stichting
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MC

AFDELING NUMERIEKE WISKUNDE NN 12/77 JULI
(DEPARTMENT OF NUMERICAL MATHEMATICS)

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NUMERICAL EXPERIMENTS WITH RUNGE-KUTTA TYPE METHODS FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 65R05

Numerical experiments with Runge-Kutta type methods for Volterra integral equations of the second kind

by

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ABSTRACT

Seven formulas of Runge-Kutta type are applied to seventeen test problems in order to test a stability theory developed for Volterra integral equations of the second kind.

KEY WORDS & PHRASES: *Integral equations, Volterra equations of second kind, Runge-Kutta, stability.*

CONTENTS

1. INTRODUCTION

In this note numerical experiments are reported with several Runge-Kutta type methods for Volterra integral equations of the second kind, i.e. equations of the form

$$(1.1)\qquad f(x) = g(x) + \int_{x_0}^{x} K(x,y,f(y))dy,$$

where g and K are given functions, and f is the unknown function. Writing (1.1) in the form

$$(1.1')\qquad \begin{aligned} f(x) &= F_n(x) + \int_{x_n}^{x} K(x,y,f(y))dy \\ F_n(x) &= g(x) + \int_{x_0}^{x_n} K(x,y,f(y))dy \end{aligned}$$

suggests the numerical formula

$$(1.2)\qquad f_{n+1} = \tilde{F}_n(x_{n+1}) + \Phi_n(K(x,y,\tilde{f}(y))), \qquad n = 0,1,\dots ,$$

where f_j, $j = 0,1,\dots,n+1$ denote numerical approximations to the solution $f(x)$ at the points $x_0, x_1, \dots, x_{n+1}$, $\tilde{F}_n(x)$ denotes a numerical approximation to $F_n(x)$, $\tilde{f}$ an interpolatory function through the values f_j, and $\Phi_n(K(x,y,\tilde{f}(y)))$ denotes some quadrature formula for the integral $\int_{x_n}^{x_{n+1}} K(x,y,\tilde{f}(y))dy$.

For $\tilde{F}_n(x)$, two approximations are tested:

$$(1.3)\qquad \begin{aligned} \tilde{F}_0(x) &= g(x), \\ \tilde{F}_n(x) &= g(x) + h \sum_{j=0}^{n}{}' K(x,x_j,f_j), \qquad n > 0 \end{aligned}$$

and

$$\tilde{F}_0(x) = g(x)$$

(1.3') $$\tilde{F}_1(x) = g(x) + h \sum_{j=0}^{1}{}' K(x,x_j,f_j)$$

$$\tilde{F}_n(x) = f_n + g(x) - g(x_n) + h \sum_{j=0}^{n}{}' [K(x,x_j,f_j) - K(x_n,x_j,f_j)], \quad n > 1.$$

Here h denotes the (constant) step length $x_{j+1} - x_j$ and $\sum'$ denotes a summation in which the first and last term have the weight factor $\frac{1}{2}$. Approximation (1.3') has the property $\tilde{F}_n(x_n) = f_n$ which makes the analysis of stability easier (cf. reference [1]) and, what is more important, the stability regions seem to be larger than those obtained when (1.3) is used (cf. reference [2]). Note that the "analytic" function F(x) satisfies the relation $F_n(x_n) = f(x_n)$ as is immediate from (1.1) and the definition of $F_n(x)$.

For $\Phi_n(K(x,y,f(y))$ we have chosen four Runge-Kutta type quadrature rules, That is formulas which are based on non-step points. In section 2, these formulas will be given.

In section 3 we give 17 test problems; 7 linear equations and 10 non-linear equations. The main purpose of this paper is to test the stability theory developed in [1]. Therefore, all experiments were arranged in such a way to demonstrate the reliability of the stability conditions.

Finally, in section 4 the results are listed and some conclusions are drawn. In summary, it turns out that the stability conditions applied are justified by the experiments, even when the kernel function K(x,y,f) does not satisfy the conditions prescribed by the stability theory.

2. FORMULAS TO BE TESTED

In this section we present the formulas we have tested, together with their stability conditions and the order of the accuracy. The derivation of these formulas and the stability conditions we apply, may be found in reference [1] and [2], respectively. These conditions are derived under the following assumptions on the kernel function K(x,y,f):

$$\frac{\partial K}{\partial f}(x,y,f) \simeq \frac{\partial K}{\partial f}(x,x_n,f_n) \quad \text{for} \quad (y,f) \in U_\varepsilon(x_n,f_n),$$

(2.0) $$\frac{\partial^2 K}{\partial x \partial f}(x,x_j,f_j) \cong \frac{\partial^2 K}{\partial x \partial f}(x_n,x_j,f_j), \quad j = 0,1,\dots,n-1 \quad \text{for } x \in U_\varepsilon(x_n),$$

$$h \left| \frac{\partial^2 K}{\partial x \partial f}(x_n,x_n,f_n) \right| << \left| \frac{\partial K}{\partial f}(x_n,x_n,f_n) \right| .$$

Here, $U_\varepsilon(\cdot)$ denotes a small neighbourhood of the point $(\cdot)$.

In the following the derivative $\partial K/\partial f$ occurring in the stability conditions will be assumed to be evaluated at the point (x_n,x_n,f_n).

2.1. An explicit two-point formula

Consider the formula

(2.1) $$f_{n+1} = \tilde{F}_n(x_n+h) + \frac{1}{2}h[K(x_n + h,\ x_n + \frac{1}{2}h\ ,\ f_n) + K\{x_n + h,\ x_n + \frac{1}{2}h\ ,\ \tilde{F}_n(x_n + h) + hK(x_n + h,\ x_n + \frac{1}{2}h\ ,\ f_n)\}.$$

For this (one- $\tilde{F}_n$ and two- K) -evaluation formula we have for the order of accuracy p and the stability condition:

(2.2) $$p = 2,\ -2 < h\frac{\partial K}{\partial f} < 0 \quad \text{if } \tilde{F}_n \text{ according to (1.3) or (1.3').}$$

2.2. An explicit three-point formula

Consider the formula

(2.3) $$f^{(1)}_{n+1} = \tilde{F}_n(x_n + h) + \lambda_1 h K(x_n + h, x_n, f_n),$$

$$f_{n+1}^{(2)} = \tilde{F}_n(x_n + h) + (1 - \lambda_2)h\, K(x_n + h,\ x_n,\ f_n) + \lambda_2 h\, K(x_n + h, x_n, f_{n+1}^{(1)}),$$

(2.3)

$$f_{n+1} = \tilde{F}_n(x_n + h) + \frac{1}{h}h\, K(x_n + h, x_n,\ f_n) + \frac{1}{2}hK(x_n + h,\ x_n + h,\ f_{n+1}^{(2)}),$$

where λ_1 and λ_2 are free parameters. This (one- $\tilde{F}_n$ and three-K) -evaluation formula requires roughly the same computational effort per integration step as formula (2.3). Accuracy and stability are given by

(2.4)

$$p = 2,\ -1.74 < h\frac{\partial K}{\partial f} < 0 \quad \text{for } \lambda_1 = \frac{2}{3},\ \lambda_2 = \frac{1}{2}, \tilde{F}_n \text{ according to (1.3)}$$

$$p = 2,\ -2.51 < h\frac{\partial K}{\partial f} < 0 \quad \text{for } \lambda_1 = \frac{2}{3},\ \lambda_2 = \frac{1}{2}, F_n \text{ according to (1.3')}.$$

2.3 A weakly implicit two-point formula

Consider the formula

(2.5)

$$f_{n+1}^{(1)} = \tilde{F}(x_n + \frac{1}{3}h) + \frac{1}{6}h[K(x_n - \frac{1}{3}h,\ x_n,\ f_n) + K(x_n + h,\ x_n + \frac{1}{3}h,\ f_{n+1}^{(1)}),$$

$$f_{n+1} = \tilde{F}(x_n + h) + \frac{1}{4}h[3K(x_n + h,\ x_n + \frac{1}{3}h,\ f_{n+1}^{(1)}) + K(x_n + h, x_n + h,\ f_{n+1}).$$

This formula requires two $\tilde{F}_n$ -evaluations, at least three K-evaluations and the solution of two equations. Hence, it involves a considerable computational effort when compared with the preceding formulas. The accuracy and stability follow from (provided that Newton-Raphson iteration is used to calculate $f_{n+1}^{(1)}$ and f_{n+1}):

(2.6)

$$p = 2,\ -2 < h\frac{\partial K}{\partial f} < 0 \quad \text{if } \tilde{F}_n \text{ according to (1.3)},$$

$$p = 2,\ -11 < h\frac{\partial K}{\partial f} < 0 \quad \text{if } \tilde{F}_n \text{ according to (1.3')},$$

$$p = 3,\ -11 < h\frac{\partial K}{\partial f} < 0 \quad \text{if } \tilde{F}_n \text{ according to (1.3') and } \partial K/\partial x \equiv 0.$$

2.4. Trapezoidal rule (a weakly implicit one-point formula)

Consider the formula

$$f_{n+1} = \tilde{F}_n(x_n+h) + \frac{1}{2} hK(x_n+h,\ x_n,\ f_n) + \frac{1}{2} hK(x_{n+h},\ x_{n+h},\ f_{n+1}). \tag{2.7}$$

This formula requires one $\tilde{F}_n$ - and two K - evaluations, and the solution of one equation. Accuracy and stability are given by

$$p = 2,\ -\infty < h\,\frac{\partial K}{\partial f} < 0,\ \tilde{F}_n \text{ according to (1.3) or (1.3').} \tag{2.8}$$

3. PROBLEMS SOLVED

In table 3.1 the problems are specified which we have solved with the formulas (2.1), (2.3), (2.5) and (2.7). Furthermore, the stability conditions for these problems are listed in table 3.2. In this table, β denotes the maximal value of $|h\,\frac{\partial K}{\partial f}|$ prescribed by the stability condition of the respective formulas. In all conditions, the function f is replaced by the analytic solution given in table 3.1. In addition to the stability conditions, the expressions for $|\partial^2 K/\partial x \partial f|^{-1} |\partial K/\partial f|$ are given. (We recall that one of the assumptions under which the stability conditions were derived, requires that h is small compared with this value).

Table 3.1 Specification of problems to be solved in the interval [0,4]

	g(x)	K(x,y,f)	$\frac{\partial K}{\partial f}$	solution
(3.1)	$x^2 + \frac{1}{7}x^7$	$-f^3$	$-3f^2$	x^2
(3.2)	$\frac{1}{10}x^2 + \frac{1}{7}x^7$	$-1000f^3$	$-3000f^2$	$\frac{1}{10}x^2$
(3.3)	$x^2 + \frac{1}{7}x^8$	$-xf^3$	$-3xf^2$	x^2
(3.4)	$\frac{1}{10}x^2 + \frac{1}{7}x^8$	$-1000xf^3$	$-3000xf^2$	$\frac{1}{10}x^2$
(3.5)	$2x + 3$	$(-2(x-y)-3)f$	$-2(x-y)-3$	$4.e^{-2x} - e^{-x}$
(3.6)	$e^{-x^2}(x^2+1)+x+1$	$x^2.e^{-xy}.f$	$x^2.e^{-xy}$	x
(3.7)	$e^{-x^2}(x^2+1)+x-1$	$x^2.e^{-xf}.f$	$(1-x^3).e^{-xf}$	x
(3.8)	$\cos x - \sin x$	$2\sin(x-y).f$	$2\sin(x-y)$	e^{-x}
(3.9)	$1 - \sin x$	$\cos(x-y).f$	$\cos(x-y)$	1
(3.10)	e^{-2x}	$e^{-2(x-y)}.f$	$e^{-2(x-y)}$	e^{-x}
(3.11)	e^{-2x}	e^{2x}/f	$-e^{-2x}/f^2$	e^{-x}
(3.12)	$(e^{-x}-1)(x-1)$	$(1+x-y).f$	$1+x-y$	$x.e^{-x}$
(3.13)	$1 - \frac{1}{2}x - \frac{4}{5}x^5$	$(\frac{1}{2} + x^4 - y^4).f$	$\frac{1}{2} + x^4 - y^4$	1
(3.14)	$x^2 + \frac{1}{7}x^9$	$-x^2f^3$	$-3x^2f^2$	x^2
(3.15)	$9x^2 + 11\frac{1}{5}x^5$	$-(x^2-f)^2$	$2(x^2-f)$	$9x^2$
(3.16)	$x + \frac{1}{4}x^3$	$-f^3/x$	$-3f^2/x$	x
(3.17)	$x + \frac{1}{3}x^4$	$-2f^5/x$	$-10f^4/x$	x

Table 3.2 Stability conditions for problem (3.1) through (3.17)

problem	stability condition	$\lvert \partial^2 K/\partial x \partial f \rvert^{-1} \lvert \partial K/\partial f \rvert$
(3.1)	$0 < 3x_n^4 h < \beta$	∞
(3.2)	$0 < 30x_n^4 h < \beta$	∞
(3.3)	$0 < 3x_n^5 h < \beta$	$\lvert x_n \rvert$
(3.4)	$0 < 30x_n^5 h < \beta$	$\lvert x_n \rvert$
(3.5)	$0 < 3h < \beta$	1.5
(3.6)	$h < 0$	$\lvert x_n (2-x_n^2)^{-1} \rvert$
(3.7)	$0 < (x_n^3 - 1)e^{-x_n^2} h < \beta$	$\lvert (x_n^3 - 1)(x^4 - 3x^2 - x)^{-1} \rvert$
(3.8)	$h > 0$	0
(3.9)	$h < 0$	∞
(3.10)	$h < 0$	.5
(3.11)	$0 < h < \beta$	.5
(3.12)	$h < 0$	1
(3.13)	$h < 0$	$\lvert 8x_n{}^3 \rvert^{-1}$
(3.14)	$0 < 3x_n^6 h < \beta$	$\frac{1}{2} \lvert x_n \rvert$
(3.15)	$0 < 16x_n^2 h < \beta$	$\frac{8}{3} \lvert x_n \rvert$
(3.16)	$0 < 3x_n h < \beta$	$\lvert x_n \rvert$
(3.17)	$0 < 10x_n^3 h < \beta$	$\lvert x_n \rvert$

4. RESULTS AND CONCLUSIONS

In the tables 4.1 through 4.17, the results are listed produced by the formulas (2.1), (2.3), (2.5) and (2.7) when applied to the 17 problems specified in table 3.1. Table 4.n refers to problem 3.n (n = 1,...,17). In each table, the reference points are indicated in the first column. The formulas used are indicated in the first row by I, II, III, and IV respectively presenting the formulas (2.1), (2.3), (2.5) and (2.7). In this row we also indicate which formula was used for the evaluation of $\tilde{F}_n$. The last row contains the point where theoretically the process is expected to become unstable, i.e. the points x_c where the quantity $h\partial K/\partial f$ becomes smaller than the stability boundary of the particular formula. When this critical point lies in the interval of integration [0,4] we have separated the theoretically stable and unstable results by a bold line. Finally, all results are presented by the number of correct digits, i.e. by the quantity

$$-{}^{10}\log \left| \frac{f_n - f(x_n)}{f_n(x_n)} \right| ;$$

negative numbers of correct digits are indicated by the minus sign.

In the left and right half of every table these results are given for $h = \frac{1}{10}$ and $\frac{1}{20}$, respectively.

An analysis of the tables of results reveals that:

(1) *the stability theory is justified by the experiments:* in the total number of 238 integrations (7 formulas, 17 examples, 2 stepsizes) only one integration became unstable at a point where the stability theory predicted a stable behaviour (cf. table 3.2, formula II/(1.3'), h = 1/20). In this connection we observe that instabilities developed in problems where $\partial K/\partial f > 0$ were not taken into account since the integral equations itself is not inherently stable.

(2) *the stability conditions are slightly pessimistic:* in 47 of the 238 experiments the integration process remained slightly longer stable than predicted (formula {I,(1.3')} 2 times, {II,(1.3)} 13 times, {II,(1.3')} 4 times, and {III,(1.3)} and {III,(1.3')} both 14 times).

(3) *in the explicit formulas* I *and* II *the use of* (1.3) *for the evaluation of* $\tilde{F}_n$ *yields more accurate results than the use of* (1.3'): *method*

{I,(1.3)} was 13 times more accurate, 2 times less accurate and method II,(1.3) was 14 times more accurate and 3 times less accurate.

(4) *the implicit formula* III *produced more accurate results when* (1.3') *is used instead of* (1.3): 21 experiments against 2 in a total of 34 integrations.

(5) *the implicit formula* IV *has a comparable accuracy compared with explicit methods.*

(6) *the implicit formulas* {III,(1.3)} *and* {III,(1.3')} *are more accurate than the formulas* {I,(1.3)}, {II,(1.3)}, {IV,(1.3)} *and* {I,(1.3')}, {II,(1.3')} *respectively:* in both cases we have 12 times more accuracy and 22 times a comparable accuracy.

(7) *Only when* $\partial K/\partial f$ *is positive and not small, instabilities are developed:* example (3.12) and (3.13) show an unstable behaviour when $\partial K/\partial f$ becomes larger than 4 and 16, respectively.

Table 4.1

Results for problem (3.1) for h = 1/10 and h = 1/20

x_n	I	I	II	II	III	III	IV	I	I	II	II	III	III	IV
	1.3	1.3'	1.3	1.3'	1.3	1.3'	1.3	1.3	1.3'	1.3	1.3'	1.3	1.3'	1.3
0.6	2.9	2.9	3.0	3.0	3.4	4.3	3.0	3.6	3.6	3.6	3.6	3.8	5.2	3.6
0.8	2.5	2.5	2.7	2.7	2.9	4.1	2.6	3.2	3.1	3.2	3.2	3.4	5.0	3.2
1.0	2.2	2.1	2.6	2.6	2.7	4.0	2.4	2.9	2.7	3.1	3.1	3.1	5.0	3.0
1.2	1.8	1.6	2.6	2.2	2.6	4.2	2.4	2.6	2.3	3.5	3.3	3.1	5.0	3.0
1.4	1.4	1.2	1.5	1.2	2.7	5.6	2.4	2.4	2.0	2.5	2.1	3.1	5.7	3.0
1.6	0.6	0.7	1.2	0.7	3.5	4.1	2.5	2.0	1.6	1.9	1.4	3.3	4.9	3.1
1.8	-	0.1	1.8	0.6	2.4	3.8	2.6	1.4	1.2	1.7	0.9	3.6	4.7	3.2
2.0	-	-	0.9	1.2	1.6	3.6	2.7	-	0.5	1.6	0.7	3.0	4.5	3.3
2.2	-	-	-	-	0.8	3.4	2.8	-	-	1.7	-	1.9	4.3	3.4
2.4	-	-	-	-	-	3.2	2.8	-	-	-	-	1.2	4.2	3.4
2.6	-	-	-	-	-	3.0	2.9	-	-	-	-	-	4.0	3.5
3.0	-	-	-	-	-	2.5	3.0	-	-	-	-	-	3.6	3.6
3.5	-	-	-	-	-	1.4	3.2	-	-	-	-	-	2.4	3.8
4.0	-	-	-	-	-	0.3	3.3	-	-	-	-	-	0.3	3.9
x_c	1.61	1.61	1.55	1.70	1.61	2.46	∞	1.91	1.91	1.84	2.02	1.91	2.92	∞

Table 4.2

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	1.8	1.8	2.2	2.2	2.5	3.4	2.1	2.5	2.4	2.7	2.7	2.9	4.4	2.7
0.8	1.1	1.0	1.3	1.2	2.4	3.7	2.0	2.0	1.7	2.5	2.1	2.7	4.8	2.6
1.0	-	0.1	1.2	0.4	2.6	3.7	2.1	1.2	1.1	1.4	0.9	3.1	4.4	2.7
1.2	-	-	0.2	-	1.5	3.2	2.2	-	-	1.4	-	2.4	4.0	2.8
1.4	-	-	-	-	0.6	2.9	2.4	-	-	0.9	-	0.9	3.7	3.0
1.6	-	-	-	-	0.1	2.6	2.5	-	-	-	-	-	3.4	3.1
1.8	-	-	-	-	-	2.2	2.6	-	-	-	-	-	3.0	3.2
2.0	-	-	-	-	-	1.8	2.7	-	-	-	-	-	2.4	3.3
2.2	-	-	-	-	-	0.8	2.8	-	-	-	-	-	1.6	3.4
2.4	-	-	-	-	-	-	2.8	-	-	-	-	-	0.2	3.4
2.6	-	-	-	-	-	-	2.9	-	-	-	-	-	-	3.5
3.0	-	-	-	-	-	-	3.0	-	-	-	-	-	-	3.6
3.5	-	-	-	-	-	-	3.2	-	-	-	-	-	-	3.8
4.0	-	-	-	-	-	-	3.3	-	-	-	-	-	-	3.9
x_c	.90	.90	0.87	0.96	.90	1.38	∞	1.07	1.07	1.03	1.37	1.07	1.64	∞

Table 4.3

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3'	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	3.2	3.2	3.2	3.2	3.6	4.1	3.2	3.8	3.8	3.8	3.8	4.0	4.6	3.8
0.8	2.6	2.6	2.7	2.7	3.0	3.6	2.7	3.3	3.2	3.3	3.3	3.5	4.2	3.3
1.0	2.1	2.1	2.5	2.6	2.7	3.3	2.4	2.9	2.7	3.1	3.1	3.1	3.9	3.0
1.2	1.7	1.5	2.1	2.0	2.6	3.3	2.3	2.5	2.2	4.0	2.9	3.0	3.8	2.9
1.4	1.0	1.0	1.1	0.9	2.9	3.6	2.4	2.1	1.8	2.0	1.7	3.1	3.9	3.0
1.6	-	0.1	1.2	0.7	2.4	3.6	2.5	1.3	1.2	1.6	0.9	3.6	4.5	3.1
1.8	-	-	1.2	-	1.5	3.0	2.6	-	0.2	1.5	0.7	2.7	4.2	3.2
2.0	-	-	-	-	0.6	2.7	2.7	-	-	0.9	0.5	1.2	3.7	3.3
2.2	-	-	-	-	-	2.4	2.8	-	-	-	-	0.7	3.4	3.4
2.4	-	-	-	-	-	2.1	2.8	-	-	-	-	-	3.0	3.4
2.6	-	-	-	-	-	1.7	2.9	-	-	-	-	-	2.5	3.5
3.0	-	-	-	-	-	-	3.0	-	-	-	-	-	1.0	3.6
3.5	-	-	-	-	-	-	3.2	-	-	-	-	-	-	3.8
4.0	-	-	-	-	-	-	3.3	-	-	-	-	-	-	3.9
x_c	1.46	1.46	1.42	1.52	1.46	2.05	∞	1.67	1.67	1.63	1.75	1.67	2.36	∞

Table 4.4

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	2.1	2.1	2.3	2.3	2.6	3.1	2.3	2.7	2.7	2.9	2.9	3.0	3.7	2.9
0.8	1.3	1.2	1.7	1.6	2.4	3.0	2.0	2.1	1.9	3.1	2.6	2.8	3.5	2.6
1.0	-	0.2	0.9	0.4	2.6	3.6	2.1	1.2	1.1	1.4	1.0	3.1	3.9	2.7
1.2	-	-	0.7	-	1.4	2.6	2.2	-	-	1.4	0.6	2.1	3.6	2.8
1.4	-	-	-	-	0.6	2.2	2,4	-	-	-	-	0.7	3.0	3.0
1.6	-	-	-	-	-	1.8	2.5	-	-	-	-	-	2.5	3.1
1.8	-	-	-	-	-	1.4	2.6	-	-	-	-	-	1.9	3.2
2.0	-	-	-	-	-	0.9	2.7	-	-	-	-	-	0.9	3.3
2.2	-	-	-	-	-	-	2.8	-	-	-	-	-	-	3.4
2.4	-	-	-	-	-	-	2.8	-	-	-	-	-	-	3.4
2.6	-	-	-	-	-	-	2.9	-	-	-	-	-	-	3.5
3.0	-	-	-	-	-	-	3.0	-	-	-	-	-	-	3.6
3.5	-	-	-	-	-	-	3.2	-	-	-	-	-	-	3.8
4.0	-	-	-	-	-	-	3.3	-	-	-	-	-	-	3.9
x_c	0.92	0.92	0.89	0.96	0.92	1.29	∞	1.05	1.05	1.03	1.10	1.05	1.48	∞

Table 4.5

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	1.7	1.7	1.8	3.8	1.8	2.5	1.9	2.4	2.3	2.5	3.2	2.5	3.2	2.5
0.8	1.5	1.6	1.6	2.6	1.7	2.2	1.8	2.2	2.3	2.3	3.8	2.3	2.9	2.4
1.0	1.3	1.6	1.4	1.9	1.5	1.9	1.6	2.1	2.4	2.1	2.5	2.2	2.5	2.2
1.2	1.0	2.4	1.2	1.3	1.2	1.5	1.4	1.8	2.2	1.9	1.9	1.9	2.1	2.0
1.4	-	-	0.2	-	0.2	0.3	0.4	0.8	0.5	0.9	0.5	0.9	0.8	1.0
1.6	1.4	0.8	1.8	1.0	1.7	1.4	2.2	2.3	1.4	2.5	1.6	2.6	1.9	2.8
1.8	2.4	0.9	2.3	1.2	2.5	1.5	2.1	3.2	1.5	2.8	1.7	2.9	2.1	2.7
2.0	2.0	0.9	1.9	1.3	1.9	1.6	1.9	2.5	1.5	2.5	1.8	2.5	2.2	2.5
2.2	1.7	0.9	1.7	1.4	1.8	1.7	1.8	2.4	1.5	2.3	1.9	2.4	2.2	2.4
2.4	1.6	0.9	1.6	1.4	1.7	1.7	1.7	2.3	1.5	2.3	1.9	2.3	2.3	2.3
2.6	1.5	0.9	1.6	1.5	1.6	1.7	1.7	2.2	1.5	2.2	1.9	2.3	2.3	2.3
3.0	1.4	0.9	1.5	1.6	1.6	1.7	1.6	2.1	1.5	2.2	2.0	2.2	2.3	2.2
3.5	1.4	0.9	1.5	1.8	1.5	1.8	1.6	2.1	1.5	2.1	2.2	2.2	2.3	2.2
4.0	1.4	0.9	1.5	2.3	1.5	1.8	1.6	2.1	1.6	2.1	2.4	2.1	2.3	2.2
x_c	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

Table 4.6

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	3.6	3.7	3.6	3.6	3.6	3.7	3.5	4.2	4.3	4.2	4.2	4.2	4.3	4.1
0.8	3.3	3.3	3.3	3.3	3.3	3.4	3.2	3.9	3.9	3.9	3.9	3.9	4.0	3.8
1.0	3.0	3.1	3.1	3.1	3.1	3.1	3.0	3.6	3.7	3.6	3.7	3.6	3.7	3.6
1.2	2.9	2.9	2.9	3.0	2.9	3.0	2.9	3.5	3.5	3.5	3.5	3.5	3.6	3.5
1.4	2.8	2.8	2.8	2.9	2.8	2.9	2.8	3.4	3.4	3.4	3.5	3.4	3.5	3.4
1.6	2.7	2.8	2.8	2.8	2.7	2.8	2.7	3.3	3.4	3.4	3.4	3.4	3.4	3.4
1.8	2.7	2.7	2.7	2.8	2.7	2.7	2.7	3.3	3.3	3.3	3.4	3.3	3.3	3.3
2.0	2.7	2.7	2.7	2.7	2.7	2.7	2.7	3.3	3.3	3.3	3.3	3.3	3.3	3.3
2.2	2.7	2.7	2.7	2.7	2.7	2.7	2.7	3.3	3.3	3.3	3.3	3.3	3.3	3.3
2.4	2.6	2.6	2.6	2.7	2.6	2.7	2.6	3.2	3.2	3.2	3.3	3.2	3.3	3.2
2.6	2.6	2.6	2.6	2.7	2.6	2.6	2.6	3.2	3.2	3.2	3.3	3.2	3.2	3.2
3.0	2.6	2.6	2.6	2.6	2.6	2.6	2.6	3.2	3.2	3.2	3.2	3.2	3.2	3.2
3.5	2.5	2.5	2.5	2.5	2.5	2.5	2.5	3.1	3.1	3.2	3.1	3.1	3.1	3.1
4.0	2.5	2.5	2.5	2.5	2.5	2.5	2.5	3.1	3.1	3.1	3.1	3.1	3.1	3.1

Table 4.7

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	3.5	3.5	3.6	3.6	3.6	3.7	3.6	4.1	4.1	4.2	4.2	4.2	4.3	4.2
0.8	3.3	3.2	3.3	3.3	3.3	3.4	3.3	3.9	3.8	3.9	3.9	3.9	4.0	3.9
1.0	3.1	3.1	3.1	3.1	3.1	3.2	3.1	3.7	3.7	3.7	3.7	3.7	3.8	3.7
1.2	3.0	3.0	3.0	3.0	3.0	3.1	3.0	3.6	3.6	3.6	3.6	3.6	3.6	3.6
1.4	2.9	2.9	2.9	2.9	2.9	3.0	2.9	3.5	3.5	3.5	3.5	3.5	3.6	3.5
1.6	2.9	2.9	2.9	2.9	2.9	2.9	2.9	3.5	3.5	3.5	3.5	3.5	3.5	3.5
1.8	2.9	2.9	2.9	2.9	2.9	2.9	2.9	3.5	3.5	3.5	3.5	3.5	3.5	3.5
2.0	2.9	2.8	2.9	2.9	2.9	2.9	2.9	3.5	3.4	3.5	3.5	3.5	3.5	3.5
2.2	2.8	2.8	2.8	2.8	2.8	2.8	2.8	3.4	3.4	3.4	3.4	3.4	3.4	3.4
2.4	2.8	2.8	2.8	2.8	2.8	2.8	2.8	3.4	3.4	3.4	3.4	3.4	3.4	3.4
2.6	2.8	2.7	2.8	2.8	2.7	2.7	2.8	3.4	3.3	3.4	3.4	3.4	3.3	3.4
3.0	2.7	2.7	2.7	2.7	2.6	2.5	2.7	3.3	3.3	3.3	3.3	3.3	3.2	3.3
3.5	2.6	2.6	2.6	2.6	2.4	2.1	2.6	3.2	3.2	3.2	3.2	3.2	2.9	3.2
4.0	2.5	2.5	2.5	2.5	2.2	1.8	2.5	3.1	3.1	3.1	3.1	3.1	2.6	3.1

Table 4.8

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	2.7	3.2	2.5	2.5	2.6	3.3	2.5	3.2	3.6	3.1	3.1	3.2	4.4	3.1
0.8	2.4	2.9	2.3	2.3	2.4	3.3	2.3	3.0	3.3	2.9	2.9	3.0	5.8	2.9
1.0	2.3	2.6	2.2	2.2	2.2	3.6	2.2	2.8	3.0	2.8	2.8	2.8	4.0	2.8
1.2	2.1	2.3	2.0	2.0	2.0	3.5	2.0	2.6	2.8	2.6	2.6	2.6	3.5	2.6
1.4	1.9	2.1	1.8	1.8	1.9	2.8	1.8	2.5	2.5	2.4	2.4	2.5	3.1	2.4
1.6	1.8	1.8	1.7	1.7	1.6	2.4	1.7	2.3	2.3	2.3	2.3	2.3	2.8	2.3
1.8	1.6	1.6	1.5	1.5	1.6	2.1	1.5	2.2	2.1	2.1	2.1	2.1	2.6	2.1
2.0	1.4	1.4	1.3	1.3	1.4	1.9	1.3	2.0	1.9	1.9	1.9	2.0	2.3	1.9
2.2	1.3	1.2	1.2	1.2	1.2	1.6	1.2	1.8	1.7	1.8	1.8	1.8	2.1	1.8
2.4	1.1	1.0	1.0	1.0	1.1	1.4	1.0	1.7	1.5	1.6	1.6	1.6	1.9	1.6
2.6	1.0	0.8	0.8	0.8	0.9	1.2	0.8	1.5	1.3	1.4	1.4	1.5	1.6	1.4
3.0	0.6	0.4	0.5	0.5	0.6	0.8	0.5	1.1	0.9	1.1	1.1	1.1	1.3	1.1
3.5	0.3	-	0.0	0.0	0.1	0.3	0.0	0.7	0.5	0.6	0.6	0.7	0.8	0.6
4.0	-	-	-	-	-	-	-	0.3	0.0	0.2	0.2	0.3	0.3	0.2

Table 4.9

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	3.4	3.8	3.1	2.9	3.3	3.9	3.2	3.9	4.2	3.7	3.5	3.8	4.9	3.8
0.8	3.2	3.5	3.0	2.7	3.1	4.0	3.0	3.7	3.9	3.6	3.3	3.7	5.4	3.6
1.0	3.0	3.2	2.9	2.6	3.0	4.2	2.9	3.6	3.8	3.5	3.2	3.6	4.9	3.5
1.2	2.9	3.1	2.8	2.5	2.9	4.6	2.8	3.5	3.6	3.4	3.1	3.5	4.4	3.4
1.4	2.8	2.9	2.7	2.4	2.8	3.8	2.7	3.4	3.4	3.3	3.0	3.4	4.1	3.3
1.6	2.8	2.8	2.6	2.3	2.7	3.4	2.7	3.3	3.3	3.3	2.9	3.3	3.8	3.3
1.8	2.7	2.6	2.6	2.2	2.7	3.2	2.6	3.3	3.2	3.2	2.8	3.3	3.7	3.2
2.0	2.7	2.5	2.5	2.2	2.6	3.0	2.6	3.2	3.1	3.2	2.8	3.2	3.5	3.2
2.2	2.7	2.4	2.5	2.1	2.6	2.8	2.6	3.2	3.0	3.1	2.7	3.2	3.3	3.2
2.4	2.7	2.3	2.5	2.1	2.6	2.7	2.6	3.2	2.9	3.1	2.7	3.2	3.2	3.2
2.6	2.7	2.2	2.5	2.0	2.6	2.5	2.6	3.2	2.8	3.1	2.6	3.2	3.1	3.2
3.0	2.8	2.1	2.6	2.0	2.7	2.3	2.7	3.3	2.7	3.2	2.6	3.3	2.9	3.3
3.5	3.5	1.9	3.1	2.0	3.3	2.1	3.2	3.9	2.5	3.8	2.6	3.9	2.7	3.9
4.0	2.7	1.8	2.6	2.1	2.7	2.0	2.6	3.3	2.4	3.2	2.8	3.3	2.6	3.3

Table 4.10

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	3.1	2.7	3.6	2.5	3.4	4.8	3.3	3.8	3.2	4.7	3.0	4.0	4.4	3.9
0.8	3.0	2.5	3.7	2.3	3.3	3.9	3.2	3.7	3.0	4.3	2.8	3.8	4.0	3.8
1.0	2.9	2.3	3.9	2.1	3.2	3.4	3.1	3.6	2.8	4.1	2.6	3.7	3.7	3.7
1.2	2.8	2.2	4.1	1.9	3.1	3.1	3.0	3.5	2.7	4.0	2.4	3.6	3.5	3.6
1.4	2.8	2.0	5.2	1.7	3.0	2.9	2.9	3.4	2.6	3.9	2.3	3.6	3.3	3.5
1.6	2.7	1.9	4.2	1.6	2.9	2.7	2.9	3.4	2.5	3.8	2.1	3.5	3.2	3.5
1.8	2.7	1.8	3.9	1.5	2.9	2.6	2.8	3.3	2.3	3.7	2.0	3.5	3.0	3.4
2.0	2.6	1.7	3.7	1.4	2.8	2.4	2.8	3.3	2.2	3.7	1.9	3.4	2.9	3.4
2.2	2.6	1.6	3.6	1.3	2.8	2.3	2.7	3.3	2.1	3.6	1.8	3.4	2.7	3.3
2.4	2.6	1.5	3.5	1.1	2.7	2.1	2.7	3.2	2.0	3.6	1.7	3.3	2.6	3.3
2.6	2.5	1.4	3.4	1.0	2.7	2.0	2.7	3.2	1.9	3.5	1.6	3.3	2.5	3.3
3.0	2.5	1.2	3.3	0.8	2.6	1.8	2.6	3.1	1.7	3.5	1.4	3.2	2.3	3.2
3.5	2.4	1.0	3.2	0.6	2.6	1.5	2.5	3.1	1.5	3.4	1.1	3.2	2.0	3.1
4.0	2.4	0.7	3.1	0.4	2.5	1.3	2.5	3.0	1.3	3.3	0.9	3.1	1.8	3.1

Table 4.11

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	3.3	2.8	3.6	3.7	3.7	3.7	3.5	4.0	3.4	4.2	4.2	4.2	4.2	4.1
0.8	3.3	2.7	3.6	3.4	3.6	3.4	3.5	4.0	3.2	4.1	4.0	4.1	4.0	4.1
1.0	3.3	2.5	3.5	3.2	3.5	3.2	3.4	4.0	3.1	4.1	3.8	4.1	3.8	4.1
1.2	3.3	2.4	3.5	3.0	3.5	3.0	3.4	3.9	3.0	4.1	3.6	4.1	3.6	4.0
1.4	3.3	2.3	3.5	2.9	3.5	2.9	3.4	3.9	2.9	4.0	3.5	4.0	3.5	4.0
1.6	3.3	2.2	3.5	2.8	3.5	2.7	3.4	3.9	2.8	4.0	3.3	4.0	3.3	4.0
1.8	3.3	2.1	3.5	2.6	3.5	2.6	3.4	3.9	2.7	4.0	3.2	4.0	3.2	4.0
2.0	3.3	2.0	3.4	2.5	3.5	2.5	3.4	3.9	2.6	4.0	3.1	4.0	3.1	4.0
2.2	3.3	1.9	3.4	2.4	3.4	2.4	3.4	3.9	2.5	4.0	3.0	4.0	3.0	4.0
2.4	3.3	1.8	3.4	2.3	3.4	2.3	3.4	3.9	2.4	4.0	2.9	4.0	2.9	4.0
2.6	3.3	1.7	3.4	2.2	3.4	2.2	3.4	3.9	2.3	4.0	2.8	4.0	2.8	4.0
3.0	3.3	1.6	3.4	2.0	3.4	2.0	3.4	3.9	2.1	4.0	2.6	4.0	2.6	4.0
3.5	3.3	1.3	3.4	1.8	3.4	1.8	3.4	3.9	1.9	4.0	2.3	4.0	2.4	4.0
4.0	3.3	1.1	3.4	1.5	3.4	1.5	3.4	3.9	1.7	4.0	2.1	4.0	2.1	4.0
x_c	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

Table 4.12

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	2.2	2.2	2.2	2.2	2.3	2.8	2.2	2.8	2.8	2.8	2.8	2.6	3.4	2.8
0.8	2.1	2.1	2.1	2.1	2.1	2.7	2.1	2.7	2.7	2.7	2.7	2.5	3.2	2.7
1.0	1.9	1.9	1.9	2.0	2.0	2.5	1.9	2.5	2.5	2.5	2.6	2.3	3.0	2.5
1.2	1.8	1.8	1.8	1.8	1.8	2.4	1.8	2.4	2.4	2.4	2.5	2.2	2.9	2.4
1.4	1.6	1.6	1.6	1.7	1.7	2.2	1.6	2.2	2.2	2.2	2.3	2.0	2.7	2.2
1.6	1.4	1.4	1.5	1.5	1.5	2.1	1.4	2.0	2.0	2.0	2.1	1.8	2.5	2.0
1.8	1.3	1.3	1.3	1.4	1.3	1.9	1.3	1.9	1.9	1.9	2.0	1.6	2.3	1.9
2.0	1.1	1.1	1.1	1.2	1.1	1.7	1.1	1.7	1.7	1.7	1.8	1.4	2.1	1.7
2.2	0.9	0.9	0.9	1.0	0.9	1.3	0.9	1.5	1.5	1.5	1.6	1.5	1.9	1.5
2.4	0.7	0.7	0.7	0.8	0.8	1.1	0.7	1.3	1.3	1.3	1.4	1.3	1.8	1.3
2.6	0.5	0.5	0.5	0.6	0.6	0.9	0.5	1.1	1.1	1.1	1.2	1.1	1.6	1.1
3.0	0.1	0.1	0.1	0.2	0.2	0.5	0.1	0.7	0.7	0.7	0.8	0.8	1.2	0.7
3.5	-	-	-	-	-	-	-	0.2	0.2	0.2	0.3	0.2	0.7	0.2
4.0	-	-	-	-	-	-	-	-	-	-	-	-	0.2	-

Table 4.13

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	3.6	3.4	3.2	3.2	3.4	5.9	3.1	3.9	4.0	3.7	3.8	3.7	6.8	3.7
0.8	3.0	3.0	2.8	2.8	2.9	4.7	2.7	3.5	3.6	3.3	3.4	3.3	5.6	3.3
1.0	2.6	2.7	2.4	2.5	2.5	4.1	2.4	3.1	3.3	3.0	3.1	3.0	5.1	3.0
1.2	2.3	2.3	2.1	2.2	2.2	3.6	2.0	2.8	2.9	2.7	2.8	2.6	4.6	2.6
1.4	1.9	2.0	1.8	1.9	1.8	3.2	1.7	2.4	2.6	2.3	2.5	2.3	4.2	2.3
1.6	1.5	1.6	1.4	1.5	1.8	2.8	1.3	2.0	2.2	1.9	2.1	1.9	3.8	1.9
1.8	1.1	1.1	0.9	1.1	1.4	2.3	0.9	1.6	1.8	1.5	1.7	1.4	3.4	1.5
2.0	0.5	0.6	0.4	0.6	1.0	1.8	0.4	1.0	1.2	1.0	1.2	0.9	2.8	1.0
2.2	-	-	-	-	0.5	1.2	-	0.5	0.6	0.4	0.6	0.3	2.2	0.4
2.4	-	-	-	-	-	0.5	-	-	-	-	-	-	1.6	-
2.6	-	-	-	-	-	-	-	-	-	-	-	-	0.8	-
3.0	-	-	-	-	-	-	-	-	-	-	-	-	-	-
3.5	-	-	-	-	-	-	-	-	-	-	-	-	-	-
4.0	-	-	-	-	-	-	-	-	-	-	-	-	-	-

Table 4.14

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	3.4	3.4	3.4	3.4	3.8	4.1	3.4	4.0	4.0	4.0	4.0	4.2	4.6	4.0
0.8	2.7	2.7	2.8	2.8	3.1	3.5	2.8	3.4	3.3	3.4	3.4	3.6	4.0	3.4
1.0	2.1	2.1	2.5	2.6	2.7	3.1	2.4	2.8	2.7	3.1	3.1	3.1	3.6	3.0
1.2	1.5	1.4	1.8	1.7	2.6	3.1	2.3	2.4	2.1	3.0	2.6	3.0	3.5	2.9
1.4	0.5	0.6	1.0	0.6	3.2	3.8	2.3	1.8	1.5	1.7	1.3	3.2	3.9	2.9
1.6	-	-	0.4	0.4	1.8	2.7	2.5	-	0.5	1.5	0.8	2.9	3.8	3.1
1.8	-	-	-	-	0.8	2.3	2.6	-	-	0.5	-	1.4	3.2	3.2
2.0	-	-	-	-	-	2.0	2.7	-	-	-	-	0.2	2.8	3.3
2.2	-	-	-	-	-	1.6	2.8	-	-	-	-	-	2.3	3.4
2.4	-	-	-	-	-	1.1	2.8	-	-	-	-	-	1.6	3.4
2.6	-	-	-	-	-	0.6	2.9	-	-	-	-	-	0.7	3.5
3.0	-	-	-	-	-	-	3.0	-	-	-	-	-	-	3.6
3.5	-	-	-	-	-	-	3.2	-	-	-	-	-	-	3.8
4.0	-	-	-	-	-	-	3.3	-	-	-	-	-	-	3.9
x_c	1.37	1.37	1.34	1.42	1.37	1.82	∞	1.53	1.53	1.50	1.59	1.53	2.04	∞

Table 4.15

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	1.5	1.4	2.9	3.0	2.3	3.5	2.0	2.3	2.1	2.8	3.1	2.7	3.8	2.6
0.8	1.2	1.1	1.5	1.3	2.4	3.5	2.1	2.2	1.8	2.7	2.1	2.8	3.9	2.7
1.0	0.8	0.8	1.2	0.7	2.7	3.7	2.3	2.0	1.6	2.0	1.5	3.0	4.1	2.9
1.2	-	0.4	1.1	0.5	2.9	4.0	2.4	1.8	1.3	1.7	1.1	3.2	4.3	3.0
1.4	-	-	1.2	0.4	2.1	4.5	2.5	1.3	1.1	1.6	0.8	3.5	4.4	3.1
1.6	-	-	1.0	-	1.4	4.6	2.6	-	0.9	1.6	0.6	3.5	4.6	3.2
1.8	-	-	-	-	0.8	4.1	2.7	-	0.1	1.7	0.5	2.5	4.8	3.3
2.0	-	-	-	-	-	3.9	2.8	-	-	1.7	-	1.5	5.2	3.4
2.2	-	-	-	-	-	3.8	2.9	-	-	1.7	-	0.9	6.9	3.5
2.4	-	-	-	-	-	3.6	3.0	-	-	0.2	-	1.1	5.3	3.6
2.6	-	-	-	-	-	3.5	3.0	-	-	-	-	-	5.0	3.6
3.0	-	-	-	-	-	3.2	3.2	-	-	-	-	-	4.6	3.8
3.5	-	-	-	-	-	2.6	3.3	-	-	-	-	-	4.3	3.9
4.0	-	-	-	-	-	1.8	3.4	-	-	-	-	-	4.1	4.0
x_c	1.11	1.11	1.04	1.25	1.11	2.62	∞	1.58	1.58	1.47	1.77	1.58	3.70	∞

Table 4.16

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	2.5	2.4	2.8	2.8	2.9	3.1	2.7	3.2	3.0	3.3	3.4	3.4	3.5	3.3
0.8	2.5	2.3	2.9	3.1	2.9	3.0	2.8	3.3	2.9	3.5	3.6	3.5	3.5	3.4
1.0	2.6	2.2	3.2	3.3	3.0	3.1	2.9	3.3	2.8	3.6	4.6	3.5	3.6	3.5
1.2	2.6	2.1	5.1	2.7	3.1	3.1	3.0	3.4	2.7	3.9	3.4	3.6	3.7	3.6
1.4	2.6	2.1	3.2	2.4	3.2	3.2	3.1	3.4	2.7	4.3	3.1	3.7	3.8	3.7
1.6	2.7	2.0	2.9	2.2	3.3	3.3	3.2	3.5	2.7	4.4	2.9	3.8	3.8	3.8
1.8	2.7	2.0	2.8	2.0	3.9	3.9	3.3	3.5	2.6	3.9	2.7	3.9	3.9	3.9
2.0	2.7	1.9	2.6	1.9	4.0	4.0	3.4	3.6	2.6	3.7	2.6	4.0	4.0	4.0
2.2	2.7	1.9	2.6	1.7	3.5	3.5	3.5	3.6	2.6	3.6	2.5	4.1	4.1	4.1
2.4	2.7	1.9	2.5	1.6	3.6	3.6	3.5	3.7	2.6	3.5	2.4	4.2	4.2	4.1
2.6	2.7	1.9	2.4	1.5	3.7	3.6	3.6	3.7	2.6	3.4	2.3	4.2	4.2	4.2
3.0	2.7	1.8	2.4	1.4	3.8	3.7	3.7	3.8	2.5	3.3	2.1	4.4	4.3	4.3
3.5	2.7	1.8	2.3	1.2	3.9	3.9	3.9	3.8	2.5	3.2	1.9	4.5	4.5	4.5
4.0	2.6	1.7	2.3	1.1	4.1	4.0	4.0	3.9	2.5	3.1	1.8	4.6	4.6	4.6
x_c	>4	>4	>4	>4	>4	>4	∞	>4	>4	>4	>4	>4	>4	∞

Table 4.17

x_n	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3	I 1.3	I 1.3'	II 1.3	II 1.3'	III 1.3	III 1.3'	IV 1.3
0.6	2.4	2.4	2.6	2.7	2.9	3.4	2.6	3.1	3.0	3.2	3.2	3.4	3.8	3.2
0.8	2.0	1.9	3.2	3.7	2.8	3.2	2.5	2.8	2.6	3.3	3.6	3.2	3.7	3.1
1.0	1.6	1.5	1.9	1.6	2.8	3.2	2.5	2.6	2.2	3.0	2.5	3.2	3.7	3.1
1.2	1.0	1.1	1.5	1.0	3.2	3.2	2.6	2.3	1.9	2.3	1.7	3.4	3.8	3.2
1.4	-	0.5	1.5	0.8	2.8	3.3	2.8	1.8	1.5	2.0	1.2	3.7	3.9	3.4
1.6	-	0.2	1.1	2.7	2.0	3.4	2.9	0.1	1.1	1.9	0.9	3.8	4.0	3.5
1.8	-	-	-	-	1.3	3.5	3.0	-	0.4	1.9	0.9	2.6	4.1	3.6
2.0	-	-	-	-	1.3	3.5	3.1	-	0.2	2.0	0.9	1.5	4.1	3.7
2.2	-	-	-	-	1.4	3.6	3.2	-	-	0.6	-	1.4	4.2	3.8
2.4	-	-	-	-	2.0	3.5	3.2	-	-	-	-	1.5	4.2	3.8
2.6	-	-	-	-	1.3	3.4	3.3	-	-	-	-	1.7	4.2	3.9
3.0	-	-	-	-	-	3.0	3.4	-	-	-	-	1.1	3.0	4.1
3.5	-	-	-	-	-	1.8	3.6	-	-	-	-	-	2.6	4.2
4.0	-	-	-	-	-	0.6	3.7	-	-	-	-	-	0.9	4.2
$\tilde{x}_c$	1.25	1.25	1.20	1.35	1.25	2.22	∞	1.58	1.58	1.51	1.71	1.58	2.80	∞

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