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ON A CONNECTION BETWEEN THE FIRST AND SECOND CONFLUENT FORMS OF THE ε-ALGORITHM †

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In previous papers [1] [2] two confluent forms

$$\{\varepsilon_{s+1}(t) - \varepsilon_{s-1}(t)\} \frac{d}{dt} \varepsilon_s(t) = 1, \qquad (s = 0, 1, \ldots) \tag{1}$$

$$\varepsilon_{-1}(t) = 0, \ \varepsilon_0(t) = f(t) \tag{2}$$

and

$$\{\varepsilon^*_{2s+1}(t) - \varepsilon^*_{2s-1}(t)\}\{\frac{d}{dt} \varepsilon^*_{2s}(t) + f(t)\} = 1, \quad (s = 0, 1, \ldots) \tag{3}$$

$$\{\varepsilon^*_{2s+2}(t) - \varepsilon^*_{2s}(t)\} \frac{d}{dt} \varepsilon^*_{2s+1}(t) = 1, \quad (s = 0, 1, \ldots) \tag{4}$$

$$\varepsilon^*_{-1}(t) = \varepsilon^*_0(t) = 0 \tag{5}$$

of the ε-algorithm are given.

If the notations

$$H_k^{(m)} = \begin{vmatrix} f^{(m)}(t) & f^{(m+1)}(t) & \ldots & f^{(m+k-1)}(t) \\ f^{(m+1)}(t) & f^{(m+2)}(t) & \ldots & f^{(m+k)}(t) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f^{(m+k-1)}(t) & f^{(m+k)}(t) & \ldots & f^{(m+2k-2)}(t) \end{vmatrix}, \ H_0^{(m)} = 1, \tag{6}$$

and

$\hat{H}_k^{(m)} = (H_k^{(m)}$ with the leading element replaced by zero)

are adopted, then it is shown that

$$\varepsilon_{2s}(t) = \frac{H_{s+1}^{(0)}}{H_s^{(2)}}, \quad \varepsilon_{2s+1}(t) = \frac{H_s^{(3)}}{H_{s+1}^{(1)}}, \qquad (s = 0, 1, \ldots) \tag{7}$$

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and

$$\varepsilon_{2s}^*(t) = \frac{\hat{H}_{s+1}^{(-1)}}{H_s^{(1)}}, \quad \varepsilon_{2s+1}^*(t) = \frac{H_s^{(2)}}{H_{s+1}^{(0)}}. \qquad (s = 0, 1, \ldots) \tag{8}$$

It has been shown that under certain conditions

$$\varepsilon_{2n}(t) = \lim_{t\to\infty} f(t) \tag{9}$$

and that

$$\varepsilon_{2n}^*(t) = \textstyle\int_t^\infty f(t)\,dt. \tag{10}$$

The question which it is the purpose of this note to answer is this: can the functions produced by one algorithm be produced by means of the other?

In order to answer this question we shall first generalise the notation used, and denote by $\varepsilon_s^{(m)}(t)$ the functions produced by applying the relationships (1) to the initial conditions

$$\varepsilon_{-1}^{(m)}(t) = 0, \ \varepsilon_0^{(m)}(t) = f^{(m)}(t); \tag{11}$$

and further denote by $\varepsilon_s^{(m)*}(t)$ the functions produced by applying the relationships

$$\{\varepsilon_{2s+1}^{(m)*}(t) - \varepsilon_{2s-1}^{(m)*}(t)\}\{\frac{d}{dt}\,\varepsilon_{2s}^{(m)*}(t) + f^{(m)}(t)\} = 1 \quad (s = 0, 1, \ldots) \tag{12}$$

in conjunction with (4) to the initial conditions (5).

Relationships (7) and (8) evolve to

$$\varepsilon_{2s}^{(m)}(t) = \frac{H_{s+1}^{(m)}}{H_s^{(m+2)}}, \ \varepsilon_{2s+1}^{(m)}(t) = \frac{H_s^{(m+3)}}{H_{s+1}^{(m+1)}}, \qquad (s = 0, 1, \ldots) \tag{13}$$

and

$$\varepsilon_{2s}^{(m)*}(t) = \frac{\hat{H}_{s+1}^{(m-1)}}{H_s^{(m+1)}}, \ \varepsilon_{2s+1}^{(m)*}(t) = \frac{H_s^{(m+2)}}{H_{s+1}^{(m)}}. \qquad (s = 0, 1, \ldots) \tag{14}$$

Introducing the notation

$$f^{(-1)}(t) = \textstyle\int_0^t f(t)\,dt \tag{15}$$

we have, by inspection of equations (13) and (14),

$$\varepsilon_{2s+1}^{(m)*}(t) = \varepsilon_{2s+1}^{(m-1)}(t), \qquad (m, s = 0, 1, \ldots) \tag{16}$$

$$\varepsilon_{2s}^{(m)*}(t) = \varepsilon_{2s}^{(m-1)} - f^{(m-1)}(t); \qquad (m, s = 0, 1, \ldots) \tag{17}$$

and from

$$\varepsilon_{2s+1}^{(m-1)}(t) = \{\varepsilon_{2s}^{(m)}(t)\}^{-1} \qquad (m, s = 0, 1, \ldots) \tag{18}$$

the further relationships

$$\varepsilon_{2s}^{(m)*}(t) = \{\varepsilon_{2s+1}^{(m-2)}(t)\}^{-1} - f^{(m-1)}(t), \qquad (m, s = 0, 1, \ldots) \qquad (19)$$

$$= \{\varepsilon_{2s+1}^{(m-1)*}(t)\}^{-1} - f^{(m-1)}(t). \qquad (m, s = 0, 1, \ldots) \qquad (20)$$

We can describe equations (16) and (17) by the following

Theorem: *If the first confluent form of the ε-algorithm is applied to the function* $f^{(m-1)}(t)$ *to produce functions* $\varepsilon_s^{(m-1)}(t)$ *and the second confluent form of the ε-algorithm is applied to the function* $f^{(m)}(t)$ *to produce functions* $\varepsilon_s^{(m)*}(t)$, *then equations* (16) *and* (17) *relate the two sequences of functions produced.*

Finally we remark that there is a consistency between the two limiting relationships (9) and (10), for

$$\int_t^\infty f^{(m)}(t)\,dt = \lim_{t\to\infty} f^{(m-1)}(t) - f^{(m-1)}(t). \qquad (m = 0, 1, \ldots) \qquad (21)$$

REFERENCES

[1] WYNN, P., A Note on a Confluent Form of the ε-Algorithm, Archiv der Math. XI (1960), 237–240.

[2] WYNN, P., Upon a Second Confluent Form of the ε–Algorithm, Proc. Glasgow Math. Ass. 5 (1962), 160–165.

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