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# Partial Differential Equations Associated with Certain Non-Linear Algorithms

by

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# I. Introduction

Recently a number of algorithms, which have important application in Numerical Analysis, have been developed. These algorithms concern themselves with quantities lying in a two dimensional array, and relate four quantities lying at the vertices of a lozenge in this array. This simple fact means that one can easily derive equations governing the propagation of error which takes place when applying the algorithms [1]; it means that one can easily derive singular rules which may be applied when the quantities involved become indeterminate [2]; it means that programming these algorithms can be done in a particularly efficient way [3],[4]; it means that confluent forms of these algorithms may easily be derived [5]; it means finally that by contracting the lozenge into a point, partial differential equations corresponding to these algorithms may easily be derived. These partial differential equations are the subject of this paper.

It transpires that in all cases which are considered in detail the lozenge algorithms are first order finite difference approximations to the derived partial differential equations. It is emphasised at the outset that in this treatment the partial differential equations are derived from the finite difference equations, and not conversely as is more often the case.

The purposes of this paper are twofold. In the following section we shall discuss some properties of the algorithms being considered and introduce certain definitions of which subsequent use will be made. In the last section we place on record the partial differential equations which have been referred to, and derive certain properties of the solutions to these equations.

The algorithms of this paper connect certain quantities by means of rational non-linear relationships: the partial differential equations to which they lead may be expressed as simultaneous non-linear partial differential equations of the first order in two dependent and two independent variables. In this paper we do not claim to give a general theory of non-linear partial differential equations nor even to give a comprehensive treatment of the type of partial differential equation being considered. This paper is merely a first step in this direction, in which we place on record certain partial differential equations which would seem to have fundamental significance in analysis and derive certain properties of their solutions. We hope that by so doing we shall stimulate the more searching inquiry which the subject undoubtedly merits.

## II. Lozenge Algorithms

### The $\phi$-array

The algorithms with which we are concerned relate members of an array of quantities the general member of which may be denoted for the purposes of exposition by $\phi_s^{(m)}$. The array may be displayed as follows:

$$
\begin{array}{ccccccc}
\phi_{-1}^{(0)} & & & & & & \\
 & \phi_{0}^{(0)} & & & & & \\
\phi_{-1}^{(1)} & & \phi_{1}^{(0)} & & & & \\
 & \phi_{0}^{(1)} & & \phi_{2}^{(0)} & & & \\
\phi_{-1}^{(2)} & & \phi_{1}^{(1)} & & \cdots & & \\
 & \phi_{0}^{(2)} & & \phi_{2}^{(1)} & & \phi_{s}^{(0)} & \\
\phi_{-1}^{(3)} & & \phi_{1}^{(2)} & & \cdots & & \phi_{s+1}^{(0)} \\
\vdots & \vdots & & \vdots & & \phi_{s}^{(1)} & \\
 & & \vdots & & & & \phi_{s+1}^{(1)} \\
 & & & & & \vdots & \\
 & & & & & & \vdots
\end{array}
$$

so that the superscript m indicates a diagonal and the suffix s a column.

The algorithms are furthermore lozenge algorithms: each quantity $\phi_s^{(m)}$ is derived from a relationship of the form

$$\Theta_s^{(m)}\left\{\phi_{s+1}^{(m)}, \phi_s^{(m)}, \phi_{s-1}^{(m+1)}, \phi_{s-2}^{(m+1)}\right\} = 0. \qquad (1)$$

The quantities occurring in this relationship are placed in the $\phi$-array thus:

$$\begin{matrix} & \phi_{s-1}^{(m)} & \\ \phi_{s-2}^{(m+1)} & & \phi_s^{(m)} \\ & \phi_{s-1}^{(m+1)} & \end{matrix}$$

i.e. they occur at the vertices of a lozenge in the $\phi$-array.

In the so-called forward use of lozenge algorithms the procedure is as follows: initial values of $\phi_{-1}^{(m)}, \phi_0^{(m)}$ $(m=0,1,\ldots)$ are prescribed and relationship (1) is solved for $\phi_s^{(m)}$. Letting s=1,2,...; m=0,1,... in the resulting formula the whole of the $\phi$-array is built up column by column.

As a simple example of a lozenge algorithm we give the following:

$$\phi_{s-1}^{(m)} + \phi_s^{(m)} - \phi_{s-1}^{(m+1)} - \phi_{s-2}^{(m+1)} = 0. \qquad (2)$$

Note: In the most general case relationship (1) is non-linear, but to simplify the exposition the examples in this explanatory section will be confined to linear cases.

In the most general case, the functional relationship $\Theta_s^{(m)}\{\ldots\}$ depends upon both m and s. However, in many cases which have found practical application, the functional relationship $\Theta_s^{(m)}\{\ldots\}$ has two differing forms which depend upon whether s is even or odd. For this reason we shall distinguish the quantities $\phi_s^{(m)}$ with even and odd suffix by writing

$$\phi^{(m)}_{2r-2} = {}_1\phi^{(m)}_r \,, \quad \phi^{(m)}_{2r-1} = {}_2\phi^{(m)}_r \tag{3}$$

and we shall replace the single relationship (1) by the two relationships

$$\left.\begin{aligned} {}_1\Theta^{(m)}_r \left\{ {}_2\phi^{(m)}_{r-1}, {}_1\phi^{(m)}_r, {}_2\phi^{(m+1)}_{r-1}, {}_1\phi^{(m+1)}_{r-1} \right\} = 0 \\ {}_2\Theta^{(m)}_r \left\{ {}_1\phi^{(m)}_r, {}_2\phi^{(m)}_r, {}_1\phi^{(m+1)}_r, {}_2\phi^{(m+1)}_{r-1} \right\} = 0 \end{aligned}\right\} \tag{4}$$

The $\phi$-array relating to dual lozenge algorithms is as follows:

$$\begin{array}{ccccccc}
{}_2\phi^{(0)}_0 & & & & & & \\
 & {}_1\phi^{(0)}_1 & & & & & \\
{}_2\phi^{(1)}_0 & & {}_2\phi^{(0)}_1 & & & & \\
 & {}_1\phi^{(1)}_1 & & {}_1\phi^{(0)}_2 & & & \\
{}_2\phi^{(2)}_0 & & {}_2\phi^{(1)}_1 & & \cdot\ \cdot & & \\
 & {}_1\phi^{(2)}_1 & & {}_1\phi^{(1)}_2 & \cdot & {}_1\phi^{(0)}_r & \\
{}_2\phi^{(3)}_0 & \cdot & {}_2\phi^{(2)}_1 & \cdot & \cdot\ \cdot & & {}_2\phi^{(0)}_r \\
\vdots & \vdots & \vdots & \vdots & & {}_1\phi^{(1)}_r & \\
 & & & & & \cdot & {}_2\phi^{(1)}_r
\end{array}$$

The quantities occurring in relationship (4) are to be found in this array as follows:

$$\begin{array}{cccc}
 & {}_2\phi^{(m)}_{r-1} & & \\
{}_1\phi^{(m+1)}_{r-1} & & {}_1\phi^{(m)}_r & \\
 & {}_2\phi^{(m+1)}_{r-1} & & {}_2\phi^{(m)}_r \\
 & & {}_1\phi^{(m+1)}_r &
\end{array}$$

An important advantage which results from this change is that in many special cases which have been considered, if the lozenge algorithm is presented in the dual form (4) tne functional relationships involved can be represented solely by means of rational operations; if the single relationship (1) is retained this is not so.

An added reason for introducing the substitutions (3) is that in the practical cases which have been studied, the numerical behaviour of the quantities $\phi_s^{(m)}$ differs markedly for odd and even values of s.

Lastly, in anticipation of later text, it is mentioned that if the lozenge algorithm relationships are applied to certain special initial values, then determinantal expressions may be derived for the quantities $\phi_s^{(m)}$ : these expressions differ in form depending upon whether s is even or odd. As a simple example of a dual lozenge algorithm we give the following:

$$\left.\begin{aligned} {}_1\phi_r^{(m)} - {}_1\phi_{r-1}^{(m+1)} &= {}_2\phi_{r-1}^{(m+1)} - {}_2\phi_{r-1}^{(m)} \\ {}_2\phi_r^{(m)} - {}_2\phi_{r-1}^{(m+1)} &= -\left({}_1\phi_r^{(m+1)} - {}_1\phi_r^{(m)}\right). \end{aligned}\right\} \quad (5)$$

It happens that relationships (5) may be so manipulated as to produce a recursion among the quantities ${}_1\phi_r^{(m)}$ themselves. Indeed we have

$${}_1\phi_{r+1}^{(m-1)} + {}_1\phi_{r-1}^{(m+1)} + {}_1\phi_r^{(m-1)} + {}_1\phi_r^{(m+1)} = 4\,{}_1\phi_r^{(m)} \quad (6)$$

and a similar relationship obtains for the quantities ${}_2\phi_r^{(m)}$.

If the single relationship (2) is to be expressed in the form of a dual lozenge algorithm relationship, then we have

$$\left.\begin{aligned} {}_1\phi_r^{(m)} - {}_1\phi_{r-1}^{(m+1)} &= {}_2\phi_{r-1}^{(m+1)} - {}_2\phi_{r-1}^{(m)} \\ {}_2\phi_r^{(m)} - {}_2\phi_{r-1}^{(m+1)} &= {}_1\phi_r^{(m+1)} - {}_1\phi_r^{(m)}. \end{aligned}\right\} \quad (7)$$

Again it transpires that relationships (7) may be made to yield recursions among the quantities ${}_1\phi_r^{(m)}$ and ${}_2\phi_r^{(m)}$ alone.

We have

$$ {}_1\phi_{r+1}^{(m-1)} - {}_1\phi_{r}^{(m-1)} + {}_1\phi_{r-1}^{(m+1)} - {}_1\phi_{r}^{(m+1)} = 0 \qquad (8)$$

and a similar relationship for the quantities ${}_2\phi_r^{(m)}$ .

II.1 The Derivation of Partial Differential Equations

We now consider the following constellation of values in the $\phi$ - array:

$$\begin{array}{ccccc}
 & & {}_1\phi_r^{(m-1)} & & \\
 & {}_2\phi_{r-1}^{(m)} & & {}_2\phi_r^{(m-1)} & \\
{}_1\phi_{r-1}^{(m+1)} & & {}_1\phi_r^{(m)} & & {}_1\phi_{r+1}^{(m-1)} \\
 & {}_2\phi_{r-1}^{(m+1)} & & {}_2\phi_r^{(m)} & \\
 & & {}_1\phi_r^{(m+1)} & &
\end{array}$$

fig.1.

These values lie at certain points, the positions of which are determined by m and r, in the plane of the paper. We now introduce new coordinates x and y (which are assumed to be continuous) and an interval h, so that the quantities lying in the above large lozenge do so at the following points in the x-y plane:

$$\begin{array}{ccccc}
 & & x,\, y-2h & & \\
 & x-h,\, y-h & & x+h,\, y-h & \\
x-2h,\, y & & x,\, y & & x+2h,\, y \\
 & x-h,\, y+h & & x+h,\, y+h & \\
 & & x,\, y+2h & &
\end{array}$$

This transformation of coordinates corresponds to the substitutions

$$x = a + 2rh, \qquad y = b + 2(m+r)h \qquad (9)$$

where a and b are two constants.

We shall derive partial differential equations from the lozenge algorithm relationships (4) in the following way: we replace the quantities ${}_1\phi_r^{(m)}$ and ${}_2\phi_r^{(m)}$ by (possibly auxiliary functions of) the functions ${}_1\phi(x,y)$ and ${}_2\phi(x+h,y+h)$, together with corresponding substitutions for the further quantities in Fig.1; we then let the interval h tend to zero.

In many cases we shall see that this process leads to the derivation of partial differential equations satisfied by the functions ${}_1\phi(x,y)$ and ${}_2\phi(x,y)$.

For example relationships (5) lead to the equations

$$\left.\begin{aligned} {}_1\phi(x,y) - {}_1\phi(x-2h,y) &= {}_2\phi(x-h,y+h) - {}_2\phi(x-h,y-h) \\ {}_1\phi(x,y+2h) - {}_1\phi(x,y) &= -\left\{{}_2\phi(x+h,y+h) - {}_2\phi(x-h,y+h)\right\} \end{aligned}\right\} \quad (10)$$

As h tends to zero we have the pair of partial differential equations

$${}_1\phi_x = {}_2\phi_y \; , \quad {}_1\phi_y = -{}_2\phi_x \qquad (11)$$

These are, of course, (allowing for a change in notation) the Cauchy-Riemann equations. Eliminating the function ${}_2\phi$ , we have

$${}_1\phi_{xx} + {}_1\phi_{yy} = 0. \qquad (12)$$

Equation (6) is a well-known first order finite-difference approximation to the Laplace equation (12).

As a second example to illustrate the above technique for deriving partial differential equations, we consider relationships (7), which lead to the partial differential equations

$${}_1\phi_x = {}_2\phi_y \; , \quad {}_1\phi_y = {}_2\phi_x \qquad (13)$$

Again the function ${}_2\phi$ may be eliminated, and we have the hyperbolic partial differential equation

$${}_1\phi_{xx} - {}_1\phi_{yy} = 0. \qquad (14)$$

Using the coordinates x and y, the finite difference relationship (8) may be written as

$${}_1\phi(x+2h,y) + {}_1\phi(x-2h,y) = {}_1\phi(x,y+2h) + {}_1\phi(x,y-2h). \qquad (15)$$

As is well known, both the partial differential equation (14) and its first order finite difference approximation (15) have the same solution

$$ {}_1\phi(x,y) = f(x+y) - g(x-y) $$

independent of the size of the interval h; we shall see that a phenomenon closely resembling this is to be found in the case of a partial differential equation derived from another lozenge algorithm.

### Non-uniqueness of the Auxiliary Substitutions

It will immediately be seen that if the functions ${}_1\phi(x,y)$, ${}_2\phi(x,y)$ in equations (10) are replaced by the two functions ${}_1\phi'(x,y)$ and ${}_2\phi'(x,y)$ where

$$ {}_1\phi'(x,y) = h^{\alpha}\,{}_1\phi(x,y), \qquad {}_2\phi'(x,y) = h^{\alpha}\,{}_2\phi(x,y) \tag{17} $$

where $\alpha$ is any constant, then the resulting partial differential equations relating ${}_1\phi'$ and ${}_2\phi'$ are the same as (11). There is a similar freedom of choice in the auxiliary substitutions leading to all the partial differential equations of this paper.

## II.2 Certain Properties of Lozenge Algorithms

### Centro-Symmetric Algorithms

At this point we mention a remarkable property which is possessed by the algorithms which have been used for the purposes of illustration, and by certain of the algorithms which have found application in practice.

We consider four quantities A,B,C,D, lying at the vertices of the lozenge of Fig.2.

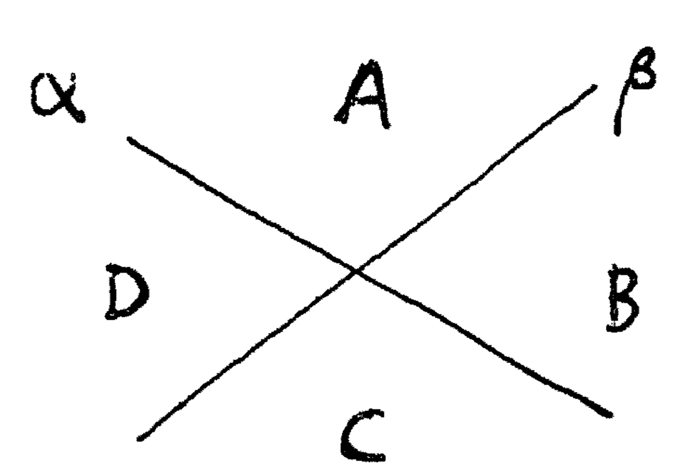

Fig.2

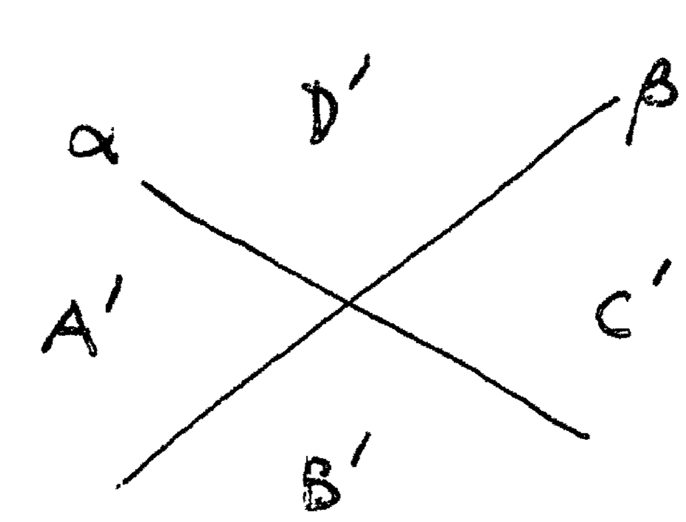

Fig.3

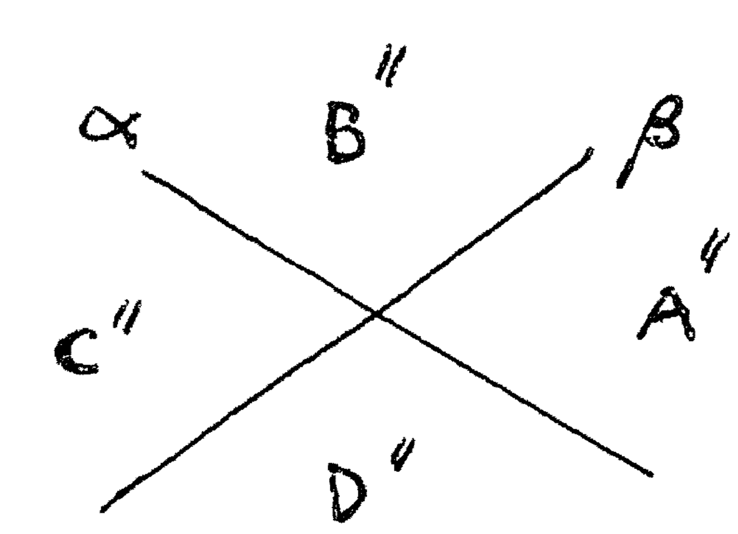

Fig.4

and connected by the relationship

$$ \phi\{A,B,C,D\} = 0. \tag{18} $$

We transpose the lozenge of Fig.2 about the line $\alpha$ and attach single dashes to the quantities involved, obtaining the lozenge of Fig.3. If we replace the quantities involved in relationship (18) by the corresponding quantities occurring in the lozenge of Fig.3, then we obtain the relationship

$$\phi\left\{D', C', B', A'\right\} = 0 \,. \tag{19}$$

Similarly we can transpose the lozenge of Fig.2 about the line $\beta$ to obtain Fig.4 and the resulting relationship

$$\phi\left\{B'', A'', D'', C''\right\} = 0 \tag{20}$$

Now in many cases it just so happens that if the dashes are removed, relationships (18), (19) and (20) are the same.

Relationship (2) may be used to illustrate this phenomenon: in the notation of Fig.2 this may be written as

$$B - D = C - A \tag{21}$$

In this particular case, relationships (19) and (20) become

$$C' - A' = B' - D' \tag{22}$$

and

$$A'' - C'' = D'' - B'' \tag{23}$$

respectively. Clearly, if the dashes are discarded, relationships (21), (22) and (23) are the same.

It is easily shown that the same property is possessed by the dual lozenge algorithm relationships (7).

Definition 1. if the relationships of a lozenge algorithm possess the property which has been described we shall refer to such an algorithm as being centro-symmetric.

Partial differential equations derived from centro-symmetric lozenge algorithms have a curious property. We first introduce

Definition 2. If a system of two simultaneous first order partial differential equations in the independent variables x and y, and dependent variables ${}_1\phi(x,y)$ and ${}_2\phi(x,y)$ is given and the system remains unchanged if x and y, and simultaneously $_1\phi(x,y)$ and $_2\phi(x,y)$ are interchanged, then this system is called self-conjugate We have the following simple

Theorem 1. A system of partial differential equations deriving from a centro-symmetric lozenge algorithm is self-conjugate.

For example, subject to the interchanges described, the partial differential equations (11) become

$$_2\phi_y = {_1\phi_x}, \quad {_2\phi_x} = -{_1\phi_y} \tag{24}$$

i.e. the system remains the same.

Multiple Lozenge Algorithms

So far we have only considered dual lozenge algorithms which concern two systems of quantities $_1\phi_r^{(m)}$ and $_2\phi_r^{(m)}$. It is not difficult to envisage systems of $k$ distinct lozenge algorithm relationships concerning $k$ systems quantities $_i\phi_r^{(m)}$ $(i=1,2,\ldots,k)$ of the form

$$\left.\begin{aligned} {_1\Theta_r^{(m)}}\left\{{_k\phi_{r-1}^{(m)}}, {_1\phi_r^{(m)}}, {_k\phi_{r-1}^{(m+1)}}, {_{k-1}\phi_{r-1}^{(m+1)}}\right\} &= 0 \\ {_2\Theta_r^{(m)}}\left\{{_1\phi_r^{(m)}}, {_2\phi_r^{(m)}}, {_1\phi_r^{(m+1)}}, {_k\phi_{r-1}^{(m+1)}}\right\} &= 0 \\ {_i\Theta_r^{(m)}}\left\{{_{i-1}\phi_r^{(m)}}, {_i\phi_r^{(m)}}, {_{i-1}\phi_r^{(m+1)}}, {_{i-2}\phi_r^{(m+1)}}\right\} &= 0, \end{aligned}\right\} \tag{25}$$

However in the examples which have been given, and it appears generally to be true, the principle under which the partial differential equations of this paper have been derived is as follows: the lozenge algorithms lead to difference equations involving for example the functions $_2\phi(x-h,y+h)$ and $_2\phi(x+y,y+h)$; when these functions are replaced in the difference equation by their equivalent Taylor series expansions, the function $_2\phi(x,t)$ is cancelled from the equation and after division throughout by $h$ there results an equation involving the derivative $_2\phi_x$.

In the case of a multiple lozenge algorithm of order higher than two, the above process leads to an equation between, for example, ${}_k\phi(x-h, y+h)$ and ${}_1\phi(x+h, y+h)$: after these functions have been replaced by their equivalent Taylor series expansions, the functions ${}_k\phi(x,y)$ and ${}_2\phi(x,y)$ do not cancel from the resulting equation.

Thus two is the highest order of the lozenge algorithms which can lead to systems of simultaneous first order partial differential equations.

II.3 Properties of Partial Differential Equations

Adjoint Partial Differential Equations

As is well known ([6],p.13 ) systems of two simultaneous first order partial differential equations involving two dependent variables do not always lead to one partial differential equation involving one of the dependent variables. However, let us assume for the purposes of the following definition that in the cases considered in this paper, this is possible.

Definition 3. If we are given a partial differential equation with the dependent variable ${}_1\phi(x,y)$, and another with the dependent variable ${}_2\phi(x,y)$, then we shall say that two equations are adjoint if they may be derived by eliminating the dependent variables ${}_2\phi(x,y)$ and ${}_1\phi(x,y)$ respectively from a system of two simultaneous first order partial differential equations which relate these dependent variables.

The reason for giving this phenomenon this name is that if we are given, for example, the partial differential equation sattisfied by ${}_1\phi(x,y)$, then we may adjoin to this equation the partial differential equation by ${}_2\phi(x,y)$ and in this way proceed to a system of two simultaneous partial differential equations of the first order which may be easier to handle than the original partial differential equation. This process is analogous to the use of the adjoint equation in the linear theory of partial differential equations.

Theorem 2. Partial differential equations deriving in the first place from single lozenge algorithms are self-adjoint.

Classes of Partial Differential Equations

Notation: We denote the argument sets

$$x, y, {}_1\phi, {}_2\phi, {}_1\phi_x, {}_2\phi_x, {}_1\phi_y, {}_2\phi_y$$
$$x, y, {}_1\phi', {}_2\phi', {}_1\phi'_x, {}_2\phi'_x, {}_1\phi'_y, {}_2\phi'_y$$

by D and D' respectively, and the argument sets

$$x, y, {}_1\phi, {}_2\phi \quad \text{and} \quad x, y, {}_1\phi', {}_2\phi'$$

by A and A' respectively

Definition 4. Suppose that two systems of simultaneous first order partial differential equations

$$_1\theta\{D\} = 0 \quad , \quad _2\theta\{D\} = 0 \tag{26}$$

and

$$_1\theta'\{D'\} = 0 \quad , \quad _2\theta'\{D'\} = 0 \tag{27}$$

exist, and furthermore that either by means of a pair of substitutions of the form

$$_1\phi = \alpha\{D'\} \quad , \quad _2\phi = \beta\{D'\} \tag{28}$$

equations (26) are transformed into equations (27), or that by means of a pair of substitutions of the form

$$_1\phi' = \alpha'\{D\} \quad , \quad _2\phi' = \beta'\{D\} \tag{29}$$

equations (27) are transformed into equations (26), then the two systems (26) and (27) are said to be members of the same differential class.

Definition 5. If in the conditions of Definition 4 the argument sets S' and S of equations (28) and (29) may be replaced by the sets A' and A respectively, then the two systems (26) and (27) are said to be members of the same analytic class.

In the derivation of the partial differential equations of this paper some degree of generalisation may be achieved by letting the lozenges be of irregular size, and replacing the substitution (9) by

$$x=a+f(r)h, \quad y=b+g(m+r)h \tag{30}$$

However the resulting system of partial differential equations belongs to the same analytic class as the original system, and in this general exposition where simplicity of presentation is to be desired, the substitutions (9) are adhered to.

III Special Lozenge Algorithms

III.1 The $\varepsilon$-algorithm [7]

The $\varepsilon$-algorithm is a single lozenge algorithm whose relationships are

$$\left(\varepsilon_{s-1}^{(m+1)}-\varepsilon_{s-1}^{(m)}\right)\left(\varepsilon_{s}^{(m)}-\varepsilon_{s-2}^{(m+1)}\right)=1. \qquad (31)$$

The main function-theoretic property of the $\varepsilon$-algorithm may be described as follows. Given a power series $\sum_{s=0}^{\infty} c_s z^s$ it is formally possible [8] to construct a double sequence $P_{i,j}$ (i,j=0,1...) of rational functions of z. The function $P_{i,j}$ is the quotient of two polynomials, the numerator of the $j^{th}$ degree, the denominator of the $i^{th}$ degree : this quotient is characterised by the property that its series expansion in ascending powers of z agrees with the series $\sum_{s=0}^{\infty} c_s z^s$ as far as the term $c_{i+j} z^{i+j}$ . Specifically

$$P_{i,j}=\frac{\sum_{s=0}^{j} p_{i,j,s} z^s}{\sum_{s=0}^{i} p'_{i,j,s} z^s}=\sum_{s=0}^{\infty} c_s z^s+\sum_{s=i+j+1}^{\infty} d_s z^s \qquad (32)$$

where $\sum_{s=i+j+1}^{\infty} d_s z^s$ is some power series expansion with (in general) non-zero coefficients.

The functions $P_{i,j}$ may be arranged in a two-dimensional array (the Padé array) in which the first suffix i indicates a row number and the second suffix j a column number.

The connection between the $\varepsilon$-algorithm and the Padé array is this: that if the $\varepsilon$-algorithm relationships (31) are applied to the initial values

$$\varepsilon_{-1}^{(m)} = 0, \quad \varepsilon_{2m+2}^{(-m-2)} = 0, \quad \varepsilon_0^{(m)} = \sum_{s=0}^{m} c_s z^s \quad (m = 0, 1, \ldots) \tag{33}$$

then [9] [10]

$$\varepsilon_{2r}^{(m)} = P_{r, m+r} \qquad r = 0, 1, \ldots; \quad m = -r, -r+1, \ldots) \tag{34}$$

We mention in passing that with the initial conditions (33) the following determinantal formulae may be given

$$\varepsilon_{2k}^{(m)} = \frac{\begin{vmatrix} S_m & S_{m+1} & \ldots & S_{m+k} \\ \Delta S_m & \Delta S_{m+1} & \ldots & \Delta S_{m+k} \\ \vdots & \vdots & & \vdots \\ \Delta S_{m+k-1} & \Delta S_{m+k} & \ldots & \Delta S_{m+2k-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \Delta S_m & \Delta S_{m+1} & \ldots & \Delta S_{m+k} \\ \vdots & \vdots & & \vdots \\ \Delta S_{m+k-1} & \Delta S_{m+k} & \ldots & \Delta S_{m+2k-1} \end{vmatrix}} = e_k(S_m), \tag{35}$$

$$\varepsilon_{2k+1}^{(m)} = \{e_k(\Delta S_m)\}^{-1}, \tag{36}$$

where

$$S_m = \sum_{s=0}^{m} c_s z^s. \tag{37}$$

Partial differential equations may be derived from relationship (31) by making the substitutions

$$\varepsilon_{2s-1}^{(m)} = h^{-1}\,{}_1\varepsilon(x,y), \quad \varepsilon_{2s}^{(m)} = h^{-1}\,{}_2\varepsilon(x,y). \tag{38}$$

As h tends to zero we have in turn

$$\left.\begin{aligned} {}_1\varepsilon_x \, {}_2\varepsilon_y &= 1 \\ {}_2\varepsilon_x \, {}_1\varepsilon_y &= 1. \end{aligned}\right\} \tag{39}$$

The functions $_1\varepsilon(x,y)$ and $_2\varepsilon(x,y)$ separately both satisfy the partial differential equation

$$\left\{\frac{1}{\varepsilon_y}\right\}_y = \left\{\frac{1}{\varepsilon_x}\right\}_x . \qquad (40)$$

As has been said the quantities $\varepsilon_{2r}^{(m)}$ are members of the Padé array; if the grid variables m and r are replaced by continuous variables x and y and the dimensions of the grid are reduced, then the Padé array may be regarded as points on a Padé surface. Equation (40), which is satisfied by the function $_2\Phi(x,y)$ is the partial differential equation of the Padé surface.

The $\varepsilon$-algorithm is centro-symmetric.

Theorem 3. Equations (39) are self-conjugate

The $\varepsilon$-algorithm is a single lozenge algorithm.

Theorem 4. Equation (40) is self-adjoint.

There is a further property which is possessed by the solutions to the partial differential equations which may be derived from the $\varepsilon$-algorithm. Subtracting the first of equations (39) from the second we see that the Jacobian

$$\begin{vmatrix} {}_2\varepsilon_x & {}_2\varepsilon_y \\ {}_1\varepsilon_x & {}_1\varepsilon_y \end{vmatrix}$$

vanishes.

Theorem 5. If the pair of functions $_1\varepsilon(x,y)$ and $_2\varepsilon(x,y)$ are any two solutions of equations (39) then a functional relationship of the form

$$f\left({}_1\varepsilon(x,y), {}_2\varepsilon(x,y)\right) = 0 \qquad (41)$$

prevails between them.

III.2 The q-d algorithm [11]

The q-d algorithm is a dual lozenge algorithm whose relationships are

$$q_p^{(m)} + e_p^{(m)} = q_p^{(m+1)} + e_{p-1}^{(m+1)}, \quad q_{p+1}^{(m)} e_p^{(m)} = q_p^{(m+1)} e_p^{(m+1)}. \tag{42}$$

In Numerical Analysis the main application of the q-d algorithm is in the spectral decomposition of a sequence of iterates. In Analysis the q-d algorithm is significant for the following reason: Given a power series $\sum_{s=0}^{\infty} c_s z^s$ it is formally possible to construct a continued fraction of the form

$$\sum_{s=0}^{m-1} c_s z^s + \frac{c_m z^m}{1-} \; \frac{q_1^{(m)} z}{1-} \; \frac{e_1^{(m)} z}{1-} \; \cdots , \frac{q_p^{(m)} z}{1-} \; \frac{e_p^{(m)} z}{1-} \; \cdots \tag{43}$$

(the so-called corresponding continued fraction) whose coefficients are uniquely determined by the power series expansion relationships

$$\sum_{s=0}^{m-1} c_s z^s + \frac{c_m z^m}{1-} \; \frac{q_1^{(m)} z}{1-} \; \frac{e_1^{(m)} z}{1-} \cdots \frac{q_p^{(m)} z}{1} \sim \sum_{s=0}^{\infty} c_s z^s + \sum_{s=m+2}^{\infty} d_s z^s$$

$$\sum_{s=0}^{m-1} c_s z^s + \frac{c_m z^m}{1-} \; \frac{q_1^{(m)} z}{1-} \; \frac{e_1^{(m)} z}{1-} \cdots \frac{e_p^{(m)} z}{1-} \sim \sum_{s=0}^{\infty} c_s z^s + \sum_{s=m+3}^{\infty} d_s' z^s \tag{44}$$

where $\sum_{s=m+2}^{\infty} d_s z^s$ and $\sum_{s=m+3}^{\infty} d_s' z^s$ are two power series expansions with (in general) non-zero coefficients. If the q--d algorithm is applied to the initial values

$$e_0^{(m)} = 0 , \quad q_1^{(m)} = c_{m+1}/c_m \qquad (m=0,1,\ldots) \tag{45}$$

then the coefficients $q_p^{(m)}, e_p^{(m)}$ of the continued fractions (43) are the quantities occurring in relationships (42).

We mention in passing that if we denote the Hankel determinant

$$\begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+k-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+k} \\ \vdots & \vdots & & \vdots \\ c_{m+k-1} & c_{m+k} & \cdots & c_{m+2k-2} \end{vmatrix} \qquad (46)$$

by $H_k^{(m)}$ with

$$H_0^{(m)} = 1 \qquad (47)$$

then it may be shown that the quantities obtained by applying the q-d algorithm relationships to the initial conditions (45) are given by

$$q_r^{(m)} = \frac{H_r^{(m+1)} H_{r-1}^{(m)}}{H_r^{(m)} H_{r-1}^{(m+1)}} , \quad e_r^{(m)} = \frac{H_{r+1}^{(m)} H_{r-1}^{(m+1)}}{H_r^{(m)} H_r^{(m+1)}} \qquad (48)$$

Partial differential equations may be derived from the relationships (42) by making the substitutions

$$q_r^{(m)} = q(x,t) , \quad e_r^{(m)} = e(x,t) \qquad (49)$$

As h tends to zero, relationships (42) become

$$\left. \begin{aligned} q_y &= e_x \\ e q_x &= q e_y \end{aligned} \right\} \qquad (50)$$

The functions q(x,y) and e(x,y) separately satisfy the partial differential equations

$$\left[ \frac{q_{yy} - (\ln q)_x q_y}{(\ln q)_{xx}} \right]_x = q_y , \qquad (51)$$

$$\left[ \frac{e_{xx} - (\ln e)_y e_x}{(\ln e)_{yy}} \right]_y = e_x ; \qquad (52)$$

these form, of course, an adjoint system.

Again the coefficients $q_r^{(m)}$ and $e_r^{(m)}$ may be regarded as points on two surfaces q(x,y) and e(x,y); we may speak of equations (51) and (52) as being the partial differential equations of the corresponding continued fraction coefficient surfaces

The q-d algorithm is centro-symmetric.

Theorem 6. The partial differential equations (50) are self-conjugate.

Note: When conducting experiments in the application of the q-d algorithm, H.Rutishauser noticed a numerical phenomenon closely akin to shock waves: this corresponds to the case in which the equations (49) form a hyperbolic system.

To conclude this section on the q-d algorithm we describe a remarkable phenomenon concerning the solutions of the partial differential equations (50) and their first order finite difference approximations (42).

Clearly, two solutions of (50) are

$$q(x,y) = e^{x+y}, \quad e(x,y) = (e^{x}-1)e^{y} \tag{53}$$

These conform to the initial conditions

$$e_0^{(m)} = 0 \tag{54}$$

in (42). In order to derive solutions of (42) corresponding to

$$q_1^{(m)} = e^{mh} \tag{55}$$

and (54), use may be made of the following result [12] : if in (46) and (48)

$$c_m = h^m q^{\alpha m^2 + \beta m + \gamma} \tag{56}$$

then

$$q_r^{(m)} = hq^{\alpha(4r+2m-3)+\beta}, \quad e_r^{(m)} = h(q^{2\alpha r}-1)q^{\alpha(2m+2r-1)+\beta}. \tag{57}$$

Substituting

$$\alpha = \frac{1}{2}\,,\ \beta = -\frac{1}{2}\,,\ q = e^{h} \tag{58}$$

in (57), the solutions of the finite difference equations (42) are seen to be

$$q(ah, bh) = e^{(a+b)h}\,,\quad e(ah, bh) = (e^{ah} - 1)e^{bh} \tag{59}$$

i.e. they agree with (53).

The meaning of this result is as follows: if we wish to solve the partial differential equations (50) subject to certain initial conditions and replace the partial differential equations by first order finite difference approximations then no matter how large the resultant truncation error may be, the agreement between the finite difference approximation and the analytic solution is exact.

Note: The same phenomenon may be observed when considering the solutions

$$q(x,y) = y \quad , e(x,y) = x \tag{60}$$

but this case is somewhat trivial, since the truncation error introduced by replacing the derivatives by first differences, is zero.

III. 3 The first g-algorithm ( [13], p.8)

This, a dual lozenge algorithm, is in essence a generalisation of the q-d algorithm; a displacement factor is introduced into the formulae which run

$$\left.\begin{aligned} (S^{(m)} - g^{(m)}_{2s-1})\, g^{(m)}_{2s} &= (S^{(m+1)} - g^{(m+1)}_{2s-1})\, g^{(m+1)}_{2s-2} \\ (1 - g^{(m)}_{2s})\, g^{(m)}_{2s+1} &= (1 - g^{(m+1)}_{2s})\, g^{(m+1)}_{2s-1} \end{aligned}\right\} \tag{61}$$

Partial differential equations may be derived from relations (61) by making the substitutions

$$g^{(m)}_{2s-1} = {}_1g(x,y),\quad g^{(m)}_{2s} = {}_2g(x,y), \tag{62}$$

As h tends to zero, there follows

$$\left.\begin{aligned} (1-{}_2\acute{g})\,{}_1\acute{g}_x &= -{}_1\acute{g}\,{}_2\acute{g}_x, \\ {}_2\acute{g}\,(S(y)-{}_1\acute{g})_y &= (S(y)-{}_1\acute{g})\,{}_2\acute{g}_x. \end{aligned}\right\} \quad (63)$$

III.4 The Second g-algorithm ([13], p.15)

This, another dual lozenge algorithm, is a variant of the q-d algorithm. Quantities $\acute{g}_s^{(m)}$ satisfy the relationships

$$\left.\begin{aligned} \acute{g}^{(m)}_{2s-1}\,\acute{g}^{(m)}_{2s} &= \acute{g}^{(m+1)}_{2s-2}\,g^{(m+1)}_{2s-1} \\ (1-\acute{g}^{(m)}_{2s})(S-\acute{g}^{(m)}_{2s+1}) &= (1-g^{(m+1)}_{2s})(S-\acute{g}^{(m+1)}_{2s-1}). \end{aligned}\right\} \quad (64)$$

Partial differential equantions may be derived from relations (64) by making the substitutions

$$\acute{g}^{(m)}_{2s-1} = {}_1\mathfrak{g}(x,y), \quad \acute{g}^{(m)}_{2s} = {}_2\mathfrak{g}(x,y). \quad (65)$$

As h tends to zero, there follow

$$\left.\begin{aligned} {}_2\mathfrak{g}\,{}_1\mathfrak{g}_y &= {}_1\mathfrak{g}\,{}_2\mathfrak{g}_x, \\ (1-{}_2\mathfrak{g})\,{}_1\mathfrak{g}_x &= (S-{}_1\mathfrak{g})\,{}_2\mathfrak{g}_x. \end{aligned}\right\} \quad (66)$$

We conclude this section on the second g-algorithm by remarking that if the q-d and the first and second g-algorithms are used to extend the $\phi$ array from sets of initial conditions which correspond, it may be shown that the quantities produced are inter-related.

More precisely, if the quantities $q_r^{(m)}, e_r^{(m)}$ of the q-d algorithm are produced from the initial conditions

$$e_0^{(m)} = 0, \quad q_1^{(m)} = c_{m+1}/c_m, \qquad (m=0,1,\ldots) \quad (67)$$

the quantities $\acute{g}_s^{(m)}$ of the first g-algorithm from

$$g_0^{(m)} = 1, \quad \acute{g}_1^{(m)} = S^{(m)} - c_{m+1}/c_m, \quad (m=0,1,\ldots) \quad (68)$$

and the quantities $\acute{g}_s^{(m)}$ of the second g-algorithm from

$$\acute{g}_0^{(m)} = 1, \quad \acute{g}_1^{(m)} = S - c_{m+1}/c_m, \tag{69}$$

then the following relationships may be shown to obtain

$$\left.\begin{array}{l} -\acute{g}_{2s-2}^{(m)}\,\acute{g}_{2s-1}^{(m)} = q_s^{(m)}, \; -\left(S - \acute{g}_{2s-1}^{(m)}\right)\left(1 - \acute{g}_{2s}^{(m)}\right) = e_s^{(m)} \\ \acute{g}_{2s-2}^{(m)}\left(S - \acute{g}_{2s-1}^{(m)}\right) = q_s^{(m)}, \; \acute{g}_{2s-1}^{(m)}\left(1 - \acute{g}_{2s}^{(m)}\right) = e_s^{(m)} \end{array}\right\} \tag{70}$$

These relationships may be transformed into functional relationships between the solutions of the partial differential equations (50), (63) and (66) which result from sets of corresponding initial conditions.

If the functions e(x,y), q(x,y) are produced from the initial conditions

$$e(0,y) = 0, \quad q(0,y) = \phi(y), \tag{71}$$

the functions ${}_1\acute{g}(x,y)$, ${}_2\acute{g}(x,y)$ from the initial conditions

$${}_1\acute{g}(0,y) = S(y) - \phi(y), \quad {}_2\acute{g}(0,y) = 1, \tag{72}$$

and the functions ${}_1\gimel(x,y)$, ${}_2\gimel(x,y)$ from the boundary conditions

$${}_1\gimel(0,y) = S - \phi(y), \quad {}_2\gimel(0,y) = 1, \tag{73}$$

then

$$-{}_2\acute{g}\,{}_1\acute{g} = q, \quad -\left(S(y) - {}_1\acute{g}\right)\left(1 - {}_2\acute{g}\right) = e \tag{74}$$

$${}_2\gimel\left(S - {}_1\gimel\right) = q, \quad {}_1\gimel\left(1 - {}_2\gimel\right) = e. \tag{75}$$

Theorem 7. The system of partial differential equations (50), (63) and (66) are members of the same analytic class.

III.5 The $\eta$-algorithm ( [13] ,p.16)

This is a dual lozenge algorithm whose relationships are

$$\left.\begin{aligned} \eta^{(m)}_{2s-1} + \eta^{(m)}_{2s} &= \eta^{(m+1)}_{2s-2} + \eta^{(m+1)}_{2s-1}, \\ \eta^{(m)\,-1}_{2s} + \eta^{(m)\,-1}_{2s+1} &= \eta^{(m+1)\,-1}_{2s-1} + \eta^{(m+1)\,-1}_{2s}. \end{aligned}\right\} \tag{76}$$

Partial differential equations may be derived from relationships (76) by making the substitutions

$$\eta^{(m)}_{2s-1} = {}_1\eta(x,y), \quad \eta^{(m)}_{2s} = {}_2\eta(x,y) \tag{77}$$

As h tends to zero there follow

$${}_2\eta_x = {}_1\eta_y, \quad \left\{\frac{1}{{}_1\eta}\right\}_x = \left\{\frac{1}{{}_2\eta}\right\}_y. \tag{78}$$

The functions ${}_1\eta$ and ${}_2\eta$ separately satisfy the partial differential equations

$$\left[\frac{-{}_2\eta_x \left\{\frac{1}{{}_2\eta}\right\}_y \pm \sqrt{({}_2\eta_x)^2({}_2\eta)^{-2} + \left\{\frac{1}{{}_2\eta}\right\}_{yy} {}_2\eta_{xx}}}{\left\{\frac{1}{{}_2\eta}\right\}_{yy}}\right]_y = {}_2\eta_x \tag{79}$$

and

$$\left[\frac{-{}_1\eta_y \left\{\frac{1}{{}_1\eta}\right\}_x \pm \sqrt{({}_1\eta_y)^2({}_1\eta)^{-2} + \left\{\frac{1}{{}_1\eta}\right\}_{xx} {}_1\eta_{yy}}}{\left\{\frac{1}{{}_1\eta}\right\}_{xx}}\right]_x = {}_1\eta_y. \tag{80}$$

The partial differential equations (79) and (80) form, of course, an adjoint system.

The $\eta$-algorithm is centro-symmetric.

Theorem 8. The partial differential equations (78) are self-conjugate

If the sequences $c_m, S_m$ (m=0,1,...) are related by ,

$$S_0 = 0, \quad S_m = \sum_{s=0}^{m-1} c_s \qquad (m=1,2,\dots) \tag{81}$$

the quantities $\eta_s^{(m)}$ of the $\eta$-algorithm are constructed from the initial conditions

$$\eta_{-1}^{(m)} = \infty \ , \quad \eta_0^{(m)} = c_m \qquad (m=0,1,\dots) \tag{82}$$

and the quantities $\varepsilon_s^{(m)}$ of the $\varepsilon$-algorithm from the initial conditions

$$\varepsilon_{-1}^{(m)} = 0 \ , \quad \varepsilon_0^{(m)} = S_m \tag{83}$$

then

$$\left.\begin{aligned} \varepsilon_{2s}^{(m+1)} - \varepsilon_{2s}^{(m)} &= \eta_{2s}^{(m)} , & -\varepsilon_{2s}^{(m+1)} + \varepsilon_{2s+2}^{(m)} &= \eta_{2s+1}^{(m)} , \\ \varepsilon_{2s+1}^{(m+1)} - \varepsilon_{2s+1}^{(m)} &= \eta_{2s+1}^{(m)\,-1} , & -\varepsilon_{2s-1}^{(m+1)} + \varepsilon_{2s+1}^{(m)} &= \eta_{2s}^{(m)\,-1} . \end{aligned}\right\} \tag{84}$$

Again these relationships may be transformed into functional relationships between certain solutions of the partial differential equations (39) and (78).

If the functions ${}_1\varepsilon(x,y)$, ${}_2\varepsilon(x,y)$ are produced from the initial conditions

$${}_1\varepsilon(0,y) = f(y) , \quad {}_2\varepsilon(0,y) = 0, \tag{85}$$

and the functions ${}_1\eta(x,y)$, ${}_2\eta(x,y)$ from the initial conditions

$${}_1\eta(0,y) = \frac{d}{dy} f(y) \ , \quad {}_2\eta(0,y) = \infty \tag{86}$$

then

$$\left.\begin{aligned} {}_2\varepsilon_y &= {}_2\eta , & {}_2\varepsilon_x &= {}_1\eta , \\ {}_1\varepsilon_y &= {}_1\eta^{-1} , & {}_1\varepsilon_x &= {}_2\eta^{-1} . \end{aligned}\right\} \tag{87}$$

Theorem 9. The partial differential equations (39) and (78) are members of the same differential class.

III.6 The $\rho$-algorithm [14]

This is a single lozenge algorithm whose relationships are

$$\left(\rho_s^{(m)} - \rho_{s-2}^{(m+1)}\right)\left(\rho_{s-1}^{(m+1)} - \rho_{s-1}^{(m)}\right) = s. \tag{88}$$

Its principal application in Numerical Analysis is in the transformation of slowly convergent series.

If the quantities $\rho_s^{(m)}$ are constructed from the initial values

$$\rho_{-1}^{(m)} = 0 \ , \ \rho_0^{(m)} = S_m, \tag{89}$$

it may be shown that

$$\rho_{2r}^{(m)} = \frac{\left| \Delta^r S_{m+s},\ \Delta^r (m+s) S_{m+s},\ \Delta^r (m+s)^2 S_{m+s},\ \ldots,\ \Delta^r (m+s)^r S_{m+s} \right|}{\left| \Delta^r S_{m+s},\ \Delta^r (m+s) S_{m+s},\ \Delta^r (m+s)^2 S_{m+s},\ \ldots,\ r! \right|} \quad (s = 0, 1, \ldots, r)$$

$$\rho_{2r+1}^{(m)} = \frac{(r+1)!\left| \Delta^{r+2} (m+s) S_{m+s},\ \Delta^{r+2} (m+s)^2 S_{m+s},\ \ldots,\ \Delta^{r+2} (m+s)^r S_{m+s} \right|}{\prod_{s=0}^{2r+1} S_{m+s} \left| \Delta^{r+1} (S_{m+s})^{-1},\ \Delta^{r+1} \left((m+s) S_{m+s}^{-1}\right),\ \ldots,\ \Delta^{r+1} \left((m+s)^{r+1} S_{m+s}^{-1}\right) \right|} \quad (s = 0, 1, \ldots, r-1) \tag{90}$$

where the numerators and denominators in these expressions represent the s$^{th}$ row of a determinant.

Partial differential equations may be derived from relation (88) by making the substitutions

$$h^r \rho_{2s-1}^{(m)} = {}_1\rho(x,y) \ , \ h\rho_{2s}^{(m)} = {}_2\rho(x,y). \tag{91}$$

As h tends to zero we obtain

$${}_1\rho_x \, {}_2\rho_y = x - a, \quad {}_1\rho_y \, {}_2\rho_x = x - a. \tag{92}$$

The functions ${}_1\rho$ and ${}_2\rho$ separately both satisfy the partial differential equation

$$\left\{ \frac{x-a}{\rho_y} \right\}_y = \left\{ \frac{x-a}{\rho_x} \right\}_x . \tag{93}$$

The $\rho$-algorithm is a single lozenge algorithm.

Theorem 10. Equation (93) is self-adjoint.

Subtracting the first from the second of equations (92) we have

Theorem 11. If the pair of functions ${}_1\rho(x,y)$ and ${}_2\rho(x,y)$ are any two solutions of equations (92) then a functional relationship of the form

$$f({}_1\rho(x,y), {}_2\rho(x,y)) = 0 \tag{94}$$

prevails between them.

IV Conclusion

The derivation of the partial differential equations resulting from certain lozenge algorithms and the description of their formal properties, which were announced in the introduction, has now been completed. We remark that a theory of the types of initial and boundary conditions which are necessary for a solution to these equations to exist, has been constructed; but at the present time this is somewhat speculative and incomplete, and we do not examine this aspect of the theory here.

When considering questions relating to the existence and uniqueness of the solutions of a partial differential equation it is often of great assistance if explicit solutions to a finite difference equation approximation to the partial differential equation can be given. In the case of all the algorithms of this paper, if the initial conditions are chosen in a certain manner, determinantal formulae for the solutions of the algorithmic relationships can be derived (in certain cases such formulae have been given). This property may well facilitate further research, and makes the algorithms of this paper particularly interesting.

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