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A Note on a Method of Bradshaw for Transforming Slowly Convergent Series and Continued Fractions

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A NOTE ON A METHOD OF BRADSHAW FOR TRANSFORMING SLOWLY CONVERGENT SERIES AND CONTINUED FRACTIONS

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1. The purpose of this note is to place in the perspective of a more general inquiry certain methods for transforming slowly convergent series and continued fractions proposed by Bradshaw [1], [2], [11].

2. The series which he treats are of two forms, viz.

$$S = \sum_{n=1}^{\infty} a_n, \tag{1}$$

and

$$S' = \sum_{n=1}^{\infty} (-1)^{n-1} a_n, \tag{2}$$

where a_n is the function

$$a_n = \left(\sum_{s=0}^{p} t_s n^s \right)^{-1}. \tag{3}$$

He modifies the series (1) and (2) by the term by term addition of the series

$$-b_0 = \sum_{n=1}^{\infty} (b_n - b_{n-1}) \tag{4}$$

and determines the quantities b_n by imposing the condition that if the magnitude of the successive terms in (1) and (2) behave like $n^{-\beta}$, the terms in the transformed series should behave like $n^{-\beta-\alpha}$, or more precisely

$$1 + \frac{b_n - b_{n-1}}{a_n} = O(n^{-\alpha}). \tag{5}$$

The functions b_n in the case of the series (1) and $(-1)^n b_n$ in the case of (2) are identified as a sequence of rational functions for $\alpha = 0, 1, 2, \cdots$ whose coefficients may be derived from a system of linear equations from those in (3) and which may be established as successive convergents of a continued fraction.

Using his method he derived the expansions

$$\sum_{r=1}^{\infty} r^{-2} = \sum_{r=1}^{n} r^{-2} + \frac{2}{2n+1+}\,\frac{1}{3(2n+1)+}\,\frac{16}{5(2n+1)+}\cdots\frac{s^4}{(2s+1)(2n+1)+}\cdots \tag{6}$$

and

$$\sum_{r=1}^{\infty} (-1)^{r-1} r^{-1} = \sum_{r=1}^{n} (-1)^{r-1} r^{-1} + (-1)^n \frac{1}{2n+1+}\,\frac{1}{2n+1+}\,\frac{4}{2n+1+}\cdots\frac{s^2}{2n+1+}\cdots. \tag{7}$$

3. Continuing the work of Stieltjes [3], [4] and Airey [5], Bickley and Miller [6], [7] have devised a method for transforming slowly convergent series which is applicable to a larger class of series than those given by (1) and (2) (as, in the event, is the method of Bradshaw).

Defining the converging factor C_n by the relation

$$R_n = u_n C_n, \tag{8}$$

where R_n is the remainder term of the slowly convergent series

$$S = \sum_{r=1}^{\infty} u_r \tag{9}$$

and is given by

$$S = \sum_{r=1}^{r-1} u_r + R_n, \tag{10}$$

they show that if it is possible to determine constants ρ_s, $s=0, 1, \cdots$, such that

$$\frac{u_{n+1}}{u_n} = \rho_0 + \rho_1 n^{-1} + \cdots + \rho_{s-1} n^{-s+1} + O(n^{-s}), \tag{11}$$

then a series

$$C_n \sim \sum_{s=-1}^{+\infty} \alpha_s n^{-s} \tag{12}$$

may be established. Again the coefficients α_s, $s = -1, 0, 1, \cdots$, are derived from a set of linear equations based upon the condition

$$C_n = \sum_{s=-1}^{n-1} \alpha_s n^{-s} + O(n^{-h}). \tag{13}$$

(Actually the formalism differs in the two cases $\rho_0 = 0$, $\rho_0 \neq 0$.)

Using their method they obtained the expansion

$$\sum_{r=1}^{\infty} r^{-2} = \sum_{r=1}^{n-1} r^{-2} + n^{-2}\left\{n - \frac{1}{2} + \sum_{r=1}^{\infty} (-1)^r B_r n^{-2r+1}\right\} \tag{14}$$

and

$$\sum_{r=1}^{\infty} (-1)^{r-1} r^{-1} = \sum_{r=1}^{n-1} (-1)^{r-1} r^{-1} + (-1)^n \sum_{s=0} (-1)^{s-1} T_s (2n)^{-2s-1}. \tag{15}$$

4. It is possible, by using a variety of methods [8], the most efficient of which is provided by the $q-d$ algorithm [9], uniquely to determine the coefficients of the Stieltjes type continued fraction

$$\frac{c_0}{z-}\,\frac{q_1^{(0)}}{1-}\,\frac{e_1^{(0)}}{z-} - \cdots \frac{q_r^{(0)}}{1-}\,\frac{e_r^{(0)}}{z-} \cdots \tag{16}$$

from those of certain formal series

$$\sum_{s=0}^{\infty} c_s z^{-s-1} \tag{17}$$

by imposing the conditions that

$$\sum_{s=0}^{\infty} c_s z^{-s-1} - \frac{c_0}{z-}\,\frac{q_1^{(0)}}{1-}\,\frac{e_1^{(0)}}{z-}\cdots\frac{q_r^{(0)}}{1} = O(z^{-2r-1}) \tag{18}$$

and

$$\sum_{s=0}^{\infty} c_s z^{-s-1} - \frac{c_0}{z-}\,\frac{q_1^{(0)}}{1-}\,\frac{e_1^{(0)}}{z-}\cdots\frac{e_r^{(0)}}{z} = O(z^{-2r-2}). \tag{19}$$

The coefficients in the continued fraction

$$\frac{c_0}{z-\alpha_0^{(0)}-}\,\frac{\beta_0^{(0)}}{z-\alpha_1^{(0)}-}\cdots\frac{\beta_{r-1}^{(0)}}{z-\alpha_r^{(0)}}\cdots \tag{20}$$

which is the even part of (16) may be determined from the relation

$$\sum_{s=0}^{\infty} c_s z^{-s-1} - \frac{c_0}{z-\alpha_0{}^{(0)}-} - \frac{\beta_0^{(0)}}{z-\alpha_1{}^{(0)}-}\cdots - \frac{\beta_{r-1}^{(0)}}{z-\alpha_r{}^{(0)}-} = O(z^{-2r-1}). \tag{21}$$

The $q-d$ algorithm relationships are

$$q_r^{(m)} + e_r^{(m)} = q_r^{(m+1)} + e_{r-1}^{(m+1)}, \qquad q_{r+1}^{(m)} e_r^{(m)} = q_r^{(m+1)} e_r^{(m+1)}, \tag{22}$$

and the starting values are

$$e_0^{(m)} = 0, \qquad q_1^{(m)} = c_{m+1}/c_m, \qquad m = 0, 1, \cdots. \tag{23}$$

The coefficients in (20) are related to those in (16) by

$$\alpha_{r+1}^{(0)} = q_{r+1}^{(0)} + e_r^{(0)}, \qquad \beta_{r-1}^{(0)} = q_r^{(0)} e_r^{(0)}. \tag{24}$$

5. From the preceding remarks it can be seen that Bradshaw's continued fractions may be obtained from the Bickley-Miller expansions by expanding $u_n C_n$ as a series in inverse powers of n and applying the $q-d$ algorithm relationship to the coefficients of this series. In general this procedure is more efficient than that proposed by Bradshaw, for the derivation of each of his rational functions necessitates the solution of a completely independent set of linear equations; the Bickley-Miller method and the $q-d$ algorithm are however recursive procedures in which the coefficients in the approximation of one degree assist in the computation of those in the next.

In particular it can be seen that the expansions (14) and (15) may be derived by expanding the integrals

$$\int_0^{\infty} e^{-nt}\operatorname{sech} t\,dt \tag{25}$$

and

$$\int_0^\infty te^{-(2n+1)t}(\sinh t)^{-1}dt \tag{26}$$

in inverse powers of n, while (6) and (7) may be derived from the Stieltjes [10] expansions

$$\int_0^\infty e^{-zt}\operatorname{sech} t dt = \frac{1}{z+}\,\frac{1^2}{z+}\,\frac{2^2}{z+}\cdots\frac{(r-1)^2}{z+}\cdots \tag{27}$$

and

$$\int_0^\infty te^{-zt}(\sinh t)^{-1}dt = \frac{1}{z+}\,\frac{1^4}{3z+}\,\frac{2^4}{5z+}\cdots. \tag{28}$$

6. Bradshaw has also applied his method to the transformation of slowly convergent continued fractions [9]. He modifies the slowly convergent continued fraction

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_r}{b_r+}\cdots \tag{29}$$

by transforming the ρth convergent

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_\rho}{b_\rho} \tag{30}$$

of (29) by the inclusion of a term d_ρ, thus

$$b_0 + \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_\rho}{d_\rho}, \tag{31}$$

where successive modifying terms d_ρ are determined by imposing the condition that

$$d_{\rho-1} - b_{\rho-1} - \frac{a_\rho}{d_\rho} = O(\rho^{-\alpha}), \qquad \alpha = 1, 2, \cdots. \tag{32}$$

Using this method he transforms the slowly convergent continued fraction

$$1 + \frac{2}{8n-4+}\,\frac{1.3}{8n-4+}\,\frac{3.5}{8n-4+}\cdots\frac{(2\rho-1)(2\rho+1)}{8n-4+}\cdots \tag{33}$$

by means of the modifying continued fraction

$$\mathrm{d}_\rho = 2\left\{\frac{2\rho+4n-1}{2+}\,\frac{(2n-1)2n}{2\rho+1+}\cdots\frac{(2n-2+\sigma)(2n-1+\sigma)}{2\rho+1+}\cdots\right\}. \tag{34}$$

7. Wynn [12] has also proposed a method for the numerical transformation of slowly convergent continued fractions of the form

$$\frac{a_0}{b_0+}\frac{c_0}{d_0+}\cdots\frac{y_0}{z_0+}\frac{a_1}{b_1+}\cdots\frac{y_1}{z_1+}\frac{a_2}{b_2+}\cdots, \tag{35}$$

where $a_n, b_n, \cdots, z_n$ are polynomials in n. He determines the coefficients in the formal expansion

$$u_n = \sum_{s=-k}^{\infty} \alpha_s n^{-s}, \tag{36}$$

where

$$u_n = \frac{a_n}{b_n+}\frac{c_n}{d_n+}\cdots\frac{y_n}{z_n+}\frac{a_{n+1}}{b_{n+1}+}\frac{c_{n+1}}{d_{n+1}+}\cdots \tag{37}$$

by imposing the condition that

$$u_n - \sum_{s=-k}^{h} \alpha_s n^{-s} = O(n^{-h-1}). \tag{38}$$

He develops a formal recursive procedure based upon use of the difference equation

$$u_n = \frac{a_n}{b_n+}\frac{c_n}{d_n+}\cdots\frac{y_n}{z_n+u_{n+1}}, \tag{39}$$

which is equivalent to

$$p(n)u_n + q(n)u_{n+1} + r(n)u_n u_{n+1} = s(n), \tag{40}$$

where $p(n)$, $q(n)$, $r(n)$, $s(n)$ are again polynomials in n.

8. Again it may be seen that Bradshaw's continued fractions may be derived more efficiently by applying the $q-d$ algorithm to the coefficients of the Wynn expansion.

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