A lower bound for the order of a partial transversal in a Latin square.

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A Lower Bound for the Order of a Partial Transversal
in a Latin Square

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ABSTRACT. The notion of partial transversal in a Latin square is defined. A proof is given of the existence of a partial transversal of order \( \frac{3}{4} N + \frac{1}{4} \) of a Latin square of order \( N (N \geq 7) \).

1. Introduction

A Latin square of order \( N \) is a square matrix, each of whose rows and columns is a permutation of the \( N \) symbols 1, 2, ..., \( N \).

A transversal of a Latin square of order \( N \) is a set of \( N \) different elements of the matrix with precisely one element in every row and column. A partial transversal of order \( k \) of a Latin square of order \( N (N \geq k) \) is a set of \( k \) different elements of the Latin square with at most one element in every row and column.

For odd \( N \), Ryser [1] conjectured that every Latin square has a transversal. As far as the author knows this conjecture is still undecided. So if we cannot prove the existence of a transversal, we can raise the question: How large may the order of a partial transversal be?

2. Formulation and Proof of the Theorem

Theorem. A Latin square of order \( N \geq 7 \) has at least one partial transversal of order \( \frac{3}{4} N + \frac{1}{4} \).

Proof: Let \( S \) be a Latin square of order \( N \). Let \( T = T_S \) be an integer for which the following holds: (i) There is no partial transversal of \( S \) of order \( \leq T \). (ii) There is at least one partial transversal of \( S \) of order \( T \). Without loss of generality, the Latin square \( S \) can be divided into submatrices \( LU, RU, LL, \) and \( RL \) as indicated in the figure, and assume that

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the main diagonal of $LU$ is a maximal partial transversal with elements $1, 2, \ldots, T$.

\[
\begin{array}{ccc}
 & L & U \\
T & & \\
 & T + l & \ \\
& T + k & R \ \\
\end{array}
\]

Condition (i) implies all elements in $RL$ are $\leq T$; therefore, each row in $LL$ must contain the elements $T + 1, \ldots, N$.

The total numbers of elements $> T$ in $LL$ is $(N - T)^2$. Suppose $T + T^{1/2} < N$; then $N - T > T^{1/2}$ and $(N - T)^2 > T$. So, in this case, there must be some column, say the $j$-th column, which contains two elements $> T$ in $LL$. Because of the maximality of $T$ we now have:

1. $j$ cannot occur in $RL$, and so occurs $N - T$ times in $RU$;
2. if $j$ occurs in $RU$ in row $p (p \leq T)$, then the $p$-th element of the $j$-th row is $\leq T$; and
3. all elements of the $j$-th row in $RU$ are $\leq T$.

Combining (1), (2), and (3) and using the fact that all elements of a row of a Latin square are different, it follows that the relation $2(N - T) + 1 \leq T$ necessarily holds; thus,

\[T \geq \frac{1}{2}N + \frac{1}{2} .\]

On the other hand, $T + T^{1/2} \geq N \geq 7$ implies (using the fact $T$ must be an integer in the case $N = 7$ and $N = 8$) immediately

\[T \geq \frac{1}{2}N + \frac{1}{2} .\]

whence the theorem follows.

**Remark.** The author has convinced himself (using trivial arguments) that the theorem also holds for $N$ in the range $3 \leq N \leq 6$; we have omitted this proof to keep this note short.

**Reference**