r}{b_r+}\cdots\right] = \text{post}\left[b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_r}{b_r+}\cdots\right]$$

in the sense that the successive convergents of both continued fractions are equal.

2. Fundamental Formulae

Let us write

$$\text{pre}[C_n = B_n^{-1} A_n] \qquad (n=0, 1, \ldots) \tag{8}$$

and determine pre$[A_n, B_n]$ in such a way as to assist in the computation of preC_n. For this we first remark that if we write in succession

$$\begin{aligned} \text{pre}[A_0 &= b_0, & B_0 &= I, \\ A_1 &= a_1 + b_1 b_0, & B_1 &= b_1, \\ A_2 &= (b_2 b_1 + a_2)\, b_0 + b_2 a_1, & B_2 &= b_2 b_1 + a_2], \end{aligned}$$

then for $n=0, 1, 2$ equation (8) is satisfied. It may easily be verified that

$$\begin{aligned} \text{pre}[A_2 &= b_2 A_1 + a_2 A_0, \\ B_2 &= b_2 B_1 + a_2 B_0], \end{aligned}$$

and this suggests that, in general,

$$\text{pre}[A_n = b_n A_{n-1} + a_n A_{n-2}, \tag{9}$$

$$B_n = b_n B_{n-1} + a_n B_{n-2}]. \tag{10}$$

But $\operatorname{pre} C_{n+1}$ is computed from the expression (5) for $\operatorname{pre} C_n$ by replacing b_n by $(b_n + b_{n+1}^{-1} a_{n+1})$. Inserting this substitution into the right-hand side of equation (9), we obtain

$$\begin{aligned}\operatorname{pre}\big[(b_n + b_{n+1}^{-1} a_{n+1}) A_{n-1} + a_n A_{n-2} \\ = b_{n+1}^{-1}\{b_{n+1}(b_n A_{n-1} + a_n A_{n-2}) + a_{n+1} A_{n-1}\} \\ = b_{n+1}^{-1}\{b_{n+1} A_n + a_{n+1} A_{n-1}\}\big].\end{aligned}$$

The right-hand side of equation (10) leads to a similar expression. Similar considerations apply to the post continued fraction. We are led to

Theorem 2.1. *Successive convergents of the continued fractions*

$$\operatorname{pre}\left[b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_r}{b_r+}\cdots\right] \quad \textit{and} \quad \operatorname{post}\left[b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_r}{b_r+}\cdots\right]$$

may be evaluated by use of the fundamental recursions

$$\begin{aligned}\operatorname{pre}[A_{-1} = I,\ A_0 = b_0,\ A_n = b_n A_{n-1} + a_n A_{n-2}, \\ B_{-1} = O,\ B_0 = I\ ,\ B_n = b_n B_{n-1} + a_n B_{n-2}]\end{aligned} \qquad (n = 1, 2, \ldots)$$

when

$$\operatorname{pre}\left[b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n} = B_n^{-1} A_n\right] \qquad (n = -1, 0, 1, \ldots)$$

and

$$\begin{aligned}\operatorname{post}[A_{-1} = I,\ A_0 = b_0,\ A_n = A_{n-1} b_n + A_{n-2} a_n, \\ B_{-1} = O,\ B_0 = I,\ B_n = B_{n-1} b_n + B_{n-2} a_n]\end{aligned} \qquad (n = 1, 2, \ldots)$$

when

$$\operatorname{post}\left[b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n} = A_n B_n^{-1}\right] \qquad (n = -1, 0, 1, \ldots).$$

The fundamental recursion formulae (9) and (10), besides providing a second definition of a continued fraction, may be used to provide a number of further important formulae.

For example, it may be proved by induction that the successive convergents of the continued fraction

$$\operatorname{pre}\left[\frac{\beta_0}{\alpha_0-}\,\frac{\beta_1\alpha_0}{\alpha_1+\beta_1-}\,\frac{\beta_2\alpha_1}{\alpha_2+\beta_2-}\cdots\frac{\beta_{r-1}\alpha_{r-2}}{\alpha_{r-1}+\beta_{r-1}-}\cdots\right]$$

are given by $B_n^{-1} A_n$, where

$$\begin{aligned}&B_{r+1} = \alpha_r\alpha_{r-1}\ldots\alpha_0, \\ &A_{r+1} = \alpha_r\alpha_{r-1}\ldots\alpha_1\beta_0 + \alpha_r\alpha_{r-1}\ldots\alpha_2\beta_1\beta_0 + \cdots + \beta_r\beta_{r-1}\ldots\beta_0 \quad (r = 0, 1, \ldots).\end{aligned}$$

This result leads to

Theorem 2.2. *The series*

$$\alpha_0^{-1}\beta_0 + \alpha_0^{-1}\alpha_1^{-1}\beta_1\beta_0 + \alpha_0^{-1}\alpha_1^{-1}\alpha_2^{-1}\beta_2\beta_1\beta_0 + \cdots$$

and the continued fraction

$$\operatorname{pre}\left[\frac{\beta_0}{\alpha_0-}\,\frac{\beta_1\alpha_0}{\alpha_1+\beta_1-}\,\frac{\beta_2\alpha_1}{\alpha_2+\beta_2-}\cdots\frac{\beta_{r-1}\alpha_{r-2}}{\alpha_{r-1}+\beta_{r-1}-}\cdots\right]$$

are equivalent [6] *in the sense that the successive partial sums of the former are equal to the successive convergents of the latter. Similarly*

$$\beta_0 \alpha_0^{-1} + \beta_0 \beta_1 \alpha_1^{-1} \alpha_0^{-1} + \beta_0 \beta_1 \beta_2 \alpha_2^{-1} \alpha_1^{-1} \alpha_0^{-1} + \cdots$$

is equivalent to

$$\text{post}\left[\frac{\beta_0}{\alpha_0 -} \; \frac{\alpha_0 \beta_1}{\alpha_1 + \beta_1 -} \; \frac{\alpha_1 \beta_2}{\alpha_2 + \beta_2 -} \cdots \frac{\alpha_{r-2} \beta_{r-1}}{\alpha_{r-1} + \beta_{r-1} -} \cdots\right].$$

Notation 2. The reader will doubtless have noticed that there is a simple rule for converting a result relating to the pre-system of continued fractions into the corresponding result relating to the post-system. It takes the following form: If

$$\text{pre}[x(a, bc, def, \ldots) = y(A, BC, DEF, \ldots)],$$

then

$$\text{post}[x(a, cb, fed, \ldots) = y(A, CB, FED, \ldots)].$$

We may thus, without significant loss, dispense with the exhibition of results relating to the post-system. But if the results derived relate only to the pre-system, there is hardly any point in including the symbols pre[...] at each stage, so that if it is perfectly clear that *from now on the formulae derived relate only to the pre-system of continued fractions,* these symbols may be omitted. Occasionally, by way of emphasis, they will be reintroduced.

To prove the next result, it is convenient to introduce the following

Lemma. *The successive denominators B_n of the continued fraction*

$$b_0 + \frac{a_1}{I -} \; \frac{a_2}{a_2 + I -} \; \frac{a_3}{a_3 + I -} \cdots \frac{a_r}{a_r + I -} \cdots$$

are equal to I.

We are immediately in a position to verify

Theorem 2.3. *The infinite product*

$$\cdots (\gamma_r + I)(\gamma_{r-1} + I) \cdots (\gamma_0 + I)$$

and the continued fraction

$$\gamma_0 + I + \frac{\gamma_1(\gamma_0 + I)}{I -} \; \frac{\gamma_2(\gamma_1^{-1} + I)}{\gamma_2(\gamma_1^{-1} + I) + I -} \cdots \frac{\gamma_r(\gamma_{r-1}^{-1} + I)}{\gamma_r(\gamma_{r-1}^{-1} + I) + I -} \cdots$$

are equivalent [6] *in the sense that the successive partial products of the former are equal to the successive convergents of the latter.*

A further exercise in the use of the fundamental recursions is the derivation of results analogous to the BAUER-MUIR relationships ([5], vol. II, p. 25). Starting from the continued fraction

$$b_0 + \frac{a_1}{b_1 +} \; \frac{a_2}{b_2 +} \cdots \frac{a_n}{b_n}$$

having successive numerators $A_{-1}, A_0, \ldots, A_n$ and denominators $B_{-1}, B_0, \ldots, B_n$, we construct a continued fraction

$$d_0 + \frac{c_1}{d_1 +} \; \frac{c_2}{d_2 +} \cdots \frac{c_n}{d_n +} \; \frac{c_{n+1}}{d_{n+1}}$$

having successive numerators $A_s + r_s A_{s-1}$ $(s = 0, 1, \ldots, n)$; σA_n and denominators $B_s + r_s B_{s-1}$ $(s = 0, 1, \ldots, n)$; σB_n $(r_s, \sigma \in N,\ \sigma \neq O)$.

To start with, we have trivially

$$d_0 = A_0 + r_0 A_{-1} = b_0 + r_0,$$
$$d_1 = B_1 + r_1 B_0 \quad = b_1 + r_1,$$

and since

$$d_1 d_0 + c_1 = A_1 + r_1 A_0,$$

then

$$c_1 = a_1 - (b_1 + r_1) r_0.$$

Subsequently

$$A_\nu + r_\nu A_{\nu-1} = d_\nu (A_{\nu-1} + r_{\nu-1} A_{\nu-2}) + c_\nu (A_{\nu-2} + r_{\nu-2} A_{\nu-3})$$

which in view of

$$A_\nu = b_\nu A_{\nu-1} + a_\nu A_{\nu-2}$$

reduces to

$$(b_\nu + r_\nu - d_\nu) A_{\nu-1} - (d_\nu r_{\nu-1} - a_\nu + c_\nu) A_{\nu-2} + c_\nu r_{\nu-2} A_{\nu-3} = 0.$$

If this is to be equivalent to

$$A_{\nu-1} = b_{\nu-1} A_{\nu-2} + a_{\nu-1} A_{\nu-3},$$

we must have

$$b_{\nu-1} = (b_\nu + r_\nu - d_\nu)^{-1} (d_\nu r_{\nu-1} - a_\nu + c_\nu)$$
$$a_{\nu-1} = (b_\nu + r_\nu - d_\nu)^{-1} c_\nu r_{\nu-2},$$

and these may be solved for d_ν and c_ν.

Finally from the equations

$$\sigma A_n = d_{n+1} (A_n + r_n A_{n-1}) + c_{n+1} (A_{n-1} + r_{n-1} A_{n-2})$$

and

$$A_n = b_n A_{n-1} + a_n A_{n-2}$$

we may derive d_{n+1} and c_{n+1}. The results are summarised in

Theorem 2.4. *If the continued fraction*

$$b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \cdots \frac{a_n}{b_n}$$

has successive numerators A_s $(s = -1, 0, 1, \ldots, n)$ *and denominators* B_s $(s = -1, 0, 1, \ldots, n)$ *and the continued fraction*

$$d_0 + \frac{c_1}{d_1+} \frac{c_2}{d_2+} \cdots \frac{c_n}{d_n+} \frac{c_{n+1}}{d_{n+1}}$$

successive numerators $A_s + r_s A_{s-1}$ $(s = 0, 1, \ldots, n)$, σA_n *and denominators* $B_s + r_s B_{s-1}$ $(s = 0, 1, \ldots, n)$, σB_n, $(r_s, \sigma \in N,\ \sigma \neq 0)$, then

$$d_0 = b_0 + r_0, \qquad d_1 = b_1 + r_1, \qquad c_1 = a_1 - (b_1 + r_1) r_0,$$
$$d_\nu = b_\nu + r_\nu - \{(b_\nu + r_\nu) r_{\nu-1} - a_\nu\} r_{\nu-2} (b_{\nu-1} r_{\nu-2} - a_{\nu-1} + r_{\nu-1} r_{\nu-2})^{-1},$$
$$c_\nu = \{(b_\nu + r_\nu) r_{\nu-1} - a_\nu\} r_{\nu-2} (b_{\nu-1} r_{\nu-2} - a_{\nu-1} + r_{\nu-1} r_{\nu-2})^{-1},$$

and

$$d_{n+1} = \sigma - r_n r_{n-1} (b_n r_{n-1} - a_n)^{-1}, \qquad c_{n+1} = r_n r_{n-1} (b_n r_{n-1} - a_n)^{-1} a_n r_{n-1}^{-1}.$$

A simpler result ([5], vol. II, p. 14), which may easily be verified by recourse to the fundamental recurrence relations, is

Theorem 2.5. *If the successive numerators and denominators of*

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_r}{b_r+}\cdots$$

are $A_0, A_1, \ldots$ *and* $B_0, B_1, \ldots$ *respectively,* $\varrho \in N$, *then the successive numerators and denominators of*

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_k}{b_k-\varrho+}\,\frac{\varrho}{1-}\,\frac{a_{k+1}\,\varrho^{-1}}{b_{k+1}+a_{k+1}\,\varrho^{-1}+}\,\frac{a_{k+2}}{b_{k+2}+}\cdots$$

are

$$A_0, A_1, \ldots, A_{k-1}, A_k - \varrho\, A_{k-1}, A_k, A_{k-1}, \ldots$$

and

$$B_0, B_1, \ldots, B_{k-1}, B_k - \varrho\, B_{k-1}, B_k, B_{k+1}, \ldots.$$

The preceding formulae enable us to insert any quantity into the sequence of convergents merely by reducing it to the quotient of $A_k - \varrho\, B_{k-1}$ and $B_k - \varrho\, B_{k-1}$. The process can quite clearly be repeated at will and without difficulty. The converse problem ([5], Vol. II, p. 10), that of constructing a continued fraction whose successive numerators and denominators are $A_{n_0}, A_{n_1}, A_{n_2}, \ldots$; $B_{n_0}, B_{n_1}, B_{n_2}, \ldots$ respectively, where $n_0, n_1, n_2, \ldots$ is an increasing integer sequence, is not so easily dealt with. We shall consider the construction of the even and odd parts of a continued fraction, *i.e.* the construction of continued fractions whose sequences of numerators and denominators are $A_0, A_2, A_4, \ldots$; $B_0, B_2, B_4, \ldots$ and $A_1, A_3, A_5, \ldots$; $B_1, B_3, B_5, \ldots$ respectively.

The construction of the even part presents no difficulty. Eliminating A_{2n-1} and A_{2n-3} between the three equations

$$A_{2n} = b_{2n} A_{2n-1} + a_{2n} A_{2n-2},$$
$$A_{2n-1} = b_{2n-1} A_{2n-2} + a_{2n-1} A_{2n-3},$$
$$A_{2n-2} = b_{2n-2} A_{2n-3} + a_{2n-2} A_{2n-4},$$

we have

$$A_{2n} = \{a_{2n} + b_{2n}(b_{2n-1} + a_{2n-1} b_{2n-2}^{-1})\}\, A_{2n-2} - b_{2n} a_{2n-1} b_{2n-2}^{-1} a_{2n-2} A_{2n-4}.$$

We may thus formulate

Theorem 2.6. *If the successive numerators and denominators of*

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_r}{b_r+}\cdots$$

are A_s, B_s $(s=0, 1, \ldots)$, *respectively, then those of*

$$b_0 + \frac{b_2 a_1}{b_2 b_1 + a_2 -}\,\frac{b_4 a_3 b_2^{-1} a_2}{a_4 + b_4(b_3 + a_3 b_2^{-1}) -}\cdots\frac{b_{2r+2} a_{2r+1} b_{2r}^{-1} a_{2r}}{a_{2r+2} + b_{2r+2}(b_{2r+1} + a_{2r+1} b_{2r}^{-1}) -}\cdots$$

are

$$A_{2s}, B_{2s} \quad (s=0, 1, \ldots).$$

Proceeding to the odd part, we obtain

$$A_{2n+1} = \{a_{2n+1} + b_{2n+1}(b_{2n} + a_{2n} b_{2n-1}^{-1})\}\, A_{2n-1} - b_{2n+1} a_{2n} b_{2n-1}^{-1} a_{2n-1} A_{2n-3}.$$

Now we observe that if the fundamental recursion formulae are to correspond to the original definition of a continued fraction, then $B_0 = I$. But in general B_1, which in the odd part of the continued fraction plays the role of B_0, is not equal to I.

We may overcome this difficulty by dividing the first numerator and denominator of the odd part throughout by b_1. This is valid only if either b_1, or b_0 and a_1, $\in S$. We obtain

Theorem 2.7. *If successive numerators and denominators of*

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_r}{b_r +} \cdots$$

are A_s, B_s $(s = 0, 1, \ldots)$, *respectively, and either* $b_1 \in S$ *or* $b_0, a_1 \in S$, *then those of*

$$b_0 + b_1^{-1} a_1 - \frac{b_3 a_2 b_1^{-2} a_1}{a_3 + b_3 (b_2 + a_2 b_1^{-1}) -} \frac{b_5 a_4 b_3^{-1} a_3}{a_5 + b_5 (b_4 + a_4 b_3^{-1}) -} \cdots \frac{b_{2r+1} a_{2r} b_{2r-1}^{-1} a_{2r-1}}{a_{2r+1} + b_{2r+1} (b_{2r} + a_{2r} b_{2r-1}^{-1}) -} \cdots$$

are $b_1^{-1} A_{2s+1}, b_1^{-} \; B_{2s+1}$ $(s = 0, 1, \ldots)$, *respectively.*

The restrictions which had to be imposed when obtaining the last result repeat themselves in

Theorem 2.8. *If* A_s, B_s $(s = 0, 1, \ldots)$ *are successive numerators and denominators of*

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_r}{b_r +} \cdots$$

and $\gamma_1, \gamma_2, \ldots \in S$, *then those of*

$$b_0 + \frac{\gamma_1 a_1}{\gamma_1 b_1 +} \frac{\gamma_2 \gamma_1 a_2}{\gamma_2 b_2 +} \cdots \frac{\gamma_r \gamma_{r-1} a_r}{\gamma_r b_r +} \cdots$$

are $A_0, \gamma_s \gamma_{s-1} \cdots \gamma_2 \gamma_1 A_s$; $B_0, \gamma_s \gamma_{s-1} \cdots \gamma_2 \gamma_1 B_s$ $(s = 1, 2, \ldots)$ *respectively* ([*4*], p. 19).

Let us conclude this section by deriving analogues of the Euler-Minding relations ([*5*], vol. I, p. 5) which provide a third definition of a continued fraction. Eliminating b_{n+1} from the two equations

$$A_{n+1} = b_{n+1} A_n + a_{n+1} A_{n-1},$$
$$B_{n+1} = b_{n+1} B_n + a_{n+1} B_{n-1},$$

we have

$$A_{n+1} A_n^{-1} - B_{n+1} B_n^{-1} = a_{n+1} (A_{n-1} A_n^{-1} - B_{n-1} B_n^{-1}).$$

Or, writing $C_s = B_s^{-1} A_s$,

$$C_{n+1} - C_n = - B_{n+1}^{-1} a_{n+1} B_{n-1} (C_n - C_{n-1})$$
$$= (-1)^n B_{n+1}^{-1} a_{n+1} B_{n-1} B_n^{-1} a_n B_{n-2} B_{n-1}^{-1} a_{n-1} B_{n-3} \cdots B_1^{-1} a_2 b_1^{-1} a_1.$$

We thus derive

Theorem 2.9. *The successive convergents of the continued fraction*

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots \frac{a_r}{b_r +} \cdots$$

are equal to the partial sums of the series

$$b_0 + \sum_{p=0}^{\infty} (-1)^p (B_{p+1}^{-1} a_{p+1} B_{p-1}) (B_p^{-1} a_p B_{p-2}) (B_{p-1}^{-1} a_{p-1} B_{p-3}) \ldots (B_2^{-1} a_2) (b_1^{-1} a_1) .$$

Finally, we use Theorems 2.2 and 2.9 to derive

Theorem 2.10. *The successive convergents of*

$$b_0 + \frac{a_1}{b_1+} \frac{a_2}{b_2+} \ldots \frac{a_r}{b_r+} \ldots$$

and

$$b_0 + \frac{b_1^{-1} a_1}{I-} \frac{B_2^{-1} a_2}{I+B_2^{-1} a_2-} \ldots \frac{B_n^{-1} a_n B_{n-2}}{I+B_n^{-1} a_n B_{n-2}-} \frac{B_{n+1}^{-1} a_{n+1} B_{n-1}}{I+B_{n+1}^{-1} a_{n+1} B_{n-1}-} \ldots$$

are equal.

3. Pre- and Post-Orthogonal Polynomials

The most important continued fractions which arise in practice are those which are associated with power series. We shall proceed to derive continued fractions which may be associated with the power series $\sum_{s=0}^{\infty} c_s z^{-s-1}$, where $c_s \in N$ and $z \in S$, but before doing so it is necessary to invoke the theory of orthogonal polynomials. (The present treatment is adapted from [7].)

Suppose that we have a sequence c_s $(s=0, 1, \ldots) \in N$; then we introduce a process $\operatorname{pre} P\{\ldots\}$ such that

$$\operatorname{pre} P\{A t^s\} = A c_s \qquad (s=0, 1, \ldots), \qquad A \in N \tag{11}$$

where t, which is a dummy variable, will be assumed scalar. The process $\operatorname{pre} P\{\ldots\}$ operates upon quantities inside the braces. These quantities consist of elements A of N, and scalar quantities including t. These scalar quantities undergo normal arithmetic inside the braces (such as addition, division, formal expansion in power series, *etc.*), but after $\operatorname{pre} P$ has been effected, the elements of N premultiply the various members of the sequence c_s $(s=0, 1, \ldots)$. Construct for some m the sequence of polynomials

$$p_n^{(m)}(t) = \sum_{s=0}^{\infty} k_{n,s}^{(m)} t^s \tag{12}$$

(again the notational operator $\operatorname{pre}\{\ldots\}$ has been omitted) from the condition

$$\operatorname{pre} P\{t^{m+s} p_n^{(m)}(t)\} = \begin{matrix} O & (s=0, 1, \ldots, n-1), \\ w_n^{(m)} & (s=n), \end{matrix} \tag{13}$$

where $w_n^{(m)}$ is chosen so that in (12)

$$k_{n,n}^{(m)} = I, \tag{14}$$

but otherwise, in general, $k_{n,s}^{(m)} \in N$. (It will be seen in Appendix I that there is at least in principle no difficulty in doing this.)

Having constructed this sequence, determine the further sequence $o_n^{(m)}(z)$ $(n=0, 1, \ldots)$ from the relation

$$o_n^{(m)}(z) = \operatorname{pre} P\{(z-t)^{-1} t^m (p_n^{(m)}(z) - p_n^{(m)}(t))\}. \tag{15}$$

(The $o_n^{(m)}(z)$ are of course polynomials, since when $z, t \in S$ then $z^s - t^s$ is divisible by $z - t$.)

Construct the third sequence of functions $r_n^{(m)}(z)$ from the relation

$$r_n^{(m)}(z) = \text{pre}\, P\{(z-t)^{-1}(p_n^{(m)}(z) - (z^{-1}t)^m p_n^{(m)}(t))\}. \tag{16}$$

The purpose of constructing these sequences is revealed by noting that a trivial consequence of (11) is

$$\text{pre}\, P\{(z-t)^{-1}\} \sim \sum_{s=0}^{\infty} c_s z^{-s-1}.$$

The sign $\sim$ of formal equivalence in this paper has nothing to do with the sum, if it exists in any sense, of the series on the right-hand side of this equation. It merely means that if we expand the left-hand side of this equation formally in descending powers of z, then we obtain the right-hand side. There then follows

$$\begin{aligned} p_n^{(m)}(z)^{-1} o_n^{(m)}(z) &= p_n^{(m)}(z)^{-1} \text{pre}\, P\{(z-t)^{-1} t^m (p_n^{(m)}(z) - p_n^{(m)}(t))\} \\ &= \text{pre}\, P\{(z-t)^{-1} t^m\} - p_n^{(m)}(z)^{-1} \text{pre}\, P\{(z-t)^{-1} t^m p_n^{(m)}(t)\} \\ &\sim \sum_{s=0}^{\infty} c_{m+s} z^{-s-1} - O(z^{-2n-1}) \end{aligned}$$

on account of (13). $O(z^{-2n-1})$ is taken to mean some function of z which when expanded as an inverse power series in z commences with a term in z^{-2n-1}. If we were to introduce the concept of a normalisable space, this latter condition could be given a more conventional asymptotic formulation.

Similarly there follows

$$p_n^{(m}(z)^{-1} r_n^{(m)}(z) = \sum_{s=0}^{\infty} c_s z^{-s-1} + O(z^{-m-2n-1}).$$

Thus we see that $p_n^{(m)}(z)^{-1} o_n^{(m)}(z)$ and $p_n^{(m)}(z)^{-1} r_n^{(m)}(z)$ are rational functions which when expanded in inverse powerse of z agree with the series $\sum_{s=0}^{\infty} c_{m+s} z^{-s-1}$ and $\sum_{s=0}^{\infty} c_s z^{-s-1}$ to $2n$ and $m+2n$ terms respectively.

It is remarked at this stage that the reason for labelling this section with the title given is that the system of polynomials $p_n^{(m)}(z)$ is orthogonal in the sense that

$$\text{pre}\, P\{p_n^{(m)}(t)\, p_r^{(m)}(t)\} = \begin{matrix} O & (r = 0, 1, \ldots, n-1) \\ w_n^{(m)} & (r = n). \end{matrix}$$

This is easily verified by appeal to equations (12) and (13). A consequence of this reveals the connection between orthogonal polynomials and continued fractions, for a three-term recursion prevails among the sequences $p_n^{(m)}(z)$, $o_n^{(m)}(z)$, $r_n^{(m)}(z)$. For expanding $p_{n+1}^{(m)}(z) - z p_n^{(m)}(z)$, a polynomial of degree n, in the form

$$p_{n+1}^{(m)}(z) - z p_n^{(m)}(z) = \sum_{s=0}^{n} b_s^{(m)} p_s^{(m)}(z) \qquad (b_s^{(m)} \in N)$$

we find from (13) that

$$b_s^{(m)} = O \qquad (s = 0, 1, \ldots, n-1).$$

There thus exists a recursion which may be written

$$p_{n+1}^{(m)}(z) = (z - \alpha_n^{(m)})\, p_n^{(m)}(z) - \beta_{n-1}^{(m)}\, p_{n-1}^{(m)}(z) \qquad (n = 1, 2, \ldots)$$

where, by inspection of (13),

$$p_0^{(m)}(z) = I, \qquad p_1^{(m)}(z) = z - c_{m+1}\, c_m^{-1}.$$

As a trivial consequence of the definitions (14) and (15) it follows that

$$\begin{aligned} o_{n+1}^{(m)}(z) &= (z - \alpha_n^{(m)})\, o_n^{(m)}(z) - \beta_{n-1}^{(m)}\, o_{n-1}^{(m)}(z) \qquad (n = 1, 2, \ldots) \\ o_0^{(m)}(z) &= 0, \qquad o_1^{(m)}(z) = c_m, \end{aligned} \tag{17}$$

and that

$$\begin{aligned} r_{n+1}^{(m)}(z) &= (z - \alpha_n^{(m)})\, r_n^{(m)}(z) - \beta_{n-1}^{(m)}\, r_{n-1}^{(m)}(z) \qquad (n = 1, 2, \ldots) \\ r_0^{(m)}(z) &= \sum_{s=0}^{\infty} c_s z^{-s-1}, \qquad r_1^{(m)}(z) = p_1^{(m)}(z)\, r_0^{(m)}(z) + c_m z^{-m}. \end{aligned} \tag{18}$$

Thus finally we are led to

Theorem 3.1. *If, given a sequence c_s $(s = 0, 1, \ldots)$ $(\in N)$, a process* pre $P\{\ldots\}$ *is defined by*

$$\text{pre}\, P\{A\, t^s\} = A\, c_s,$$

the polynomial sequence

$$p_n^{(m)}(z) = \sum_{s=0}^{n} k_{n,s}^{(m)} z^s \qquad (k_{n,s}^{(m)} \in N)$$

is determined from

$$\text{pre}\, P\{t^{m+s}\, p_n^{(m)}(t)\} = \begin{matrix} O & (s = 0, 1, \ldots, n-1) \\ w_n^{(m)} & (s = n) \end{matrix} \qquad k_{n,n}^{(m)} = I,$$

and coefficients in the recursion system

$$p_{n+1}^{(m)}(z) = (z - \alpha_n^{(m)})\, p_n^{(m)}(z) - \beta_{n-1}^{(m)}\, p_{n-1}^{(m)}(z) \tag{19}$$

computed, then the series expansion in inverse powers of z of the n^{th} convergent of

$$\frac{c_m}{z - \alpha_0^{(m)} -}\; \frac{\beta_0^{(m)}}{z - \alpha_1^{(m)} -} \cdots \frac{\beta_{r-1}^{(m)}}{z - \alpha_r^{(m)} -} \cdots \tag{20}$$

agrees with the series $\sum_{s=0}^{\infty} c_{m+s} z^{-s-1}$ as far as the term $c_{m+2n-1} z^{-2n}$, and a similar expansion of the n^{th} convergent of

$$\sum_{s=0}^{m-1} c_s z^{-s-1} + \frac{z^{-m} c_m}{z - \alpha_0^{(m)} -}\; \frac{\beta_0^{(m)}}{z - \alpha_1^{(m)} -} \cdots \frac{\beta_{r-1}^{(m)}}{z - \alpha_r^{(m)} -} \cdots \tag{21}$$

agrees with the series $\sum_{s=0}^{\infty} c_s z^{-s-1}$ as far as the term $c_{m+2n-1} z^{-m-2n}$.

For the sake of completeness it is mentioned at this point that equations (13) and (14) do not provide the most economical means for computing the coefficients in the polynomials $p_n^{(m)}(z)$ since they involve at each stage the solution of a set of linear equations.

Using (17), in conjunction with (14), there follows

$$\begin{aligned} \operatorname{pre} P\{t^{m+r}\, p_r^{(m)}(t)\} &= \beta_{r-1}^{(m)} \operatorname{pre} P\{t^{m+r-1}\, p_{r-1}^{(m)}(t)\} \\ &= \beta_{r-1}^{(m)} \beta_{r-2}^{(m)} \cdots \beta_0^{(m)} c_m . \end{aligned} \tag{22}$$

Again

$$\begin{aligned} \operatorname{pre} P\{t^{m+r+1}\, p_r^{(m)}(t)\} &= \alpha_r^{(m)} \operatorname{pre} P\{t^{m+r}\, p_r^{(m)}(t)\} + \beta_{r-1}^{(m)} \operatorname{pre} P\{t^{m+r}\, p_{r-1}^{(m)}(t)\} \\ &= \alpha_r^{(m)} \beta_{r-1}^{(m)} \cdots \beta_0^{(m)} c_m + \beta_{r-1}^{(m)} \alpha_{r-1}^{(m)} \beta_{r-2}^{(m)} \cdots \beta_0^{(m)} c_m \\ &\quad + \beta_{r-1}^{(m)} \beta_{r-2}^{(m)} \alpha_{r-2}^{(m)} \beta_{r-3}^{(m)} \cdots \beta_0^{(m)} c_m + \cdots + \beta_{r-1}^{(m)} \beta_{r-2}^{(m)} \cdots \beta_1^{(m)} \alpha_0^{(m)} c_m . \end{aligned} \tag{23}$$

If $p_r^{(m)}(z)$ is known, $\beta_{r-1}^{(m)}$ and $\alpha_r^{(m)}$ are computed recursively from (22) and (23) and $p_{r+1}^{(m)}(z)$ from (19).

4. The q—d Algorithm

For any one value of m, the systems of equations (22), (23), (19), (17) and (18) offer the most economical processes for computing the continued fractions (20) and (21).

However, for several consecutive values of m it is more efficient to proceed as follows ([8], p. 13): In (20), write

$$\alpha_r^{(m)} = q_{r+1}^{(m)} + e_r^{(m)}, \qquad \beta_r^{(m)} = e_{r+1}^{(m)} q_{r+1}^{(m)} \qquad (r = 0, 1, \ldots), \qquad e_0^{(m)} = O,$$

and denote by $F_m(z)$ the power series $\sum_{s=0}^{\infty} c_{m+s} z^{-s-1}$. Then

$$\frac{c_m}{z - q_1^{(m)} -}\; \frac{e_1^{(m)} q_1^{(m)}}{z - q_2^{(m)} - e_1^{(m)} -} \cdots \frac{e_r^{(m)} q_r^{(m)}}{z - q_{r+1}^{(m)} - e_r^{(m)} -} \cdots \sim F_m(z) \tag{24}$$

is plainly the even part of

$$\frac{c_m}{z -}\; \frac{q_1^{(m)}}{I -}\; \frac{e_1^{(m)}}{z -} \cdots \frac{q_r^{(m)}}{I -}\; \frac{e_r^{(m)}}{z -} \cdots,$$

the odd part of which is (z is of course scalar)

$$c_m z^{-1} + \frac{q_1^{(m)} c_m z^{-1}}{z - e_1^{(m)} - q_1^{(m)} -}\; \frac{q_2^{(m)} e_1^{(m)}}{z - e_2^{(m)} - q_2^{(m)} -} \cdots \frac{q_{r+1}^{(m)} e_r^{(m)}}{z - e_{r+1}^{(m)} - q_{r+1}^{(m)} -} \cdots.$$

However

$$F_m(z) = c_m z^{-1} + z^{-1} F_{m+1}(z),$$

and thus it follows that

$$\frac{q_1^{(m)} c_m}{z - e_1^{(m)} - q_1^{(m)} -}\; \frac{q_2^{(m)} e_1^{(m)}}{z - e_2^{(m)} - q_2^{(m)} -} \cdots \frac{q_{r+1}^{(m)} e_r^{(m)}}{z - e_{r+1}^{(m)} - q_{r+1}^{(m)} -} \cdots \sim F_{m+1}(z).$$

Comparing (25) and (24) with superscripts advanced by unity, we have

Theorem 4.1. *The coefficients in the continued fraction*

$$\operatorname{pre}\left[\frac{c_m}{z - \alpha_0^{(m)} -}\; \frac{\beta_0^{(m)}}{z - \alpha_1^{(m)} -} \cdots \frac{\beta_{r-1}^{(m)}}{z - \alpha_r^{(m)} -} \cdots\right] \sim F_m(z)$$

are given by

$$\operatorname{pre}\left[\beta_r^{(m)} = e_{r+1}^{(m)} q_{r+1}^{(m)},\; \alpha_r^{(m)} = q_{r+1}^{(m)} + e_r^{(m)} \qquad (r = 1, 2, \ldots)\right]$$

where

$$\mathrm{pre}\,[e_r^{(m+1)} q_r^{(m+1)} = q_{r+1}^{(m)} e_r^{(m)}, \tag{25}$$

$$q_r^{(m+1)} + e_{r-1}^{(m+1)} = e_r^{(m)} + q_r^{(m)}]\,. \tag{26}$$

These relations (the $q-d$ algorithm relationships) are used to construct the sequences $e_r^{(m)}$, $q_{r+1}^{(m)}$ by means of the formulae

$$\mathrm{pre}\,[e_r^{(m)} = q_r^{(m+1)} + e_{r-1}^{(m+1)} - q_r^{(m)}, \tag{27}$$

$$q_{r+1}^{(m)} = e_r^{(m+1)} q_r^{(m+1)} e_r^{(m)-1}] \tag{28}$$

from the initial sequences

$$\mathrm{pre}\,[e_0^{(m)} = O, \qquad q_1^{(m)} = c_{m+1}\, c_m^{-1} \qquad (m = 0, 1, \ldots)]\,. \tag{29}$$

An immediate consequence of the $q-d$ algorithm relations is that the polynomials $p_n^{(m)}(z)$ satisfy further recursion systems themselves ([8], p. 11); indeed in particular

$$p_n^{(m)}(z) = z\, p_{n-1}^{(m+1)}(z) - q_n^{(m)}\, p_{n-1}^{(m)}(z)\,, \tag{30}$$

$$p_{n+1}^{(m)}(z) = p_n^{(m)}(z) - e_n^{(m)}\, p_{n-1}^{(m+1)}(z)\,. \tag{31}$$

Assume that equations (30) and (31) are true with $n-1$ in place of n; then use of (19), (25) and (26) leads to

$$\begin{aligned} & z\, p_{n-1}^{(m+1)}(z) - p_n^{(m)}(z) - q_n^{(m)}\, p_{n-1}^{(m)}(z) \\ & = z\left\{p_{n-1}^{(m+1)}(z) - p_{n-1}^{(m)}(z) + e_{n-1}^{(m)}\, p_{n-2}^{(m+1)}(z)\right\} - e_{n-1}^{(m)}\left\{z\, p_{n-2}^{(m+1)}(z) + q_{n-1}^{(m)}\, p_{n-2}^{(m)}(z) + p_{n-1}^{(m)}(z)\right\}, \end{aligned}$$

and equation (30) is thus true for n. Similarly we obtain

$$\begin{aligned} & p_n^{(m+1)}(z) - p_n^{(m)}(z) + e_n^{(m)}\, p_{n-1}^{(m+1)}(z) \\ & \quad = z\, p_{n-1}^{(m+1)}(z) - p_n^{(m)}(z) - q_n^{(m)}\, p_{n-1}^{(m)}(z) - q_n^{(m)}\left\{p_{n-1}^{(m+1)}(z) - e_{n-1}^{(m)}\, p_{n-2}^{(m+1)}(z) - p_{n-1}^{(m)}(z)\right\}, \end{aligned}$$

and thus equation (31) is true for n. But equations (30) and (31) are manifestly true when $n=1$; they are thus generally true.

It is again a trivial consequence of the definitions (15) and (16) that

$$z\, o_n^{(m)}(z) - o_n^{(m+1)}(z) - e_n^{(m)}\, o_{n-1}^{(m+1)}(z) = c_m\, p_n^{(m)}(z)\,,$$

$$o_n^{(m)}(z) + q_n^{(m)}\, o_{n-1}^{(m)}(z) - o_{n-1}^{(m+1)}(z) = c_m\, p_{n-1}^{(m+1)}(z)\,,$$

$$r_n^{(m+1)}(z) = r_n^{(m)}(z) - e_n^{(m)}\, r_{n-1}^{(m+1)}(z)\,, \tag{32}$$

$$r_n^{(m)}(z) = z\, r_{n-1}^{(m+1)}(z) - q_n^{(m)}\, r_{n-1}^{(m)}(z)\,. \tag{33}$$

Inspection of (30), (31), (32) and (33) reveals that

$$\begin{aligned} \mathrm{pre}\,[& r_0^{(m)}(z),\ p_1^{(m)}(z)^{-1} r_1^{(m)}(z),\ p_1^{(m+1)}(z)^{-1} r_1^{(m+1)}(z), \ldots, \\ & p_s^{(m)}(z)^{-1} r_s^{(m)}(z),\ p_s^{(m+1)}(z)^{-1} r_s^{(m+1)}(z), \ldots] \end{aligned}$$

are successive convergents of

$$\mathrm{pre}\left[\sum_{s=0}^{m-1} c_s z^{-s-1} + \frac{c_m z^{-m}}{z-}\,\frac{q_1^{(m)}}{I-}\,\frac{e_1^{(m)}}{z-} \cdots \frac{q_r^{(m)}}{I-}\,\frac{e_r^{(m)}}{z-} \cdots\right].$$

It is quite clear that the system of the equations (25), (26), (30), (31), (32) and (33) may be manipulated to produce further recursion systems, and a little reflection reveals that continued fractions may be constructed which have as successive convergents pre $[p_n^{(m)}(z)^{-1} r_n^{(m)}(z)]$ not necessarily following the order m, n; $m+1, n$; $m+1, n+1$; ..., but these will not be considered. Instead we produce some fundamental transformations.

The Reciprocal of a Corresponding Continued Fraction. Suppose that we are given the expansion

$$\beta(z) \sim s + \frac{c}{z-} \frac{q_1}{I-} \frac{e_1}{z-} \cdots \frac{q_r}{I-} \frac{e_r}{z-} \cdots \tag{34}$$

and that we wish to produce a similar continued fraction for $\{\beta(z)\}^{-1}$ ([5], vol. II, p. 136). We observe that if successive numerators and denominators of (34) are $A_0, A_1, \ldots$; $B_0, B_1, \ldots$, then those of

$$\frac{I}{s+} \frac{c}{z-} \frac{q_1}{I-} \frac{e_1}{z-} \cdots \frac{q_r}{I-} \frac{e_r}{z-} \cdots \tag{35}$$

are $O, B_0, B_1, \ldots$; $I, A_0, A_1, \ldots$. The odd part of (35) is, however,

$$s^{-1} + \frac{c\,s^{-1}}{z-c-q_1-} \frac{e_1 q_1}{z-e_1-q_2-} \cdots \frac{e_r q_r}{z-e_r-q_{r+1}-} \cdots.$$

But this is plainly the even part of a continued fraction having the same form as (34), and we have

Theorem 4.2 *If*

$$\beta(z) \sim s + \frac{c}{z-} \frac{q_1}{I-} \frac{e_1}{z-} \cdots \frac{q_r}{I-} \frac{e_r}{z-} \cdots$$

and

$$\{\beta(z)\}^{-1} \sim s' + \frac{c'}{z-} \frac{q'_1}{I-} \frac{e'_1}{z-} \cdots \frac{q'_r}{I-} \frac{e'_r}{z-} \cdots,$$

then

$$s' = s^{-1},$$

$$s c' = c s^{-1},$$

$$q'_1 = c + q_1,$$

and thereafter

$$e'_r q'_r = e_r q_r, \qquad q'_{r+1} + e'_r = q_{r+1} + e_r \qquad (r = 1, 2, \ldots).$$

Translation of the Origin. Once both the artifice of taking the even part of a continued fraction and reexpanding in terms of another variable has been comprehended, and due note has been taken of Theorem 3.1, then it is quite easy ([5], vol. II, p. 141) to construct the proof of

Theorem 4.3 *If*

$$F(z) \sim \frac{c}{z-} \frac{q_1}{I-} \frac{e_1}{z-} \cdots \frac{q_r}{I-} \frac{e_r}{z-} \cdots$$

$$z = \xi + \lambda,$$

then

$$F(z) \sim \frac{c}{\xi-} \frac{q'_1}{I-} \frac{e'_1}{\xi-} \cdots \frac{q'_r}{I-} \frac{e'_r}{\xi-} \cdots$$

where

$$q'_1 = q_1 + \lambda,$$

$$e'_r q'_r = e_r q_r, \qquad q'_{r+1} + e'_r = q_{r+1} + e_r + \lambda \qquad (r = 1, 2, \ldots).$$

Addition of Continued Fractions. Suppose that we have an expansion such as

$$F(z) \sim \frac{c}{z-}\,\frac{q_1}{I-}\,\frac{e_1}{z-}\cdots\frac{q_r}{I-}\,\frac{e_r}{z-}\cdots$$

and we wish to obtain a similar expansion for $sz^{-1}+F(z)$ ([5], vol. II, p. 139). This is easily accomplished; we take the odd part of the expansion for $F(z)$, add sz^{-1} to the first term, and reexpand to obtain

Theorem 4.4. *If*

$$F(z) \sim \frac{c}{z-}\,\frac{q_1}{I-}\,\frac{e_1}{z-}\cdots\frac{q_r}{I-}\,\frac{e_r}{z-}\cdots$$

and

$$F(z)+sz^{-1} \sim \frac{c'}{z-}\,\frac{q'_1}{I-}\,\frac{e'_1}{z-}\cdots\frac{q'_r}{I-}\,\frac{e'_r}{z-}\cdots,$$

then

$$c'=c+s, \qquad q'_1c'=q_1c$$

and

$$e'_r+q'_r=e_r+q_r, \qquad q'_{r+1}e'_r=q_{r+1}e_r \qquad (r=1,2,\ldots).$$

As a comprehensive exercise in these manipulations we adapt an addition theorem of RUTISHAUSER ([8], p. 23). We are concerned to express

$$\frac{c}{z-\lambda}+\frac{s}{z-}\,\frac{q_1}{I-}\,\frac{e_1}{z-}\cdots\frac{e_{n-2}}{z-}\,\frac{q_{n-1}}{I}, \tag{36}$$

where $\lambda \in S$, as a single continued fraction.

First we transform (36) into a continued fraction in $\xi = z-\lambda$ as in Theorem 4.3. (36) becomes

$$c\xi^{-1}+\frac{s}{\xi-}\,\frac{q'_1}{I-}\,\frac{e'_1}{\xi-}\cdots\frac{e'_{n-2}}{\xi-}\,\frac{q'_{n-1}}{I} \tag{37}$$

where

$$q_1=q'_1-\lambda, \qquad e'_rq'_r=e_rq_r, \qquad q'_{r+1}+e'_r=q_{r+1}+e_r-\lambda \qquad (r=1,2,\ldots,n-2).$$

As in Theorem 4.4, the odd part of (37) is

$$\xi^{-1}\left\{c+s+\frac{sq'_1}{\xi-q'_1-e'_1-}\,\frac{q'_2e'_1}{\xi-q'_2-e'_2-}\cdots\frac{q'_{n-1}e'_{n-1}}{\xi-q_{n-1}}\right\},$$

and expanding this again as a single continued fraction, we obtain

$$\frac{s^*}{\xi-}\,\frac{q^*_1}{I-}\,\frac{e^*_1}{\xi-}\cdots\frac{e^*_{n-1}}{\xi} \tag{38}$$

where

$$s^*=c+s, \qquad q_1s^*=q'_1s,$$

$$q^*_{r+1}e^*_r=q'_{r+1}e'_r, \qquad q^*_r+e^*_r=q'_r+e'_r \qquad (r=1,2,\ldots,n-2),$$

$$q^*_{n-1}+e^*_{n-1}=q'_{n-1}.$$

We now expand (38) as a continued fraction in z, taking the even part, writing $\xi=z-\lambda$, and reexpanding; we obtain

$$\frac{s^*}{z-}\,\frac{q''_1}{I-}\,\frac{e''_1}{z-}\cdots\frac{e''_{n-1}}{z-}\,\frac{q''_n}{I}$$

where

$$q_1'' = q_1^* + \lambda,$$
$$e_r'' q_r'' = e_r^* q_r^*, \qquad q_{r+1}'' + e_r'' = q_{r+1}^* + e_r^* + \lambda \qquad (r = 1, 2, \ldots, n-2),$$
$$q_n'' + e_{n-1}'' = e_{n-1}^*.$$

The above result, apart from being of interest in itself, has a most important consequence. Before stating it we interpolate a

Lemma. *If* $F(z) = c(z-\chi)^{-n}$, $\chi \in S$, *then, in the notation of the* $q-d$ *scheme,* $e_n^{(m)} = O$.

For this series $c_m = \binom{-n}{m} \chi^m$, and it is easily verified that

$$q_r^{(m)} = -\frac{(n+r+m-1)(r+m-1)}{(2r+m-2)(2r+m-1)}\chi,$$
$$e_r^{(m)} = \frac{-r(r-n)\chi}{(2r+m-1)(2r+m)} \qquad (m = 0, 1, \ldots; r = 1, 2, \ldots, n).$$

Theorem 4.5. *If* $\sum_{s=0}^{\infty} c_s z^{-s-1}$ *is the formal power series expansion of the rational function*

$$F(z) = \left(\sum_{i=0}^{n} b_i z^i\right)^{-1} \left(\sum_{i=0}^{n-1} a_i z^i\right) \tag{39}$$

where $b_i \in S$ $(i = 0, 1, \ldots, n)$, *then in the notation of the* $q-d$ *scheme*

$$e_n^{(m)} = O \qquad (m = 0, 1, \ldots).$$

Suppose that the above statement is true for $n-1$. We then add the function $\frac{c}{z-\lambda_r}$ to $F(z)$ and obtain a similar rational fraction with numerator and denominator of one degree higher. Performing, for some m, the continued fraction addition

$$\sum_{s=0}^{m-1} c_s z^{-s-1} + \sum_{s=0}^{m-1} c\,\lambda^s z^{-s-1} + \frac{z^{-m}\lambda^m c}{z-c} + \frac{z^{-m} c_m}{z-} + \frac{q_1^{(m)}}{I-} \cdots \frac{e_{n-2}^{(m)}}{z-} \frac{q_{n-1}^{(m)}}{I},$$

we obtain a continued fraction of the form

$$\sum_{s=0}^{m-1} c_s'' z^{-s-1} + \frac{z^{-m} c_m''}{z-} \frac{q_1^{(m)''}}{I-} \frac{e_1^{(m)''}}{z-} \cdots \frac{e_{n-1}^{(m)''}}{z-} \frac{q_n^{(m)''}}{I}$$

i.e. in which $e_n^{(m)''} = 0$. This process we repeat for all values of m, thus if the above statement is true for $n-1$, it is true for n. But it is manifestly true for $n = 1$, and thus the theorem is generally true. The only difficulty occurs when the denominator of (39) has multiple roots, but this is taken care of by the preceding lemma.

A consequence of this result and equations (31) and (32) is

Theorem 4.6. *If* $\sum_{s=0}^{\infty} c_s z^{-s-1}$ *is the formal power series expansion of* (39), *then the polynomials* $p_n^{(m)}(z)$ *and functions* $r_n^{(m)}(z)$ *(the latter are in the event polynomials) are constant for all* m. *They are in fact the denominator and numerator of* (39) *normalised so that* $b_n = I$.

Theorem 4.5 will again be brought into play in the next section. We terminate this section by giving an invariant property of the $q-d$ algorithm.

Theorem 4.7. *If* $\sum_{s=0}^{\infty} c_s z^{-s-1}$ *is the formal power series expansion of* (36) *and quantities* $e_r^{(m)}$, $q_r^{(m)}$ *are constructed by means of relationships* (27) *and* (28) *from the initial conditions* (29), *then the sum*

$$\sum_{s=1}^{n} q_s^{(m)} + \sum_{s=1}^{n-1} e_s^{(m)}$$

is invariant with respect to m.

5. The g, π, η and ϵ-Algorithms

With Theorem 4.5 in hand we are in a position to prove a fundamental result in the theory of the ε-algorithm, but rather than do this immediately we shall proceed by a more indirect but nevertheless more instructive and in the final analysis more elegant route.

Using Theorem 2.2, we note that the continued fraction

$$\frac{c_m}{z-}\ \frac{q_1^{(m)}}{I-}\ \frac{e_1^{(m)}}{z-} \cdots \frac{q_r^{(m)}}{I-}\ \frac{e_r^{(m)}}{z-} \cdots \tag{40}$$

and the series

$$\begin{aligned}(c_m z^{-1}) &+ p_1^{(m)}(z)^{-1} q_1^{(m)} (c_m z^{-1}) + \left(p_1^{(m+1)}(z)^{-1} e_1^{(m)} p_0^{(m+1)}(z)\right)\left(p_1^{(m)}(z)^{-1} q_1^{(m)}\right)(c_m z^{-1}) + \\ &\cdots + \left(p_n^{(m)}(z)^{-1} q_n^{(m)} p_{n-1}^{(m)}(z)\right) \cdots \left(p_2^{(m)}(z)^{-1} q_2^{(m)} p_1^{(m)}(z)\right) \times \\ &\times \left(p_1^{(m+1)}(z)^{-1} e_1^{(m)} p_0^{(m+1)}(z)\right)\left(p_1^{(m)}(z)^{-1} q_1^{(m)}\right)(c_m z^{-1}) + \\ &+ \left(p_n^{(m+1)}(z)^{-1} e_n^{(m)} p_{n-1}^{(m+1)}(z)\right)\left(p_n^{(m)}(z)^{-1} q_n^{(m)} p_{n-1}^{(m)}(z)\right) \cdots \\ &\cdots \left(p_1^{(m+1)}(z)\, e_1^{(m)} p_0^{(m+1)}(z)\right)\left(p_1^{(m)}(z)^{-1} q_1^{(m)}\right)(c_m z^{-1}) + \cdots\end{aligned} \tag{41}$$

are equivalent in the sense that successive convergents of the former are equal to successive partial sums of the latter.

We shall derive an algorithm relating the ratios between the successive denominators of (40), then we shall derive an algorithm relating the ratios of successive terms in (41), then an algorithm relating successive terms in (41), and finally an algorithm relating successive partial sums of (41).

The second g-Algorithm. The successive denominators of (40) are of course

$$z p_0^{(m+1)}(z),\ p_1^{(m)}(z),\ z\, p_1^{(m+1)}(z), \ldots, z\, p_r^{(m+1)}(z),\ p_{r+1}^{(m)}(z), \ldots .$$

Accordingly we write

$$\overset{(m)}{g}_{2r} = p_r^{(m)}(z)^{-1} p_r^{(m+1)}(z), \qquad \overset{(m)}{g}_{2r+1} = p_r^{(m+1)}(z)^{-1} p_{r+1}^{(m)}(z) \tag{42}$$

and note, by use of relations (30) and (31), that

$$I - \overset{(m)}{g}_{2r} = -\, p_r^{(m)}(z)^{-1} e_r^{(m)} p_{r-1}^{(m+1)}(z), \qquad z - \overset{(m)}{g}_{2r-1} = p_{r-1}^{(m+1)}(z)^{-1} q_r^{(m)} p_{r-1}^{(m)}(z). \tag{43}$$

From the definitions (42) and the $q-d$ algorithm we have immediately ([9], p. 13)

$$\overset{(m+1)}{g}_{2r}\ \overset{(m+1)}{g}_{2r+1} = \overset{(m)}{g}_{2r+1}\ \overset{(m)}{g}_{2r+2},$$

$$\left(z - \overset{(m)}{g}_{2r+1}\right)\left(I - \overset{(m)}{g}_{2r}\right) = \left(I - \overset{(m+1)}{g}_{2r}\right)\left(z - \overset{(m+1)}{g}_{2r-1}\right),$$

with

$$\overset{(m)}{g}_0 = I, \qquad \overset{(m)}{g}_1 = z - c_{m+1}\, c_m^{-1}. \tag{44}$$

From Theorem 4.5 and (43) we have

Theorem 5.1. *If in* (44)

$$\sum_{i=0}^{n} c_{m+i}\, b_i = O \qquad (b_i \in S) \quad (m = 0, 1, \ldots), \tag{45}$$

then

$$\overset{(m)}{g}_{2n} = I.$$

We note, using (43), that the series (41) may be written as

$$\begin{aligned} & c_m z^{-1} + \overset{(m)}{g}{}_1^{-1} (z - \overset{(m)}{g}_1)\, c_m z^{-1} + \overset{(m)}{g}{}_2^{-1} (I - \overset{(m)}{g}_2)\, \overset{(m)}{g}{}_1^{-1} (z - \overset{(m)}{g}_1)\, c_m z^{-1} + \cdots \\ & \quad + \overset{(m)}{g}{}_{2r-1}^{-1} (z - \overset{(m)}{g}_{2r-1}) \ldots \overset{(m)}{g}{}_2^{-1} (I - \overset{(m)}{g}_2)\, \overset{(m)}{g}{}_1^{-1} (z - \overset{(m)}{g}_1)\, c_m z^{-1} + \\ & \quad + \overset{(m)}{g}{}_{2r}^{-1} (I - \overset{(m)}{g}_{2r})\, \overset{(m)}{g}{}_{2r-1}^{-1} (z - \overset{(m)}{g}_{2r-1}) \ldots \overset{(m)}{g}{}_2^{-1} (I - \overset{(m)}{g}_2)\, \overset{(m)}{g}{}_1^{-1} (z - \overset{(m)}{g}_1)\, c_m z^{-1} + \cdots. \end{aligned}$$

The π-Algorithm. We wish to relate successive ratios of the terms in this series. Accordingly write

$$\pi_{2r}^{(m)} = \overset{(m)}{g}{}_{2r}^{-1} (I - \overset{(m)}{g}_{2r}), \qquad \pi_{2r+1}^{(m)} = \overset{(m)}{g}{}_{2r+1}^{-1} (z - \overset{(m)}{g}_{2r+1}), \tag{46}$$

and from the g-algorithm we have ([9], p. 13)

$$\begin{aligned} (I + \pi_{2r}^{(m)})\,(I + \pi_{2r-1}^{(m)}) &= (I + \pi_{2r-1}^{(m+1)})\,(I + \pi_{2r-2}^{(m+1)}), \\ (I + \pi_{2r}^{(m)^{-1}})\,(I + \pi_{2r+1}^{(m)^{-1}}) &= (I + \pi_{2r-1}^{(m+1)^{-1}})\,(I + \pi_{2r}^{(m+1)^{-1}}), \\ \pi_0^{(m)} = O, \qquad \pi_1^{(m)} &= (z\, c_m\, c_{m+1}^{-1} - I)^{-1}. \end{aligned} \tag{47}$$

From Theorem 5.1 and (46) we have

Theorem 5.2. *If in* (47) *equation* (45) *obtains, then*

$$\pi_{2n}^{(m)} = O \qquad (m = 0, 1, \ldots).$$

The η-Algorithm. We now investigate the terms in (41) and accordingly write

$$\eta_s^{(m)} = \pi_s^{(m)}\, \eta_{s-1}^{(m)} \qquad (s = 1, 2, \ldots). \tag{48}$$

The η-algorithm relationships then evolve to the form

$$\begin{aligned} (\eta_{2r}^{(m)} + \eta_{2r-1}^{(m)})\,(\eta_{2r}^{(m)^{-1}} + \eta_{2r+1}^{(m)^{-1}}) &= (\eta_{2r-1}^{(m+1)} + \eta_{2r-2}^{(m+1)})\,(\eta_{2r-1}^{(m+1)^{-1}} + \eta_{2r}^{(m+1)^{-1}}), \\ (\eta_{2r}^{(m)} + \eta_{2r-1}^{(m)})\,(\eta_{2r-1}^{(m)^{-1}} + \eta_{2r-2}^{(m)^{-1}}) &= (\eta_{2r-1}^{(m+1)} + \eta_{2r-2}^{(m+1)})\,(\eta_{2r-2}^{(m+1)^{-1}} + \eta_{2r-3}^{(m+1)^{-1}}); \end{aligned}$$

a simple inductive proof suffices to show that ([9], p. 14)

$$z\{\eta_{2r}^{(m)} + \eta_{2r-1}^{(m)}\} = \eta_{2r-1}^{(m+1)} + \eta_{2r-2}^{(m+1)}, \tag{49}$$

$$\eta_{2r}^{(m)^{-1}} + \eta_{2r+1}^{(m)^{-1}} = z\{\eta_{2r-1}^{(m+1)^{-1}} + \eta_{2r}^{(m+1)^{-1}}\}, \tag{50}$$

$$\eta_{-1}^{(m)^{-1}} = O, \qquad \eta_0^{(m)} = c_m z^{-1}. \tag{51}$$

From (48) and Theorem 5.2 we have

Theorem 5.3. *If in* (51) *equation* (45) *obtains, then*

$$\eta_{2n}^{(m)} = O \qquad (m = 0, 1, \ldots).$$

The ε-Algorithm. Now, in terms of the quantities $\eta_s^{(m)}$

$$p_r^{(m)}(z)^{-1} r_r^{(m)}(z) = \sum_{s=0}^{m-1} z^{-s} \eta_0^{(s)} + z^{-m} \sum_{s=0}^{2r-1} \eta_s^{(m)};$$

accordingly let us define

$$\varepsilon_{2r}^{(m)} = \sum_{s=0}^{m-1} z^{-s} \eta_0^{(s)} + z^{-m} \sum_{s=0}^{2r-1} \eta_s^{(m)} \tag{52}$$

and

$$\varepsilon_{2r+1}^{(m)} = z^m \sum_{s=0}^{n} \eta_s^{(m)^{-1}}. \tag{53}$$

Then by multiple use of relationships (52) and (53) we have

$$\varepsilon_{2r}^{(m+1)} - \varepsilon_{2r}^{(m)} = z^{-m} \eta_{2r}^{(m)}, \qquad \varepsilon_{2r+2}^{(m)} - \varepsilon_{2r}^{(m+1)} = z^{-m} \eta_{2r+1}^{(m)},$$
$$\varepsilon_{2r+1}^{(m+1)} - \varepsilon_{2r+1}^{(m)} = z^m \eta_{2r+1}^{(m)^{-1}}, \qquad \varepsilon_{2r+1}^{(m)} - \varepsilon_{2r-1}^{(m+1)} = z^m \eta_{2r}^{(m)^{-1}}.$$

That is ([2], [9], p. 14),

$$\begin{aligned} &\varepsilon_{s+1}^{(m)} - \varepsilon_{s-1}^{(m+1)} = (\varepsilon_s^{(m+1)} - \varepsilon_s^{(m)})^{-1}, \\ &\varepsilon_{-1}^{(m)} = O, \qquad \varepsilon_0^{(m)} = \sum_{s=0}^{m-1} c_s z^{-s-1}. \end{aligned} \tag{54}$$

If the quantities c_m satisfy a linear recursion of the form

$$\sum_{i=0}^{n} b_i c_{m+i} = O \qquad (b_i \in S) \ (m = 0, 1, \ldots),$$

then $\varepsilon_{2n}^{(m)}$ $(m = 0, 1, \ldots)$ are the same rational function of z and may be given the partial fraction decomposition

$$\varepsilon_{2n}^{(m)} = \sum_{s=1}^{n} A_s (z - \lambda_s)^{-1} \qquad (A_s \in N)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ $(\in S)$ are the roots of the polynomial equation

$$\sum_{i=0}^{n} b_i \lambda^i = O. \tag{55}$$

Writing

$$S_m = \sum_{s=0}^{m-1} c_s z^{-s-1},$$

we have

$$S_m = \sum_{s=1}^{s=n} A_s (z - \lambda_s)^{-1} - \sum_{s=1}^{n} A_s \lambda_s^m (z - \lambda_s)^{-1}$$

and

$$\sum_{i=0}^{i=n} b_i \left(S_{m+i} - \sum_{s=1}^{s=n} A_s (z - \lambda_s)^{-1} \right) = 0$$

or, in conclusion, we have

Theorem 5.4. *If the relationships* (54) *of the ε-algorithm are applied to the initial conditions*

$$\varepsilon_{-1}^{(m)} = 0, \qquad \varepsilon_1^{(m)} = S_m \qquad (m = 0, 1, \ldots) \tag{56}$$

and

$$\sum_{i=0}^{n} b_i S_{m+i} = \left(\sum_{i=0}^{n} b_i\right) A \qquad (m=0, 1, \ldots), \qquad b_i \in S \qquad (i=0, 1, \ldots, n), \tag{57}$$

then

$$\varepsilon_{2n}^{(m)} = A .$$

Note. For simplicity in presentation it was assumed in the above proof that the roots of equation (55) are distinct, but this restriction is not essential.

This is perhaps an appropriate point at which to introduce

Theorem 5.5. *If the ε-algorithm relationships* (54) *are applied to the initial values*

$$\varepsilon_{-1}^{(m)} = O, \qquad \varepsilon_0^{(m)} = S_m \qquad (m=0, 1, \ldots)$$

to produce quantities $\varepsilon_s^{(m)}$, to the initial values

$$\varepsilon_{-1}^{(m)'} = O, \qquad \varepsilon_0^{(m)'} = A + B S_m, \qquad (A, B \in N) \qquad (m=0, 1, \ldots),$$

to produce quantities $\varepsilon_s^{(m)'}$, and to the initial values

$$\varepsilon_{-1}^{(m)''} = O, \qquad \varepsilon_0^{(m)''} = A + S_m B \qquad (A, B \in N) \qquad (m=0, 1, \ldots)$$

to produce quantities $\varepsilon_s^{(m)''}$, then

$$\varepsilon_{2s}^{(m)'} = A + B\,\varepsilon_{2s}^{(m)}, \qquad \varepsilon_{2s+1}^{(m)''} = \varepsilon_{2s+1}^{(m)} B^{-1}, \qquad (m, s=0, 1, \ldots),$$
$$\varepsilon_{2s}^{(m)''} = A + \varepsilon_{2s}^{(m)} B, \qquad \varepsilon_{2s+1}^{(m)''} = B^{-1}\,\varepsilon_{2s+1}^{(m)} \qquad (m, s=0, 1, \ldots).$$

It will have been noted that the η- and ε-algorithm relations are free from multiplication, and further that the same relations and same initial conditions are obtained for both the pre- and post-continued fraction systems. We have the important

Theorem 5.6. *In the notation of Section 4*

$$\operatorname{pre}[p_r^{(m)}(z)^{-1} o_r^{(m)}(z)] = \operatorname{post}[o_r^{(m)}(z)\, p_r^{(m)}(z)^{-1}] \qquad (m, r=0, 1, \ldots),$$
$$\operatorname{pre}[p_r^{(m)}(z)^{-1} r_r^{(m)}(z)] = \operatorname{post}[r_r^{(m)}(z)\, p_r^{(m)}(z)^{-1}] \qquad (m, r=0, 1, \ldots).$$

We note in passing the following two invariants associated with the η- and ε-algorithms.

Theorem 5.7. *If condition* (45) *is assumed and the η-algorithm relations* (49) *and* (50) *are applied to the initial conditions* (51), *then the sum*

$$\sum_{s=0}^{2n-1} \eta_s^{(m)}$$

is invariant with respect to m.

For the ε-algorithm there are two Bauer [*10*] invariants.

Theorem 5.8. *If the ε-algorithm relations* (54) *are applied to the initial conditions* (56), *and in equation* (57) $A=0$, *then the two sums*

$$\varepsilon_0^{(m)}\varepsilon_1^{(m)} - \varepsilon_2^{(m)}\varepsilon_1^{(m)} + \varepsilon_2^{(m)}\varepsilon_3^{(m)} - \varepsilon_4^{(m)}\varepsilon_3^{(m)} + \cdots + \varepsilon_{2n-2}^{(m)}\varepsilon_{2n-1}^{(m)}$$

and

$$\varepsilon_1^{(m)}\varepsilon_0^{(m)} - \varepsilon_1^{(m)}\varepsilon_2^{(m)} + \varepsilon_3^{(m)}\varepsilon_2^{(m)} - \varepsilon_3^{(m)}\varepsilon_4^{(m)} + \cdots + \varepsilon_{2n-1}^{(m)}\varepsilon_{2n-2}^{(m)}$$

are invariant with respect to m.

The algorithms of this section have been derived by considering the series (41) and further series equivalent to it. We remark that there is a further equivalent system of continued fraction decompositions. Indeed we have ([9], p. 14)

Theorem 5.9. *The continued fractions*

$$\frac{c_m}{z-}\,\frac{q_1^{(m)}}{I-}\,\frac{e_1^{(m)}}{z-}\cdots\frac{q_r^{(m)}}{I-}\,\frac{e_r^{(m)}}{z-}\cdots$$

$$\frac{c_m z^{-1}}{I-}\,\frac{p_1^{(m)}(z)^{-1}q_1^{(m)}}{I+p_1^{(m)}(z)^{-1}q_1^{(m)}-}\,\frac{p_1^{(m+1)}(z)^{-1}e_1^{(m)}p_0^{(m+1)}(z)}{I+p_1^{(m+1)}(z)^{-1}e_1^{(m)}p_0^{(m+1)}(z)-}\cdots$$
$$\cdots\frac{p_r^{(m)}(z)^{-1}q_r^{(m)}p_{r-1}^{(m)}(z)}{I+p_r^{(m)}(z)^{-1}q_r^{(m)}p_{r-1}^{(m)}(z)-}\,\frac{p_r^{(m+1)}(z)^{-1}e_r^{(m)}p_{r-1}^{(m+1)}(z)}{I+p_r^{(m+1)}(z)^{-1}e_r^{(m)}p_{r-1}^{(m+1)}(z)-}\cdots$$

$$\frac{c_m z^{-1}}{I-}\,\frac{\overset{(m)}{g}{}_1^{-1}(z-\overset{(m)}{g}_1)}{I+\overset{(m)}{g}{}_1^{-1}(z-\overset{(m)}{g}_1)-}\,\frac{\overset{(m)}{g}{}_2^{-1}(I-\overset{(m)}{g}_2)}{I+\overset{(m)}{g}{}_2^{-1}(I-\overset{(m)}{g}_2)-}\cdots$$
$$\cdots\frac{\overset{(m)}{g}{}_{2r-1}^{-1}(z-\overset{(m)}{g}_{2r-1})}{I+\overset{(m)}{g}{}_{2r-1}^{-1}(z-\overset{(m)}{g}_{2r-1})-}\,\frac{\overset{(m)}{g}{}_{2r}^{-1}(I-\overset{(m)}{g}_{2r})}{I+\overset{(m)}{g}{}_{2r}^{-1}(I-\overset{(m)}{g}_{2r})-}\cdots$$

$$\frac{c_m z^{-1}}{I-}\,\frac{\pi_1^{(m)}}{I+\pi_1^{(m)}-}\cdots\frac{\pi_s^{(m)}}{I+\pi_s^{(m)}-}\cdots$$

$$\frac{c_m z^{-1}}{I-}\,\frac{\eta_1^{(m)}\eta_0^{(m)-1}}{I+\eta_1^{(m)}\eta_0^{(m)-1}-}\cdots\frac{\eta_s^{(m)}\eta_{s-1}^{(m)-1}}{I+\eta_s^{(m)}\eta_{s-1}^{(m)-1}-}\cdots$$

$$\frac{c_m z^{-1}}{I-}\,\frac{(\varepsilon_2^{(m)}-\varepsilon_0^{(m+1)})\,\varepsilon_1^{(m)}}{I+(\varepsilon_2^{(m)}-\varepsilon_0^{(m+1)})\,\varepsilon_1^{(m)}-}\cdots\frac{(\varepsilon_{2r+2}^{(m)}-\varepsilon_{2r}^{(m+1)})\,(\varepsilon_{2r+1}^{(m)}-\varepsilon_{2r-1}^{(m+1)})}{I+(\varepsilon_{2r+2}^{(m)}-\varepsilon_{2r0}^{(m+1)})\,(\varepsilon_{2r+1}^{(m)}-\varepsilon_{2r-1}^{(m+1)})-}\cdots$$
$$\cdots\frac{(\varepsilon_{2r+2}^{(m+1)}-\varepsilon_{2r+2}^{(m)})\,(\varepsilon_{2r+1}^{(m+1)}-\varepsilon_{2r+1}^{(m)})}{I+(\varepsilon_{2r+2}^{(m+1)}-\varepsilon_{2r+2}^{(m)})\,(\varepsilon_{2r+1}^{(m+1)}-\varepsilon_{2r+1}^{(m)})-}\cdots$$

are equivalent in the sense that they have the same convergents.

In the preceding development, the relations of the $q-d$ algorithm were assumed valid, and from these the g-, π-, η- and ε-algorithms were developed. In fact, however, it would have been possible, using the continued fraction decompositions of Theorem 5.9, to have proved any one of these algorithms from the beginning in a manner similar to that in which the $q-d$ algorithm was established. We illustrate this by outlining ([9], p. 4) the proof of

The First g-Algorithm. Suppose that given $c^{(m)}(\in S)$ and a function $f_m(z)$ $(\in N)$ we have derived a continued fraction expansion of the form

$$f_m(z)\sim\frac{c_m}{z-}\,\frac{(c^{(m)}-g_1^{(m)})\,g_0^{(m)}}{I-}\,\frac{(I-g_2^{(m)})\,g_1^{(m)}}{z-}\,\frac{(c^{(m)}-g_3^{(m)})\,g_2^{(m)}}{I-}\cdots \tag{58}$$

(where $g_0^{(m)}=I$) and we wish to obtain a similar expansion for the function

$$f_{m+1}(z)=(z-c^{(m)})\,f_m(z)-c_m.$$

The even part of (58) is

$$\frac{c_m}{z-(c^{(m)}-g_1^{(m)})\,g_0^{(m)}-}\,\frac{(I-g_2^{(m)})\,g_1^{(m)}\,(c^{(m)}-g_1^{(m)})\,g_0^{(m)}}{z-(I-g_2^{(m)})\,g_1^{(m)}-(c^{(m)}-g_3^{(m)})\,g_2^{(m)}-}$$
$$\frac{(I-g_4^{(m)})\,g_3^{(m)}\,(c^{(m)}-g_3^{(m)})\,g_2^{(m)}}{z-(I-g_4^{(m)})\,g_3^{(m)}-(c^{(m)}-g_5^{(m)})\,g_4^{(m)}-}\cdots,$$

which can be rearranged as

$$\frac{c_m}{z-c^{(m)}+g_1^{(m)}\,g_0^{(m)}-}\;\frac{(I-g_2^{(m)})\,(c^{(m)}-g_3^{(m)})\,g_3^{(m)}\,g_2^{(m)}}{z-c^{(m)}+(I-g_2^{(m)})\,(c^{(m)}-g_1^{(m)})+g_3^{(m)}\,g_2^{(m)}-}$$
$$\frac{(I-g_4^{(m)})\,(c^{(m)}-g_3^{(m)})\,g_3^{(m)}\,g_2^{(m)}}{z-c^{(m)}+(I-g_4^{(m)})\,(c^{(m)}-g_3^{(m)})+g_5^{(m)}\,g_4^{(m)}-}\;\cdots.$$

This is the even part of

$$\frac{c_m}{z-c^{(m)}+}\;\frac{g_1^{(m)}\,g_0^{(m)}}{I+}\;\frac{(I-g_2^{(m)})\,(c^{(m)}-g_1^{(m)})}{z-c^{(m)}+}\;\frac{g_3^{(m)}\,g_2^{(m)}}{I+}\;\frac{(I-g_4^{(m)})\,(c^{(m)}-g_3^{(m)})}{z-c^{(m)}+}\;\cdots,$$

the odd part of which is

$$c_m(z-c^{(m)})^{-1}+\frac{(z-c^{(m)})^{-1}\,g_1^{(m)}\,c_{(m)}}{z-c^{(m)}+g_1^{(m)}\,g_0^{(m)}+(I-g_2^{(m)})\,(c^{(m)}-g_1^{(m)})-}$$
$$\frac{g_3^{(m)}\,g_2^{(m)}\,(I-g_2^{(m)})\,(c^{(m)}-g_1^{(m)})}{z-c^{(m)}+g_3^{(m)}\,g_2^{(m)}+(I-g_4^{(m)})\,(c^{(m)}-g_3^{(m)})-}\;\frac{g_5^{(m)}\,g_4^{(m)}\,(I-g_4^{(m)})\,(c^{(m)}-g_3^{(m)})}{z-c^{(m)}+g_5^{(m)}\,g_4^{(m)}+(I-g_6^{(m)})\,(c^{(m)}-g_5^{(m)})-}\;\cdots$$

This we may rearrange as

$$c_m(z-c^{(m)})^{-1}+\frac{(z-c^{(m)})^{-1}\,g_1^{(m)}\,c_{(m)}}{z-g_2^{(m)}\,(c^{(m)}-g_1^{(m)})-}\;\frac{g_3^{(m)}\,(I-g_2^{(m)})\,g_2^{(m)}\,(c^{(m)}-g_1^{(m)})}{z-g_4^{(m)}\,(c^{(m)}-g_3^{(m)})-g_3^{(m)}\,(I-g_2^{(m)})-}$$
$$\frac{g_5^{(m)}\,(I-g_4^{(m)})\,g_4^{(m)}\,(c^{(m)}-g_3^{(m)})}{z-g_6^{(m)}\,(c^{(m)}-g_5^{(m)})-g_5^{(m)}\,(I-g_4^{(m)})-}\;\cdots,$$

which is the even part of

$$c_m(z-c^{(m)})^{-1}-\frac{(z-c^{(m)})^{-1}\,g_1^{(m)}\,c_m}{z-}\;\frac{g_2^{(m)}\,(c^{(m)}-g_1^{(m)})}{I-}\;\frac{g_3^{(m)}\,(I-g_2^{(m)})}{z-}\;\frac{g_4^{(m)}\,(c^{(m)}-g_3^{(m)})}{I-}\;\cdots$$

We are thus led to

Theorem 5.10. *If*

$$f_m(z)\sim\frac{c_m}{z-}\;\frac{(c^{(m)}-g_1^{(m)})\,g_0^{(m)}}{I-}\;\frac{(I-g_2^{(m)})\,g_1^{(m)}}{z-}\;\frac{(c^{(m)}-g_3^{(m)})\,g_2^{(m)}}{I-}\;\cdots,$$

$$f_{m+1}(z)\sim\frac{c_{m+1}}{z-}\;\frac{(c^{(m+1)}-g_1^{(m+1)})\,g_0^{(m+1)}}{I-}\;\frac{(I-g_2^{(m+1)})\,g_1^{(m+1)}}{z-}\;\frac{(c^{(m+1)}-g_3^{(m+1)})\,g_2^{(m+1)}}{I-}\;\cdots,$$

$$g_0^{(m)}=g_0^{(m+1)}=I,\ c^{(m)},\qquad c^{(m+1)}\in S$$

and

$$f_{m+1}(z)=(z-c^{(m)})\,f_m(z)-c_m, \tag{59}$$

then

$$g_{2r}^{(m)}(c^{(m)}-g_{2r-1}^{(m)})=(c^{(m+1)}-g_{2r-1}^{(m+1)})\,g_{2r-2}^{(m+1)},$$
$$g_{2r+1}^{(m)}(I-g_{2r}^{(m)})=(I-g_{2r}^{(m+1)})\,g_{2r-1}^{(m+1)},$$

where

$$c_{m+1}=-g_1^{(m)}\,c_m.$$

The first g-algorithm may be repeated for increasing values of m. When $c^{(m)}=O$ $(m=0,1,\ldots)$ the transformation (59) corresponds to the relations between the series $F_m(z)$ of the $q-d$ algorithm, and it is easily verified that in this case

$$g_{2r-1}^{(m)}\,g_{2r-2}^{(m)}=-q_r^{(m)},$$
$$(I-g_{2r}^{(m)})\,g_{2r-1}^{(m)}=e_r^{(m)}.$$

6. Gauss- and Euler-Type Continued Fraction Expansions

We shall now depart from the consideration of continued fractions specifically associated with power series and turn to further classes of expansions.

Gauss Type Continued Fractions ([4], ch. XVIII). Suppose that we have a sequence of functions pre $[A_n, B_n, C_n, D_n, \ldots, O_n, P_n]$ $(n=0, 1, \ldots)$ which satisfy the set of equations

$$\begin{aligned} \text{pre}\,[A_n &= A_{n+1} b_n + B_n c_n, \\ A_{n+1} &= B_n d_n + C_n e_n, \\ B_n &= C_n f_n + D_n g_n, \\ &\cdots\cdots \\ O_n &= P_n w_n + A_{n+1} x_n, \\ P_n &= A_{n+1} y_n + A_{n+2} z_n] \qquad (n=0, 1, \ldots). \end{aligned} \tag{60}$$

These may be written as

$$\begin{aligned} \text{pre}\,[A_{n+1}^{-1} A_n &= b_n + (B_n^{-1} A_{n+1})^{-1} c_n, \\ B_n^{-1} A_{n+1} &= d_n + (C_n^{-1} B_n)^{-1} e_n, \\ C_n^{-1} B_n &= f_n + (D_n^{-1} C_n)^{-1} g_n, \\ &\cdots\cdots \\ P_n^{-1} O_n &= w_n + (A_{n+1}^{-1} P_n)^{-1} x_n, \\ A_{n+1}^{-1} P_n &= y_n + (A_{n+2}^{-1} A_{n+1})^{-1} z_n] \qquad (n=0, 1, \ldots). \end{aligned}$$

Bearing in mind our first definition of a continued fraction we have

Theorem 6.1. *If the sequence of functions* pre $[A_n, B_n, C_n, D_n, \ldots, O_n, P_n]$ $(n=0, 1, \ldots)$ *satisfy relations* (60), *then*

$$\text{pre}\left[A_1^{-1} A_0 = b_0 + \frac{c_0}{d_0+}\,\frac{e_0}{f_0+}\cdots\frac{x_0}{y_0+}\,\frac{z_0}{b_1+}\,\frac{c_1}{d_1+}\cdots\frac{x_n}{y_n+}\,\frac{z_n}{A_{n+2}^{-1} A_{n+1}}\right].$$

Euler Type Continued Fractions ([5], vol. II, p. 274). These continued fractions, which constitute a subset of those considered above, are associated with linear homogeneous differential equations of the second order. As a preliminary we shall assume that there exists a point t_0 of the complex plane and a real positive non zero number δ such that to every point t' for which $|t_0 - t'| \leqq \delta$ there corresponds a member t of S given by

$$t = t' I.$$

Suppose that a function pre $[y]$ $(\in N)$ of t satisfies the equation

$$\text{pre}\,[y = y' Q_0 + y'' P_1] \qquad (Q_0, P_1 \in N) \tag{61}$$

where dashes denote differentiation with respect to t. Then as is easily verified

$$\text{pre}\,[y^{(\nu)} = y^{(\nu+1)} Q_\nu + y^{(\nu+2)} P_{\nu+1}] \qquad (\nu = 1, 2, \ldots),$$

where

$$\text{pre}[Q_\nu = (Q_{\nu-1} + P'_\nu)(I - Q'_{\nu-1})^{-1}, \quad P_{\nu+1} = P_\nu (I - Q'_{\nu-1})^{-1}] \qquad (\nu = 1, 2, \ldots). \tag{62}$$

We are led to

Theorem 6.2. *If the function* pre$[y]$ *of* t *satisfies equation* (61) *and the sequences* $Q_\nu, P_{\nu+1}$ $(\nu = 1, 2, \ldots)$ *are constructed by means of equations* (62), *then*

$$\text{pre}\left[y'^{-1} y = Q_0 + \frac{P_1}{Q_1+} \frac{P_2}{Q_2+} \cdots \frac{P_{n-1}}{Q_{n-1}+} \frac{P_n}{y^{(n+1)-1} y^{(n)}}\right].$$

7. Interpolatory Continued Fractions

Inverse Differences. Suppose that we have a sequence of pairs x_r, f_r (both $\in N$) $(r = 0, 1, \ldots)$, then construct the table of inverse differences $v_r(x)$ by means of the relations ([*11*], p. 396)

$$\text{pre}[v_0(x_r) = f_r, \tag{63}$$

$$v_{r+1}(x) = (x - x_r)\{v_r(x) - v_r(x_r)\}^{-1}] \quad (x = x_{r+1}, x_{r+2}, \ldots) \quad (r = 0, 1, \ldots). \tag{64}$$

Then from (64) we have

$$\text{pre}[v_r(x) = v_r(x_r) + v_{r+1}(x)^{-1}(x - x_r)] \quad (x = x_{r+1}, x_{r+2}, \ldots; \ r = 0, 1, \ldots).$$

Thus, bearing in mind the first definition (5) of a continued fraction, we have

Theorem 7.1. *If the table of inverse differences* $v_r(x)$ $(x = x_{r+1}, x_{r+2}, \ldots;$ $r = 0, 1, \ldots)$ *is constructed in accordance with relations* (63) *and* (64), *then*

$$f(x) = \text{pre}\left[f_0 + \frac{x - x_0}{v_1(x_1)+} \frac{x - x_1}{v_2(x_2)+} \cdots \frac{x - x_h}{v_{h+1}(x_{h+1})+} \cdots \frac{x - x_r}{v_{r+1}(x)}\right].$$

Clearly

$$\text{pre}\left[f_0 + \frac{x_h - x_0}{v_1(x_1)+} \frac{x_h - x_1}{v_2(x_2)+} \cdots \frac{x_h - x_{h-1}}{v_h(x_h)}\right] = f(x_h).$$

Reciprocal Differences. Define new quantities $\varrho_r(x)$ by the relation

$$\text{pre}[\varrho_{-1}(x) = O$$
$$\varrho_k(x) = v_k(x) + v_{k-2}(x_{k-2}) + v_{k-4}(x_{k-4}) + \cdots].$$

We have

Theorem 7.2. *If a table of reciprocal differences* pre$[\varrho_k(x)]$ $(x = x_k, x_{k+1}, \ldots;$ $k = 0, 1, \ldots)$ *is constructed according to the relations*

$$\text{pre}[\varrho_{-1}(x_k) = O, \ \varrho_0(x_k) = f_k, \qquad k = 0, 1, \ldots$$
$$\varrho_{k+1}(x) = \varrho_{k-1}(x_{k-1}) + (x - x_k)\{\varrho_k(x) - \varrho_k(x_k)\}^{-1} \tag{65}$$
$$(x = x_{k+1}, x_{k+2}, \ldots; \ k = 0, 1, \ldots),$$

then [*12*]

$$f(x) = \text{pre}\left[f_0 + \frac{x - x_0}{\varrho_1(x_1)+} \frac{x - x_1}{\varrho_2(x_2) - f_0+} \cdots \frac{x - x_h}{\varrho_{n+1}(x_{h+1}) - \varrho_{h-1}(x_{n-1})+} \cdots \right.$$
$$\left. \cdots \frac{x - x_r}{\varrho_{r+1}(x) - \varrho_{r-1}(x_{r-1})}\right].$$

We now proceed to the restriction $x, x_0, x_1, \ldots \in S$ and from Theorem 1.1 we have immediately

Theorem 7.3. *If* $x, x_0, x_1, \ldots \in S$ *inverse differences are computed according to* (64) *and reciprocal differences according to* (65), *then*

$$\operatorname{pre}\left[f_0 + \frac{x-x_0}{v_1(x_1)+} \frac{x-x_1}{v_2(x_2)+} \cdots \frac{x-x_r}{v_{r+1}(x_{r+1})+} \cdots\right] \tag{66}$$

$$= \operatorname{post}\left[f_0 + \frac{x-x_0}{v_1(x_1)+} \frac{x-x_1}{v_2(x_2)+} \cdots \frac{x-x_r}{v_{r+1}(x_{r+1})+} \cdots\right]$$

$$= \operatorname{pre}\left[f_0 + \frac{x-x_0}{\varrho_1(x_1)+} \frac{x-x_1}{\varrho_2(x_2)-f_0+} \cdots \frac{x-x_r}{\varrho_{r+1}(x_{r+1})-\varrho_{r-1}(x_{r-1})+} \cdots\right]$$

$$= \operatorname{post}\left[f_0 + \frac{x-x_0}{\varrho_1(x_1)+} \frac{x-x_1}{\varrho_2(x_2)-f_0+} \cdots \frac{x-x_r}{\varrho_{r+1}(x_{r+1})-\varrho_{r-1}(x_{r-1})+} \cdots\right] \tag{67}$$

in the sense that the r^{th} $(r=0, 1, \ldots)$ *convergents of all these continued fractions are equal.*

Quite clearly $A_r(x)$, $B_r(x)$, the successive numerators and denominators of the continued fractions (66) to (67), are polynomials in x, and a simple inductive proof based on the recursions of Theorem 2.1 suffices for

Theorem 7.4. *If* $A_r(x)$, $B_r(x)$ *are the successive numerators and denominators of the expansions* (66) *to* (67), *then*

$$A_{2n}(x) = \sum_{s=0}^{n} a_{2n,s}\, x^s, \qquad A_{2n+1}(x) = \sum_{s=0}^{n+1} a_{2n+1,s}\, x^s,$$

$$B_{2n}(x) = \sum_{s=0}^{n} b_{2n,s}\, x^s, \qquad B_{2n+1}(x) = \sum_{s=0}^{n} b_{2n+1,s}\, x^s \qquad (n=0, 1, \ldots).$$

We remark that the convergent $\{B_{2n}(x)\}^{-1} A_{2n}(x)$ is the quotient of two n^{th} degree polynomials and that

$$f(x) = \operatorname{pre}\left[f_0 + \frac{x-x_0}{v_1(x_1)+} \frac{x-x_1}{v_2(x_2)+} \cdots \frac{x-x_{2n-1}}{v_{2n}(x)}\right]$$

is an identity. We wish to inquire into that property of $v_{2n}(x)$ which must obtain if $f(x)$ is to be the quotient of two n^{th} degree polynomials. We write $\{B_{2n}(x)\}^{-1} A_{2n}(x)$ as

$$\{v_{2n}(x)\, B_{2n-1}(x) + (x - x_{2n-1})\, B_{2n-2}(x)\}^{-1} \{v_{2n}(x)\, A_{2n-1}(x) + \\ + (x - x_{2n-1})\, A_{2n-2}(x)\}.$$

Examining the term in x^n in the numerator, we arrive at

Theorem 7.5. *If* $f(x) = \left(\sum_{s=0}^{n} b_s\, x^s\right)^{-1} \left(\sum_{s=0}^{n} a_s\, x^s\right)$ *and inverse differences are computed by means of the relations*

$$v_0(x_r) = f(x_r), \; v_{r+1}(x) = (x - x_r)\{v_r(x) - v_r(x_r)\}^{-1} \quad (x = x_{r+1}, x_{r+2}, \ldots; r = 0, 1, \ldots)$$

and if reciprocal differences are computed by means of the relations

$$\varrho_{-1}(x_r)=O, \qquad \varrho_0(x_r)=f(x_r),$$

$$\varrho_{k+1}(x)=\varrho_{k-1}(x_{k-1})+(x-x_k)\{\varrho_k(x)-\varrho_k(x_k)\}^{-1}$$

$$(x=x_{k+1},\, x_{k+2},\, \ldots;\ k=0,1,\ldots),$$

then

$$v_{2n}(x)=k,\ \textit{independent of } x,$$

$$\varrho_{2n}(x)=k',\ \textit{independent of } x.$$

Rational Function Extrapolation: The ρ-Algorithm [*13*]. Let us now change slightly the sequences used in the construction of the reciprocal differences, still with $x, x_0, x_1, \ldots \in S$, compute quantities $\varrho'_r(x_m, x_{m+1}, \ldots, x_{m+r})$ $(r, m = 0, 1, \ldots)$ according to

$$\varrho_{-1}(x_m)=0, \qquad \varrho_0(x_m)=f_m,$$

$$\varrho'_{r+1}(x_m)=\varrho'_{r+1}(x_m, x_{m+1}, \ldots, x_{m+r+1})$$

$$=\varrho'_{r-1}(x_{m+1}, x_{m+2}, \ldots, x_{m+r})+(x_{m+1+r}-x_m)\{\varrho'_r(x_{m+1}, x_{m+2}, \ldots, x_{m+r+1})-$$

$$-\varrho'_r(x_m, x_{m+1}, \ldots, x_{m+r})\}^{-1}.$$

Then we have

$$f(x)=f_m+\frac{x-x_m}{\varrho'_1(x_m)+}\ \frac{x-x_{m+1}}{\varrho'_2(x_m)-f_m+}\cdots\frac{x-x_{m+r}}{\varrho'_{r+1}(x, x_m, x_{m+1}, \ldots, x_{m+r})-\varrho'_{r-1}(x_m)}. \tag{68}$$

Now let $x_m=mI$ $(m=0,1,\ldots)$, $x=hI$ where h is a positive integer, and let h tend to infinity in (68), which becomes

$$f(\infty I)\sim f_m+\frac{I}{0+}\ \frac{I}{\varrho'_2(x_m)-f_m+}\ \frac{I}{0+}\cdots\frac{I}{0+}\ \frac{I}{\varrho'_{2n}(x_m)-\varrho'_{2n-2}(x_m)+}\cdots.$$

The even-order convergents of this continued fraction are of course $\varrho'_{2n}(x_m)$ $(n=0,1,\ldots)$. We are thus led to

Theorem 7.6. *Suppose that we are given a sequence of values of the function*

$$f(x)=\Big(\sum_{s=0}^{n} b_{n,s}\, x^s\Big)^{-1}\Big(\sum_{s=0}^{n} a_{n,s}\, x^s\Big) \qquad (a_{ns}, b_{ns}\in N;\ s=0,1,\ldots,n)$$

for $x=(\overline{m}+m)I$, $(m=0,1,\ldots)$, *where* $\overline{m}$ *is an integer, then we may determine the value of* $f(x)$ *when* x *tends to* ∞I *by constructing the table of reciprocal differences* $\varrho_s^{(m)}$ *according to*

$$\varrho_{-1}^{(m)}=0, \qquad \varrho_0^{(m)}=f_{\overline{m}+m} \qquad (m=0,1,\ldots),$$

$$\varrho_{s+1}^{(m)}=\varrho_{s-1}^{(m+1)}+(s+1)\,I\,\{\varrho_s^{(m+1)}-\varrho_s^{(m)}\}^{-1} \qquad (m,s=0,1,\ldots),$$

for

$$\varrho_{2n}^{(m)}=b_{n,n}^{-1}\,a_{n,n} \qquad (m=0,1,\ldots).$$

8. Confluent Forms

We now introduce the assumption that to every complex number t' there corresponds one member t of S given by

$$t=t'I.$$

Suppose that we are given an infinitely differentiable function $\varphi(t)$ $(\in N)$ of the scalar variable t and that we wish to obtain a continued fraction,

$$F(z)=\int_0^\infty \varphi(a+t)\,e^{-zt}dt \sim \operatorname{pre}\left[\frac{\varphi(a)}{z-Q_1(a)-}\;\frac{E_1(a)}{z-Q_2(a)-}\cdots\frac{E_{r-1}(a)}{z-Q_r(a)-}\cdots\right] \tag{69}$$

for its Laplace transform. We may, of course, do this with the apparatus already available. We obtain the formal expansion $\sum_{s=0}^{\infty}\varphi^{(s)}(a)\,z^{-s-1}$, apply the $q-d$ algorithm to this series, and obtain the required continued fraction expansion by means of the formulae

$$Q_r(a)=q_r^{(0)}+e_{r-1}^{(0)},\qquad E_r(a)=q_r^{(0)}\,e_r^{(0)}\qquad (r=1,2,\ldots).$$

We can, following RUTISHAUSER [14], proceed in another way. We apply the $q-d$ algorithm to the series $h\sum_{m=0}^{\infty}\varphi(a+mh)\,e^{-zmh}$ $(h\in S)$ to obtain the continued fraction

$$\operatorname{pre}\left[\frac{h c_0'}{e^{hz}-q_1^{(0)}-}\;\frac{e_1^{(0)\prime}\,q_1^{(0)\prime}}{e^{hz}-q_2^{(0)\prime}-e_1^{(0)\prime}-}\cdots\frac{e_r^{(0)\prime}\,q_r^{(0)\prime}}{e^{hz}-q_{r+1}^{(0)\prime}-e_r^{(0)\prime}-}\cdots\right] \tag{70}$$

and then let h tend to zero. In order to see what happens to the quantities $q_r^{(m)\prime}$, $e_r^{(m)\prime}$, let us introduce the continuous variable t defined by

$$t=a+mh,$$

and the auxilary functions $E_r(t)$, $Q_r(t)$ defined by

$$\operatorname{pre}\left[E_r(t)=e_r^{(m)\prime}\,h^{-2},\; Q_r(t)=(q_r^{(m)\prime}-I)\,h^{-1}\right].$$

As h tends to zero, the continued fraction (70) becomes that of (69), and the $q-d$ algorithm relations for the $q_r^{(m)\prime}$, $e_r^{(m)\prime}$ lead to

$$\operatorname{pre}\left[\frac{d}{dt}Q_r(t)=E_r(t)-E_{r-1}(t)\right.$$
$$\left.\frac{d}{dt}E_r(t)=Q_{r+1}(t)\,E_r(t)-E_r(t)\,Q_r(t)\right].$$

We may thus transform Theorem 4.5 and obtain finally

Theorem 8.1 *If $\varphi(t)$ $(\in N)$ satisfies the differential equation*

$$\sum_{s=0}^{n} b_s\,\varphi^s(t)=O\quad(b_s\in S,\ (s=0,1,\ldots,n)), \tag{71}$$

and if the functions $E_r(t)$, $Q_r(t)$ are constructed by means of the relations

$$\operatorname{pre}\left[E_r(t)=E_{r-1}(t)+\frac{d}{dt}Q_r(t),\right. \tag{72}$$

$$\left.Q_{r+1}(t)=\left\{\frac{d}{dt}E_r(t)-E_r(t)\,Q_r(t)\right\}\left(E_r(t)^{-1}\right)\right] \tag{73}$$

from the initial conditions

$$\operatorname{pre}\left[E_0(t)=O,\; Q_1(t)=\varphi'(t)\{\varphi(t)\}^{-1}\right],$$

then

$$\operatorname{pre}\left[E_n(t)=O\right].$$

Further, the sum

$$\mathrm{pre}\left[\sum_{s=1}^{n-1} E_s(t)\right]$$

is independent of t.

We can follow the development outlined in [15] and obtain confluent forms of further algorithms.

Making the substitutions

$$\mathrm{pre}\,[g_{2s}^{(m)} - I = {}_2g_r(t)\,h^2, \qquad g_{2s-1}^{(m)} + I = {}_1g_r(t)\,h, \qquad c^{(m)} = c(t)\,h],$$

we obtain

Theorem 8.2. *If* $\varphi(t)$ *satisfies* (71) *and if functions* ${}_1g_r(t)$, ${}_2g_r(t)$ *are obtained by means of the relations*

$$\mathrm{pre}\Big[{}_1g_{r+1}(t)\,{}_2g_r(t) - {}_2g_r(t)\,{}_1g_r(t) = \frac{d}{dt}\,{}_2g_r(t),$$
$${}_2g_r(t) - {}_2g_{r-1}(t) = \frac{d}{dt}\{c(t) - {}_1g_r(t)\},\Big]$$

from the initial conditions

$$\mathrm{pre}\,[{}_2g_0(t) = O,\ {}_1g_0(t) = -\varphi'(t)\{\varphi(t)\}^{-1} + c(t)],$$

then

$$\mathrm{pre}\,[{}_2g_n(t) = O].$$

Furthermore, in terms of the functions $E_r(t)$, $Q_r(t)$ *derived from* (72) *and* (73)

$$\mathrm{pre}\,[{}_2g_r(t) = E_r(t),\ c(t) - {}_1g_r(t) = Q_r(t)]. \tag{74}$$

Substituting

$$\mathrm{pre}\,[I - \overset{(m)}{g}_{2s} = {}_2\mathfrak{g}_r(t)\,h^2,\ z - \overset{(m)}{g}_{2s-1} + I = {}_1\mathfrak{g}_r(t)\,h],$$

we obtain

Theorem 8.3. *If* $\varphi(t)$ *satisfies* (71), *and if the functions* ${}_1\mathfrak{g}_r(t)$, ${}_2\mathfrak{g}_r(t)$ *are constructed by means of the relations*

$$\mathrm{pre}\Big[\frac{d}{dt}\,{}_2\mathfrak{g}_r(t) = {}_2\mathfrak{g}_r(t)\,{}_1\mathfrak{g}_r(t) - {}_1\mathfrak{g}_{r+1}(t)\,{}_2\mathfrak{g}_r(t),$$
$$(z + I)\{{}_2\mathfrak{g}_{r+1}(t) - {}_2\mathfrak{g}_r(t)\} = \frac{d}{dt}\,{}_1\mathfrak{g}_{r+1}(t)\Big]$$

from the initial condition

$$\mathrm{pre}\,[{}_2\mathfrak{g}_0(t) = O,\ {}_1\mathfrak{g}_1(t) = \varphi'(t)\{\varphi(t)\}^{-1}],$$

then

$$\mathrm{pre}\,[{}_2\mathfrak{g}_n(t) = O].$$

The relation corresponding to (74) *is*

$$\mathrm{pre}\,[{}_1\mathfrak{g}_r(t) = Q_r(t),\ (I + z)\,{}_2\mathfrak{g}_r(t) = E_r(t)].$$

From the substitution

$$\eta_{2s-1}^{(m)} = {}_1\eta_r(t), \qquad \eta_{2s}^{(m)} = {}_2\eta_r(t)\,h,$$

we obtain

Theorem 8.4. *If $\varphi(t)$ satisfies (71), and if the functions ${}_1\eta_r(t)$, ${}_2\eta_r(t)$ are constructed by means of the relations*

$$ {}_2\eta_r(t) - {}_2\eta_{r-1}(t) = \frac{d}{dt}\,{}_1\eta_r(t), \tag{75} $$

$$ {}_1\eta_r(t)^{-1} - {}_1\eta_{r+1}(t)^{-1} = {}_2\eta_r(t)^{-1}\left\{\frac{d}{dt}\,{}_2\eta_r(t)\right\}{}_2\eta_r(t)^{-1} \tag{76} $$

from the initial conditions

$$ {}_2\eta_{-1}(t)^{-1} = O, \qquad {}_2\eta_0(t) = \varphi(t), $$

then

$$ {}_2\eta_n(t) = O, $$

and the sum

$$ \sum_{s=1}^{n-1} {}_1\eta_s(t) $$

is independent of t.

From the substitution

$$ \varepsilon_{2s+1}^{(m)} = \varepsilon_{2s+1}(t)\,h^{-1}, \qquad \varepsilon_{2s}^{(m)} = \varepsilon_{2s}(t), $$

we obtain

Theorem 8.5. *If functions $\varepsilon_r(t)$ are obtained by means of the relations*

$$ \varepsilon_{r+1}(t) = \varepsilon_{r-1}(t) + \left\{\frac{d}{dt}\,\varepsilon_r(t)\right\}^{-1} $$

from the initial conditions

$$ \varepsilon_{-1}(t) = O, \qquad \varepsilon_0(t) = f(t) \qquad (\in N), $$

where

$$ \sum_{s=0}^{n} a_s f^{(s)}(t) = b \qquad (a_s\,(s=0,1,\ldots,n) \in S,\ b \in N), $$

then

$$ \varepsilon_{2n}(t) = a_0^{-1} b. $$

Further, if in (78) $b=0$, then the function

$$ \varepsilon_0(t)\,\varepsilon_1(t) - \varepsilon_2(t)\,\varepsilon_1(t) + \varepsilon_2(t)\,\varepsilon_3(t) - \varepsilon_4(t)\,\varepsilon_3(t) + \cdots + \varepsilon_{2n-2}(t)\,\varepsilon_{2n-1}(t) $$

and

$$ \varepsilon_1(t)\,\varepsilon_0(t) - \varepsilon_1(t)\,\varepsilon_2(t) + \varepsilon_3(t)\,\varepsilon_2(t) - \varepsilon_3(t)\,\varepsilon_4(t) + \cdots + \varepsilon_{2n-1}(t)\,\varepsilon_{2n-2}(t) $$

are independent of t.

Theorem 8.6. *If $\varepsilon_r(t)$ are derived from*

$$ \varepsilon_{-1}(t) = O, \qquad \varepsilon_0(t) = g(t) \qquad (\in N), $$

$$ \varepsilon_r^*(t) \text{ from } \varepsilon_{-1}^*(t) = 0, \qquad \varepsilon_1^*(t) = A + B\,g(t), $$

$$ \varepsilon_r'(t) \text{ from } \varepsilon_{-1}'(t) = 0, \qquad \varepsilon_0'(t) = A + g(t)\,B \qquad (A, B \in N), $$

then

$$ \varepsilon_{2r}^*(t) = A + B\,\varepsilon_{2r}(t), \qquad \varepsilon_{2r+1}^*(t) = e_{2r+1}(t)\,B^{-1}, $$

$$ \varepsilon_{2r}'(t) = A + \varepsilon_{2r}(t)\,B, \qquad \varepsilon_{2r+1}'(t) = B^{-1}\varepsilon_{2r+1}(t). $$

The relations between the confluent forms of the η- and ε-algorithms is given by

Theorem 8.7. *If functions $\eta_r(t)$ are derived by means of relations (75) and (76) from the initial conditions*

$$_1\eta_{-1}(t)^{-1}=0, \qquad _2\eta_0(t)=\frac{d}{dt}g(t) \qquad (\in N)$$

and the functions $\varepsilon_r(t)$ by means of relations (77) from the initial conditions

$$\varepsilon_{-1}(t)=0, \qquad \varepsilon_0(t)=g(t) \qquad (\in N),$$

then

$$\frac{d}{dt}\varepsilon_{2r}(t)={}_2\eta_r(t), \qquad \varepsilon_{2r+2}(t)-\varepsilon_{2r}(t)={}_1\eta_{r+1}(t),$$

$$\frac{d}{dt}\varepsilon_{2r+1}(t)={}_1\eta_{r+1}(t)^{-1}, \qquad \varepsilon_{2r+1}(t)-\varepsilon_{2r-1}(t)={}_2\eta_r(t).$$

Finally, introducing the substitutions

$$\varrho_{2r}^{(m)}=\varrho_{2r}(t), \qquad \varrho_{2r+1}^{(m)}=\varrho_{2r-1}(t)\,h^{-1},$$

we have

Theorem 8.8. *If functions $\varrho_r(t)$ are constructed by means of the relations*

$$\varrho_{r+1}(t)-\varrho_{r-1}(t)=(r+1)\left\{\frac{d}{dt}\varrho_r(t)\right\}^{-1}$$

from the initial conditions

$$\varrho_{-1}(t)=0, \qquad \varrho_0(t)=f(t) \qquad (\in N), \tag{77}$$

then ([*16*], p. 453)

$$f(x)=f(a)+\frac{x-a}{\varrho_1(a)+}\;\frac{x-a}{\varrho_2(a)-f(a)+}\cdots\frac{x-a}{\varrho_{r+1}(x)-\varrho_{r-1}(a)}.$$

Further, if in (77)

$$f(t)=\left(\sum_{s=0}^{n}b_s t^s\right)^{-1}\left(\sum_{s=0}^{n}a_s t^s\right) \qquad (a_s, b_s\in N,\ s=0,1,\dots,n),$$

then

$$\varrho_{2n}(t)=b_n^{-1}a_n;$$

and if

$$f(t)=\left(\sum_{s=0}^{n}a_s t^s\right)\left(\sum_{s=0}^{n}b_s t^s\right)^{-1} \qquad (a_s, b_s\in N,\ s=0,1,\dots,n),$$

then

$$\varrho_{2n}(t)=a_n b_n^{-1}.$$

There is a further theorem relating to the confluent form of the ϱ-algorithm analogous to Theorem 8.6.

9. Conclusion

The adaptive formulation announced in the introductory paragraphs has now been completed. There are nevertheless two points which deserve further comment.

The first of these concerns the fact that the variable z in the expansions derived from power series is restricted to be scalar. This is brought about mainly by two considerations. First, the proof of the $q-d$ algorithm which we have given stands only if z is scalar (otherwise the odd part of the continued fraction

$$\frac{c_m}{z-}\frac{q_1^{(m)}}{I-}\frac{e_1^{(m)}}{z-}\cdots\frac{q_r^{(m)}}{I-}\frac{e_r^{(m)}}{z-}\cdots$$

does not exist). Further difficulties arise with regard to the functions $o_r^{(m)}(z)$ and $p_r^{(m)}(z)$. Now both of these, in satisfying three-term recursions of a certain form, are polynomials in z. Admittedly the term which, if z were scalar would correspond to $k_{n,s}^{(m)} z^s$, is compounded of a number of products in which the position of z varies, but nevertheless within this wider definition $o_r^{(m)}(z)$ and $p_r^{(m)}(z)$ are polynomials. But from the definition (15) $o_r^{(m)}(z)$ cannot possibly be a polynomial, and the formalism therefore breaks down completely.

The second point concerns the difficulties to be expected in constructing a convergence theory for the continued fractions of this paper. Let us examine one example in particular: The continued fraction

$$\mathrm{pre}\left[B+\frac{A}{B+}\frac{A}{B+}\cdots\frac{A}{B+}\cdots\right] \tag{78}$$

has as fundamental recursions

$$\mathrm{pre}[A_n = B A_{n-1} + A A_{n-2},$$
$$B_n = B B_{n-1} + A B_{n-2}].$$

Since, when $A, B \in S$, the roots of the equation

$$x^2 - Bx - A = 0 \tag{79}$$

may be determined and the initial values A_{-1}, A_0, B_{-1}, B_0 are known, closed expressions may be derived for A_n and B_n (and hence for C_n), and thus a complete description of the convergence behavior of (78) may be given.

When A and B are general elements of N, however, equation (79) may not have a finite number of solutions (consider, for example, the case in which A and B are 2×2 matrices), and thus the conventional treatment can not immediately be adapted.

Such a theory, as has been said, undoubtedly exists, and its derivation must be regarded as the next and most important project in this field of research.

Appendix I. The Theory of Determinants

Since the theory of determinants was well established before the theory of continued fractions, it was quite natural that the former should be used as a tactical weapon in the development of the latter, and it is indeed by inspection of various determinantal formulae that certain results become very obvious. This has not been the case with continued fractions of the type considered in this paper, and it is important to discuss the reasons for this.

The determinantal expression

$$\left(\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}\right)$$

in which the $a_{i,j}$ $(i, j = 1, 2, \ldots, n) \in N$ may be considered from two points of view.

First Definition. First, simply as a determinant for which we must additionally specify the order in which the elements are multiplied together. This presents no difficulty. We can impose arbitrarily the condition that the elements are to be multiplied according to the direct order (pre-system) or inverse order (post-system) of their row suffix, and this leads to

$$\text{pre}'\left(\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}\right) = \Sigma \pm a_{1\alpha}\, a_{2\beta} \ldots a_{n\nu} \tag{80}$$

where $\alpha, \beta, \ldots, \nu$ are all possible permutations of the integers $1, 2, \ldots, n$ and the signs attached to the products depend upon the class of the permutations. (The dash is used to distinguish these determinants from others which will be defined later.) A number of the properties of such determinants follow immediately. Determinants having two identical rows are not necessarily equal to 0, but those having two identical columns are; we may thus add scalar multiples of various columns to each other without changing the value of the determinant. We must remember, when discussing the expansion of a determinant, that each term has a pre-cofactor and a post-cofactor. With this qualification in mind, analogues of the Cauchy and Laplace expansions may be established.

Such determinants do find some application in the theory of the continued fractions of this paper. For example, by use of the fundamental recursion formulae, we have

$$\text{post}\, A_n = \text{pre}'\left(\begin{vmatrix} b_0 & -I & 0 & 0 & \cdot & \cdot & \cdot \\ a_1 & b_1 & -I & 0 & \cdot & \cdot & \cdot \\ 0 & a_2 & b_2 & -I & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a_{n-2} & b_{n-2} & -I & 0 \\ \cdot & \cdot & \cdot & 0 & a_{n-1} & b_{n-1} & -I \\ \cdot & \cdot & \cdot & 0 & 0 & a_n & b_n \end{vmatrix}\right)$$

and

$$\text{post}\, B_n = \frac{\partial}{\partial b_0}\, \text{post}\, A_n;$$

further,

$$\text{pre}\, A_n = \text{pre}'\left(\begin{vmatrix} b_n & a_n & 0 & 0 & \cdot & \cdot & \cdot \\ -I & b_{n-1} & a_{n-1} & 0 & \cdot & \cdot & \cdot \\ 0 & -I & b_{n-2} & a_{n-2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & b_2 & a_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & -I & b_1 & a_1 \\ \cdot & \cdot & \cdot & \cdot & 0 & -I & b_0 \end{vmatrix}\right)$$

and

$$\mathrm{pre}\, B_n = \frac{c}{b_0}\, \mathrm{pre}\, A_n .$$

Using these formulae we have

Theorem I.1. *If A'_n refers to the continued fraction*

$$b_n + \frac{a_n}{b_{n-1}+} \frac{a_{n-1}}{b_{n-2}+} \cdots \frac{a_1}{b_0},$$

then

$$\mathrm{post}\, A'_n = \mathrm{pre}\, A_n ,$$
$$\mathrm{pre}\, A'_n = \mathrm{post}\, A_n .$$

Also we have

Theorem I.2. *In the notation of Theorem* 1.I

$$\mathrm{pre}\left[b_n + \frac{a_n}{b_{n-1}+} \frac{a_{n-1}}{b_{n-2}+} \cdots \frac{a_1}{b_0}\right] = \mathrm{post}\,[A_{n-1}^{-1} A_n],$$

$$\mathrm{post}\left[b_n + \frac{a_n}{b_{n-1}+} \frac{a_{n-1}}{b_{n-2}+} \cdots \frac{a_1}{b_0}\right] = \mathrm{pre}\,[A_n A_{n-1}^{-1}],$$

$$\mathrm{pre}\left[b_n + \frac{a_n}{b_{n-1}+} \frac{a_{n-1}}{b_{n-2}+} \cdots \frac{a_2}{b_1}\right] = \mathrm{pre}\,[B_{n-1}^{-1} B_n],$$

$$\mathrm{post}\left[b_n + \frac{a_n}{b_{n-1}+} \frac{a_{n-1}}{b_{n-2}+} \cdots \frac{a_2}{b_1}\right] = \mathrm{pre}\,[B_n B_{n-1}^{-1}].$$

The difficulty in the further application of such determinants lies simply in the fact that there is no such thing as pivotal condensation. One of the more pernicious consequences of this is that there is no analogue of the theorem on compound determinants (see *e.g.* [3], p. 49) which has such fruitful application in this domain of inquiry.

Consider for example the expression

$$\mathrm{pre}'\left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\right) \mathrm{pre}'\left(\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ d_1 & d_2 & d_4 \end{vmatrix}\right) - \\ - \mathrm{pre}'\left(\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}\right) \mathrm{pre}'\left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\right).$$

This is the Laplace expansion (by the first three rows) of

$$\mathrm{pre}'\left(\begin{vmatrix} a_1 & a_2 & a_3 & 0 & 0 & a_4 \\ b_1 & b_2 & b_3 & 0 & 0 & b_4 \\ c_1 & c_2 & c_3 & 0 & 0 & c_4 \\ 0 & 0 & a_3 & a_1 & a_2 & a_4 \\ 0 & 0 & b_3 & b_1 & b_2 & b_4 \\ 0 & 0 & d_3 & d_1 & d_2 & d_4 \end{vmatrix}\right). \qquad (81)$$

Now if $a_s, b_s, \ldots \in S$, then (81) may be transformed into

$$\text{pre}'\left(\begin{vmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_1 & c_2 & c_4 \\ 0 & 0 & a_3 & a_1 & a_2 & a_4 \\ 0 & 0 & b_3 & b_1 & b_2 & b_4 \\ 0 & 0 & d_3 & d_1 & d_2 & d_4 \end{vmatrix}\right) \tag{82}$$

or

$$\text{pre}'\left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right) \text{pre}'\left(\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}\right);$$

that is, from the cross product of determinants of the third order, one of fourth order has been built up, and this result is, of course, easily generalised. But if the $a_s, b_s, \ldots \in N$ and are not scalar, then the step from (81) to (82) is not possible, and the matter ends there.

Now in the conventional theory of continued fractions it is, for example, possible to express the coefficients of a continued fraction as determinantal quotients involving the coefficients in an associated power series. As the suffix of the continued fraction coefficient increases by steps of unity the order of the determinants in these quotients does likewise. Certain non-linear recursion systems exist which relate the coefficients in a power series and those in associated continued fractions; they may be proved by appeal to the theory of compound determinants. But we see that in the present treatment this is not possible with expressions of the form (80).

Second Definition. We might also consider the determinantal expressions which arise in the solution of linear equations, for example

$$\begin{aligned} a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,n} x_n &= h_1, \\ a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,n} x_n &= h_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n,1} x_1 + a_{n,2} x_2 + \cdots + a_{n,n} x_n &= h_n. \end{aligned} \tag{83}$$

If $a_{i,j}, h_i, x_j \in N$, in general there is no difficulty in principle in determining a solution for x_j of the form

$$x_j = \{\text{pre}''\,(|a_{i,1}, a_{i,2}, \ldots, a_{i,j-1}, a_{i,j}, a_{i,j+1}, \ldots, a_{1,n}|)\;\; i = 1, 2, \ldots, n\}^{-1} \times$$
$$\times \{\text{pre}''\,(|(a_{i,1}, a_{i,2}, \ldots, a_{i,j-1}, h_i, a_{i,j+1}, \ldots, a_{1,n}|)\;\; i = 1, 2, \ldots, n\}.$$

(By reason of the regularity of the determinants in question, it has been possible to introduce a condensed notation which uses the elements of the i^{th} row only.)

However, using the r^{th} equation, we may eliminate the variable x_s from the set (83). This leads to the recursive definition

$$\begin{aligned} &\mathrm{pre}''(|a_{i,1}, a_{i,2}, \ldots, a_{i,s-1}, a_{i,s}, a_{i,s+1}, \ldots, a_{i,n}|) \quad i=1,2,\ldots,n \\ &= \mathrm{pre}''(|(a_{i,1} - a_{i,s}\, a_{r,s}^{-1}\, a_{r,1}), (a_{i,2} - a_{i,s}\, a_{r,s}^{-1}\, a_{r,2}), \ldots \\ &\quad \ldots, (a_{i,s-1} - a_{i,s}\, a_{r,s}^{-1}\, a_{r,s-1}), (a_{i,s+1} - a_{i,s}\, a_{r,s}^{-1}\, a_{r,s+1}), \ldots \\ &\quad \ldots, (a_{i,n} - a_{i,s}\, a_{r,s}^{-1}\, a_{r,n})|) \qquad i=1,2,\ldots,r-1, r+1,\ldots,n \end{aligned}$$

with

$$\mathrm{pre}''(|a_{1,1}|) = a_{1,1}$$

for the determinants of the second definition; pivotal condensation does exist (indeed it is the essence of the definition). We may proceed to build up a general theory of such determinants in analogy with the conventional theory, but this time there is no immediately tangible expansion of the form (80). They are, therefore, unsuited to our purpose. We were forced at each step to develop the theory of the continued fractions of this paper using recursion systems alone, and if at any point this had been impossible, the theory would immediately have collapsed.

Appendix II. Applications

That part of this paper which has so far found application concerns the epsilon algorithm. It will be recalled that the fundamental result of this algorithm may be stated as follows: If we have a sequence $S_m \in N$ $(m=0, 1, \ldots)$ which obeys a recursion of the form

$$\sum_{i=0}^{n} b_i\, S_{m+i} = \left(\sum_{i=0}^{n} b_i\right) A \quad (m=0,1,\ldots) \qquad b_i \in S \ (i=0,1,\ldots,n), \qquad A \in N,$$

and if we put

$$\varepsilon_{-1}^{(m)} = O, \qquad \varepsilon_0^{(m)} = S_m \qquad (m=0,1,\ldots)$$

and determine further quantities $\varepsilon_s^{(m)}$ by applying the relation

$$\varepsilon_s^{(m)} = \varepsilon_{s-2}^{(m+1)} + (\varepsilon_{s-1}^{(m+1)} - \varepsilon_{s-1}^{(m)})^{-1}, \tag{84}$$

then

$$\varepsilon_{2n}^{(m)} = A \qquad (m=0,1,\ldots).$$

Let us discuss a problem to which this result may be applied. Suppose that we are to solve the set of n' linear equations in n' unknown quantities x_s $(s=1,2,\ldots,n')$ expressible in conventional matrix-vector notation as

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{h}.$$

One method of treating this is as follows. Write

$$\boldsymbol{B} = \boldsymbol{L} + \boldsymbol{U}$$

where $\boldsymbol{L}$ is a lower triangular matrix of order n' and $\boldsymbol{U}$ is an upper triangular matrix of the same order in which the principal diagonal contains zero elements only. From an initial estimate $\boldsymbol{x}^{(0)}$ of the solution vector $\boldsymbol{x}$, obtain further iterated estimates $\boldsymbol{x}^{(m+1)}$ $(m=0,1,\ldots)$ by means of the scheme

$$\boldsymbol{L}\boldsymbol{x}^{(m+1)} = \boldsymbol{h} - \boldsymbol{U}\boldsymbol{x}^{(m)} \qquad (m=0,1,\ldots).$$

(This is easily done, the determination of each component of $\boldsymbol{x}^{(m+1)}$ requiring one division and $n'-1$ additions or subtractions.) As is easily verified, $\boldsymbol{x}^{(m)}$ is given by

$$\boldsymbol{x}^{(m)} = \{\boldsymbol{L}^{-1}\boldsymbol{U}\}^m \boldsymbol{k} + \{\boldsymbol{L}+\boldsymbol{U}\}^{-1}\boldsymbol{h}$$

where the vector $\boldsymbol{k}$ depends upon $\boldsymbol{A}$ and $\boldsymbol{x}^{(0)}$. This is the Gauss-Seidel iteration scheme. It converges if the moduli of all the eigenvalues of $\boldsymbol{L}^{-1}\boldsymbol{U}$ are less than unity.

Now in general the matrix $\boldsymbol{L}^{-1}\boldsymbol{U}$ is of rank $n'-1$ and satisfies its own characteristic equation; *i.e.* there exists a relation of the form

$$\sum_{s=0}^{n'-1} b_s \{\boldsymbol{L}^{-1}\boldsymbol{U}\}^s = 0,$$

where the b_s $(s=0, 1, \ldots, n'-1)$ are scalar. Using this, we have

$$\sum_{s=0}^{n'-1} b'_s\, \boldsymbol{x}^{(m+s)} = \left(\sum_{s=0}^{n'-1} b'_s\right)(\boldsymbol{L}+\boldsymbol{U})^{-1}\boldsymbol{h}.$$

Thus if we write

$$\varepsilon_{-1}^{(m)} = 0, \qquad \varepsilon_0^{(m)} = \boldsymbol{x}^{(m)} \qquad (m=0, 1, \ldots)$$

and determine further vectors $\varepsilon_s^{(m)}$ by applying the ε-algorithm relations (54), then we should expect that

$$\varepsilon_{2n'-2}^{(m)} = \boldsymbol{x} \qquad (m=0, 1, \ldots),$$

and this (*cf.* [*1*]) turns out to be the case. (For the inverse of a vector a suggestion* of K. Samelson has been exploited. He defines the inverse of the vector $(y_1, y_2, \ldots, y_n) = \boldsymbol{y}$ by

$$\boldsymbol{y}^{-1} = (y_1, y_2, \ldots, y_n)^{-1} = \left(\sum_{i=1}^{n} y_i \bar{y}_i\right)^{-1} (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n).$$

($\boldsymbol{y}^{-1}$ is thus the inverse point of $\boldsymbol{y}$ with respect to the unit sphere in n-space.)

We introduce the second example by recalling that if the quantities $\varepsilon_0^{(m)}$ $(m=0, 1, \ldots)$ are the partial sums of the series $\sum_{s=0}^{n} c_s z^{-s-1}$, then the quantities $\varepsilon_{2s}^{(m)}$ $(m=0, 1, \ldots)$ $(s=1, 2, \ldots)$ are convergents of various continued fractions which may be associated with this power series. We should expect then (although no convergence theory has been given) that the sequence $\varepsilon_{2s}^{(0)}$ $(s=0, 1, \ldots)$, for example, should converge more rapidly in certain cases than the sequence $\varepsilon_0^{(m)}$ $(m=0, 1, \ldots)$.

Consider the following problem: We are concerned with solving the boundary-value problem set by seeking the solution of

$$\varphi_{xy} = \tfrac{1}{2}\{\varphi_{xx} + \varphi_{yy}\} \tag{85}$$

subject to given values of $\varphi(x, y)$ on the lines parallel to the axes joining the points in the $x-y$ plane

$$(1, 1), \quad (1+[n+1]\,d, 1), \quad (1+[n+1]\,d, 1+[n+1]\,d), \quad (1, 1+[n+1]\,d)$$

* This is equivalent to a suggestion of C. Lanczos [*17*].

in the following way. We render the problem discrete by considering the function values $\varphi(1+jd, 1+[n+1-i]d)$ $(i, j=1, 2, \ldots, n)$ and obtain a finite difference approximation to (85) by use of the well known grid formulae

$$\begin{array}{|ccc|}\hline -1 & \cdot & 1 \\ \cdot & 0 & \cdot \\ 1 & \cdot & -1 \\ \hline \end{array} \equiv 4d^2 \frac{\partial^2}{\partial x\, \partial y}, \quad \begin{array}{|ccc|}\hline 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \\ \hline \end{array} \equiv 6d^2 \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\}.$$

The problem is solved iteratively as follows: we determine $\varphi^{(0)}$ by solving the equation

$$\varphi_{xy}^{(0)} = 0$$

with the given boundary conditions and thereafter obtain a sequence of iterated solutions $\varphi^{(m)}$ $(m=1, 2, \ldots)$ from the equation

$$\varphi_{xy}^{(m)} = \tfrac{1}{2}\left\{\varphi_{xx}^{(m-1)} + \varphi_{yy}^{(m-1)}\right\}. \tag{86}$$

What we obtain in effect is of course a sequence of $n \times n$ matrices. With appropriate boundary conditions, equation (85) has the simple analytic solution $x/(x+y)$; we are thus in a position to furnish an indication of the accuracy of various estimates of the solution by considering the norm

$$n^{-2} \sum_{i,j}^{n} |\varphi^{(m)}(1+jd, 1+[n+1-i]d) - (1+jd)/(2+[n+1+j-i]d)|, \tag{87}$$

and similarly for the transformed estimates.

The functions $\varepsilon_s^{(m)}$ of the ε-algorithm may be placed in the scheme (the ε-array)

$$\begin{array}{ccccccccc}
 & \varepsilon_0^{(0)} & & & & & & & \\
\varepsilon_{-1}^{(1)} & & \varepsilon_1^{(0)} & & & & & & \\
 & \varepsilon_0^{(1)} & & \varepsilon_2^{(0)} & & & & & \\
\varepsilon_{-1}^{(2)} & & \varepsilon_1^{(1)} & & \varepsilon_3^{(0)} & & & & \\
 & \varepsilon_0^{(2)} & & \varepsilon_2^{(1)} & & \ddots & & & \\
\varepsilon_{-1}^{(3)} & & \varepsilon_1^{(2)} & & \ddots & & \varepsilon_s^{(0)} & & \\
 & \varepsilon_0^{(3)} & & \ddots & & & & \varepsilon_{s+1}^{(0)} & \\
 & & \ddots & & & & \varepsilon_s^{(1)} & & \ddots \\
 & & & & & & & \varepsilon_{s+1}^{(1)} & \\
 & & & & & & \varepsilon_s^{(2)} & & \ddots \\
 & & & & & & & \varepsilon_{s+1}^{(2)} & \\
 & & & & & & & & \ddots
\end{array}$$

in which the quantities in (84) occur at the vertices of a lozenge. As has been said, only the even order columns of the ε-array provide useful estimates of the required limit, and for this reason we give in the Table the means (87) applied to the even columns of the ε-array derived from the iterative scheme (86) when $n=4$, $d=0{,}25$.

Table

m	s 0	2	4	6	8	10
0	.62 (−3)					
1	.77 (−2)	.41 (−3)				
2	.15 (0)	.24 (−2)	.40 (−3)			
3	.29 (+1)	.12 (−1)	.13 (−2)	.53 (−4)		
4	.56 (+2)	.96 (−1)	.25 (−1)	.11 (−3)	.90 (−5)	
5	.11 (+4)	.52 (0)	.56 (−1)	.21 (−3)	.21 (−4)	.34 (−5)
6	.21 (+5)	.40 (+1)	.13 (+1)	.70 (−3)	.12 (−3)	
7	.40 (+6)	.23 (+2)	.23 (+2)	.19 (−2)		
8	.77 (+7)	.17 (+3)	.37 (+2)			
9	.15 (+9)	.99 (+3)				
10	.28 (+10)					

(The numbers in this table are normalised; the bracketed figure is the power of ten by which each entry must be multiplied.)

It will be observed that the original sequence (corresponding to the sequence $\varepsilon_0^{(m)}$ $(m=0, 1, \ldots)$) diverges strongly, but that the transformed sequence $\varepsilon_{2s}^{(0)}$ $(s=0, 1, \ldots)$ appears to converge quite reasonably to the finite difference approximation to the required solution of (85).

This type of numerical behaviour is precisely the same as that which attends the application of the ε-algorithm to divergent and asymptotically convergent series. The described example serves therefore to illustrate the use of the ε-algorithm as a device for accelerating the convergence of a slowly convergent sequence of elements which satisfy a non-commutative law of multiplication, and to furnish an indication that the continued fractions which have been the subject of this paper indeed possess a convergence theory.

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