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ON THE STABILITY OF DIRECT QUADRATURE RULES FOR SECOND KIND VOLTERRA INTEGRAL EQUATIONS

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On the stability of direct quadrature rules for second kind Volterra integral equations

by

P.J. van der Houwen & P.H.M. Wolkenfelt

ABSTRACT

The purpose of this note is to analyze the stability properties of a class of direct quadrature rules for second kind Volterra integral equations with an arbitrary (non linear) kernel function. The main idea proposed here is to imbed the integral equation into a differential equation containing a parameter. The solution of the integral equation is then found as part of the solution of the differential equation. The stability analysis follows the stability approach for numerical methods applied to differential equations.

KEY WORDS & PHRASES : *Numerical analysis, Volterra integral equations of the second kind, stability.*

1. INTRODUCTION

In this note a possible approach is proposed to investigate the stability behaviour of numerical methods for non-linear Volterra integral equations of the form

$$(1.1) \qquad f(x) = g(x) + \int_{x_0}^{x} K(x,y,f(y))dy$$

where $g(x)$ and $K(x,y,f)$ are arbitrary vector functions belonging to a class of sufficient differentiability and $f(x)$ is the unknown function.

Recently, several papers have been published in which stability results are stated for particular classes of methods and kernel functions varying from simple linear functions such as

$$(1.2) \qquad K = af \text{ and } K = (ax+b)f, \text{ a and b constant}$$

(cf. [1,2]) to rather general kernel functions of the form (cf. [3])*

$$(1.3) \qquad K = \sum_{i=1}^{r} A_i(x)B_i(y,f).$$

In essence, the approach presented in these papers turns out to be the analysis of a numerical method for ordinary differential equations to which the integral equation method is more or less equivalent when applied to kernel functions of the form (1.2) or (1.3).

This suggests to start with converting the integral equation (1.1) into a differential equation without restricting the kernel function K. By identifying the integral equation method to an integration method for this (rather peculiar) differential equation and by carrying out a stability analysis of the integration method it seems possible to derive stability conditions for general kernel functions. In this note we apply this approach to a class of direct quadrature rules.

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* See also C.T.H. Baker: Structure of Recurrence Relations in the Study of Stability in the Numerical Treatment of Volterra Equations, Techn. Rep. No. 36, Depart. of Mathematics, Univ. of Manchester, 1979.

2. DERIVATION OF THE DIFFERENTIAL EQUATION

Let the definition of the kernel function $K(x,y,f)$ be extended for $y > x$ (e.g. by defining $K(x,y) = K(y,x)$ if $y > x$) and define the function

$$(2.1) \qquad F(t,x) = g(x) + \int_{x_0}^{t} K(x,y,f(y))dy, \qquad x_0 \le t,\ x \le X$$

where $f(x)$ satisfies the integral equation (1.1). Since we obviously have

$$(2.2) \qquad f(x) = F(x,x)$$

we may write (2.1) as

$$(2.1') \qquad F(t,x) = g(x) + \int_{x_0}^{t} K(x,y,F(y,y))dy.$$

This equation contains $f(x)$ as a part of its solution. Let us differentiate with respect to t, then we are led to the initial value problem

$$(2.3) \qquad \begin{aligned} &\frac{d}{dt} F(t,x) = K(x,t,F(t,t)) \\ &F(x_0,x) = g(x) \end{aligned} \quad , \quad x_0 \le x,\ t \le X.$$

Note that here x is considered as a parameter and t as the integration variable.

3. INTEGRATION BY LINEAR MULTISTEP METHODS

Suppose that (2.3) is integrated by a linear k-step method with coefficients $\{a_\ell, b_\ell\}$, then we obtain a scheme of the form

$$(3.1) \qquad \sum_{\ell=0}^{k} [a_\ell F_{n+1-\ell}(x) + b_\ell h_n K(x, t_{n+1-\ell}, F_{n+1-\ell}(t_{n+1-\ell}))] = 0,$$

$$n = k-1, k, \ldots, N-1,$$

where $h_n = t_{n+1}-t_n$ denotes the integration step and $F_n(x)$ denotes the numerical approximation to $F(t_n,x)$, $n = 0,1,\dots,N$. In order to start scheme (3.1) we need apart from $F_0(x) = g(x)$ the functions $F_1(x),\dots,F_{k-1}(x)$. In [7] methods may be found to compute these starting functions. $F_{n+1}(x)$, $n = k-1,k,\dots,$ can be computed by first solving $F_{n+1}(t_{n+1})$ from the equation

$$(3.2)\qquad a_0F_{n+1}(t_{n+1})+b_0h_nK(t_{n+1},t_{n+1},F_{n+1}(t_{n+1})) = -\sum_{\ell=1}^{k}[a_\ell F_{n+1-\ell}(t_{n+1})+ +b_\ell h_nK(t_{n+1},t_{n+1-\ell},F_{n+1-\ell}(t_{n+1-\ell}))],$$

and then writing

$$(3.3)\qquad F_{n+1}(x) = -a_0^{-1}\left\{\sum_{\ell=1}^{k}a_\ell F_{n+1-\ell}(x)+h_n\sum_{\ell=0}^{k}b_\ell K(x,t_{n+1-\ell},F_{n+1-\ell}(t_{n+1-\ell}))\right\}.$$

Finally, by putting $f_{n+1} = F_{n+1}(x_{n+1})$ a numerical approximation to $f(x_{n+1})$ is obtained. Since we are only interested in f_{n+1} it suffices to evaluate (3.3) for $x = t_{n+1},t_{n+2},\dots,t_N$. Thus, scheme (3.1) roughly requires the solution of N equations of the form (3.2) and $N^2/2$ evaluations of the kernel function K.

3.1. Relation with direct quadrature rules

In this section we indicate a relation between the scheme (3.1) and the direct quadrature rules frequently used for the integration of (1.1), that is formulas of the form

$$(3.4)\qquad f_{n+1} = g(x_{n+1}) + \sum_{j=0}^{n+1} w_{n+1,j}K(x_{n+1},x_j,f_j),$$

where $w_{n,j}$, $j=0,1,\dots,n$; $n = 0,1,\dots,N$ are given weight parameters and f_n denotes the numerical approximation to $f(x_n)$. We have the following theorem.

THEOREM 3.1. *If the weights* $w_{n,j}$ *in formula* (3.4) *are such that an integer* k *and coefficients* a_ℓ *can be found such that*

$$(3.5)\qquad \begin{aligned} &\sum_{\ell=0}^{k} a_\ell w_{n+1-\ell,j} = 0 \\ &\sum_{\ell=0}^{k} a_\ell = 0 \end{aligned} \quad , \quad j = 0,1,\ldots,n-k, \quad n \geq k-1,$$

and if we define the coefficients b_ℓ *by*

$$(3.6)\qquad b_\ell = -\frac{1}{h_n}\sum_{i=0}^{k} a_i w_{n+1-i,n+1-\ell}, \quad \ell = 0,1,\ldots,k, \quad n \geq k-1$$

then $F_{n+1}(t_{n+1})$ *as defined by* (3.1) *and* f_{n+1} *defined by* (3.4) *are identical provided that* $t_n = x_n$, $n=0,1,\ldots,N$, *and that the starting functions* $F_n(x)$ *are defined by*

$$(3.7)\qquad F_n(x) = g(x) + \sum_{j=0}^{n} w_{n,j} K(x,t_j,F_j(t_j)), \quad n = 0,1,\ldots,k-1.$$

PROOF. The proof is straightforward by writing $F_{n+1}(x)$ in the form

$$(3.8)\qquad F_{n+1}(x) = g(x) + \sum_{j=0}^{n+1} w_{n+1,j} K(x,t_j,F_j(x_j)),$$

by substitution into (3.1), and by equating corresponding terms. It then appears that (3.8) is correct provided that (3.5) and (3.6) are satisfied. From (3.8) it is immediate that $F_{n+1}(x_{n+1})$ satisfies the same scheme as f_{n+1} so that $f_{n+1} = F_{n+1}(x_{n+1})$. □

EXAMPLE 3.1. Consider a Gregory scheme of order 3 which is generated by the matrix

$$(3.9) \qquad (w_{nj}) = \frac{h}{12}\begin{bmatrix} 6 & 6 & & & & & & & \\ 5 & 14 & 5 & & & & & & \\ 5 & 13 & 13 & 5 & & & & & \\ 5 & 13 & 12 & 13 & 5 & & & & \\ & & \cdots & & & & & & \\ 5 & 13 & 12 & 12 & .. & 12 & 12 & 13 & 5 \end{bmatrix}$$

Evidently, conditions (3.5) are satisfied when we choose $k=2$, $a_0=-1$, $a_1=1$, $a_2=0$. For the coefficients b_ℓ we obtain $b_0 = 5/12$, $b_1 = 8/12$ and $b_2 = -1/12$. These coefficients are easily recognized as those of the third order Adams-Moulton formula for ordinary differential equations. It can be proved that a k-th order Gregory scheme for the integral equation (1.1) may be embedded in a k-th order Adams-Moulton method for equation (2.3). Of course, this correspondence is a consequence of the link between Gregory quadrature rules and the Adams-Moulton integration methods. For further details of the relation between direct quadrature formulas and linear multistep methods we refer to [7]. □

EXAMPLE 3.2. Consider an integration scheme based on the repeated Simpson rule and using the trapezoidal rule for the first interval when n is odd. The matrix of weights then becomes

$$(3.10) \qquad (w_{n,j}) = \frac{h}{6}\begin{bmatrix} 3 & 3 & & & & & & & & & & \\ 2 & 8 & 2 & & & & & & & & & \\ 3 & 5 & 8 & 2 & & & & & & & & \\ 2 & 8 & 4 & 8 & 2 & & & & & & & \\ & & & & \cdots & & & & & & & \\ 3 & 5 & 8 & 4 & 8 & 4 & 8 & \ldots & 4 & 8 & 2 & \\ 2 & 8 & 4 & 8 & 4 & 8 & 4 & \ldots & 8 & 4 & 8 & 2 \end{bmatrix}$$

For all values of $n \geq 1$ conditions (3.5) can be satisfied by $k = 2$, $a_0 = -1$, $a_1 = 0$, $a_2 = 1$. The coefficients b_ℓ follow from (3.6): $b_0 = 1/3$, $b_1 = 4/3$ and $b_2 = 1/3$. These coefficients define the fourth order Milne method. □

3.2. Backward differentiation formulas

A particularly interesting class of multistep methods are based on backward differentiation formulas (also known as Curtiss-Hirschfelder formulas). These formulas are recommended in the literature because of their excellent stability properties [5]. Since stability may be a problem in the integration of Volterra integral equations, we pay special attention to the use of backward differentiation formulas for integration of (1.1). The coefficients of these formula are defined by (using constant integration steps)

$$a_0 = -1, \quad \sum_{\ell=1}^{k} (1-\ell)^j a_\ell + j b_0 = 1, \qquad j = 0,1,\ldots,k. \tag{3.11}$$

The corresponding scheme of the form (3.4) may be found by solving (3.5) and (3.6) for $w_{n+1,j}$ (cf. [7]). The resulting scheme is discussed in [4,7]. From a practical point of view, however, it may be more convenient to base the implementation of the algorithm on (3.1) instead of (3.4).

3.3. Stability

The first order variational equation of scheme (3.1) assumes the form

$$\sum_{\ell=0}^{k} [a_\ell \Delta F_{n+1-\ell}(x) + b_\ell h_n \frac{\partial K}{\partial f}(x,t_{n+1-\ell},f_{n+1-\ell})\Delta F_{n+1-\ell}(t_{n+1-\ell})] = 0. \tag{3.12}$$

Let us define the operators $B_\ell = B_\ell(t_{n+1})$ by

$$B_\ell\colon F(x) \to a_\ell F(x) + b_\ell h_n J_{n+1-\ell}(x) F(t_{n+1-\ell}), \tag{3.13}$$

where

$$\frac{\partial K}{\partial f}(x,t_{n+1-\ell},f_{n+1-\ell}) = J_{n+1-\ell}(x). \tag{3.14}$$

Then (3.12) may be written in the form

$$(3.12') \qquad \Delta F_{n+1}(x) = -B_0^{-1} \sum_{\ell=1}^{k} B_\ell F_{n+1-\ell}(x)$$

or equivalently

$$(3.15) \qquad \Delta\vec{V}_{n+1} = A_n \Delta\vec{V}_n,$$

where $\Delta\vec{V}_{n+1}$ is the vector function

$$\Delta\vec{V}_{n+1}(x) = (\Delta F_{n+1}(x), \Delta F_n(x), \ldots, \Delta F_{n+2-k}(x))^T$$

and

$$(3.16) \qquad A_n = \begin{bmatrix} -B_0^{-1}B_1 & -B_0^{-1}B_2 & \cdots & -B_0^{-1}B_{k-1} & -B_0^{-1}B_k \\ I & & & \bigcirc & \\ & I & & & \\ \bigcirc & & \ddots & & \\ & & & I & 0 \end{bmatrix}$$

I denoting the unit matrix.

DEFINITION 3.1. Scheme (3.1) will be called stable when there exists for A_n a norm $\| \ \|$ independent of n such that $\|A_n\| \leq 1$ for all n.

A *necessary* condition for stability in the sense of this definition is the requirement that (a) all eigenvalues of A_n are on the unit disk; (b) an eigenvalue which is on the unit circle has μ independent eigenfunctions, μ being the multiplicity of that eigenvalue. For *matrix operators* A_n these requirements would be *sufficient* for the existence of a norm $\| \ \|_n$ such that $\|A_n\|_n \leq 1$ (cf. [6]), which may be interpreted as a form of *local stability*. This has led us to investigate under what conditions the operator A_n defined by (3.16) satisfies the requirements (a) and (b), and to use the resulting conditions as stability conditions for scheme (3.1).

THEOREM 3.2. *Let* $\rho(\zeta)$ *be the polynomial defined by*

$$\rho(\zeta) = \sum_{\ell=0}^{k} a_\ell \zeta^{k-\ell},$$

let $R(\zeta)$ *be the matrix with elements*

$$r_{j\ell} = \rho(\zeta)\delta_{j,\ell} I_s + b_\ell h_n J_{n+1-\ell}(t_{n+1-j})\zeta^{k-\ell}, \qquad j,\ell = 0,1,\ldots,k,$$

where s *is the dimension of the system* (1.1) *and let* $\tilde{R}(\zeta)$ *be the* $(k+1-\mu)*(k+1-\mu)$*-matrix obtained from* $R(\zeta)$ *by omitting the* j-th *row and the* j-th *column for all* j *with* $b_j = 0$, μ *being the number of vanishing coefficients* b_j. *Then, the following holds*:

(i) *The eigenvalues of* A_n *satisfy the characteristic equation*

$$[\rho(\zeta)]^{\mu s}\det[\tilde{R}(\zeta)] = 0; \tag{3.17}$$

(ii) *A necessary condition for scheme* (3.1) *to be stable in the sense of definition* 3.1 *is the requirement that* $\rho(\zeta)$ *has its zeroes within or on the unit circle those on the unit circle having multiplicity* 1((*the so-called root condition*), *and that* $\det[\tilde{R}(\zeta)]$ *has its zeroes within the unit circle.*

PROOF. Without loss of generality we prove the theorem for s = 1. Let $\vec{e}(x) = (e_1(x),\ldots,e_k(x))^T$ be an eigenfunction of A_n with eigenvalue ζ. We first construct $\vec{e}$ and ζ. The equation

$$A_n\vec{e} = \zeta\vec{e}$$

leads to the equations

$$e_1(x) = \zeta^{k-1}e_k(x),\ldots,e_{k-1}(x) = \zeta e_k(x) \tag{3.18a}$$

and

$$\sum_{\ell=0}^{k} B_\ell \zeta^{k-\ell} e_k(x) = 0. \tag{3.18b}$$

Substitution of (3.13) yields

$$(3.18b') \quad \sum_{\ell=0}^{k} [a_\ell I e_k(x) + b_\ell h_n J_{n+1-\ell}(x) e_k(t_{n+1-\ell})]\zeta^{k-\ell} = 0.$$

This relation determines $e_k(x)$, and by (3.18a) the eigenfunction $\vec{e}$, if we can find ζ and the $e_k(t_{n+1-\ell})$, $\ell = 0,1,\dots,k$. Consider the relations

$$(3.18b'') \quad \sum_{\ell=0}^{k} [a_\ell e_k(t_{n+1-j}) + b_\ell h_n J_{n+1-\ell}(t_{n+1-j}) e_k(t_{n+1-\ell})]\zeta^{k-\ell} = 0,$$

$$j = 0,1,\dots,k,$$

which are obtained from (3.18b') by substituting $x = t_{n+1-j}$. These relations represent a linear homogeneous system of k+1 equations for the k+1 components $e_k(t_{n+1-\ell})$. A non-trivial solution is obtained if the matrix of coefficients has a zero-determinant, i.e. if the eigenvalue ζ satisfies the equation

$$(3.17') \quad \det\{(\rho(\zeta)\delta_{j,\ell} I + b_\ell h_n J_{n+1-\ell}(t_{n+1-j})\zeta^{k-\ell})\} = 0,$$

where $\delta_{j,\ell}$ is the Kronecker symbol. Solving ζ from this equation and the $e_k(t_{n+1-j})$ from (3.18b"), and substitution into (3.18b') yields the component $e_k(x)$ from which $\vec{e}$ can be derived by virtue of (3.17).

The first part of the theorem follows from (3.17') which is easily verified to be equivalent to (3.17).

The second part is based on the requirements (a) and (b) formulated above and the eigenvalue equation (3.17). It suffices to show that to each zero $\tilde{\zeta}$ of $\rho(\zeta)$ with $|\tilde{\zeta}| = 1$ there correspond μ independent eigenfunctions of A_n. This means that we have to show that (3.18b") with $\zeta = \tilde{\zeta}$, i.e. the equations

$$\sum_{\ell=0}^{k} b_\ell J_{n+1-\ell}(t_{n+1-j}) e_k(t_{n+1-\ell})\tilde{\zeta}^{k-\ell} = 0, \quad j = 0,1,\dots,k,$$

has μ linearly independent solution vectors $\vec{v} = (e_k(t_{n+1}),\ldots,e_k(t_{n+1-k}))^T$. Since μ coefficients b_ℓ are zero, this system of k+1 equations contains only k+1-μ unknowns $e_k(t_{n+1-\ell})$ and therefore does have μ independent solutions which proves the second part of the theorem. □

3.4. Stability conditions for the trapezoidal rule

In order to illustrate the use of theorem 3.2 we apply it to the trapezoidal rule which is generated by

$$k = 1,\ a_0 = -a_1 = -1,\ b_0 = b_1 = \tfrac{1}{2}.$$

The polynomial $\rho(\zeta)$ is given by $\rho(\zeta) = 1-\zeta$, which trivially satisfies the root condition. The determinantal equation in (3.17) is given by

$$\text{(3.19)}\qquad \det[\tilde{R}(\zeta)] = \det\begin{pmatrix} I-\zeta I+\tfrac{1}{2}hJ_{n+1}(t_{n+1})\zeta & \tfrac{1}{2}hJ_n(t_{n+1}) \\ \tfrac{1}{2}hJ_{n+1}(t_n)\zeta & I-\zeta I+\tfrac{1}{2}hJ_n(t_n) \end{pmatrix} = 0.$$

Let the matrices $J_i(t_\ell)$ occurring in (3.19) have the same (complete) set of eigenvectors Q. That is, they can be reduced to diagonal matrices (containing the eigenvalues) by the same transformation Q (containing the eigenvectors). In this case (3.19) can be written as

$$\text{(3.19')}\qquad \det\Big[\zeta^2(I-\tfrac{1}{2}hJ_{n+1}(t_{n+1})) -\zeta(2I+\tfrac{1}{2}h\{J_n(t_n)-J_{n+1}(t_{n+1})\}+\frac{h^2}{4}\{J_{n+1}(t_n)J_n(t_{n+1})-J_{n+1}(t_{n+1})J_n(t_n)\}) +I+\tfrac{1}{2}hJ_n(t_n)\Big] = 0,$$

by virtue of the following lemma.

LEMMA 3.1. *Suppose* A, B, C *and* D *are square matrices of the same order. If*

$$DC = CD$$

then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD-BC).$$

For a proof of this lemma we refer to [4].

Due to our assumptions the matrices occurring in (3.19') can be reduced to diagonal matrices without affecting the value of the determinant. Let eigenvalues of the matrices $hJ_i(t_\ell)$ corresponding to the same eigenvector be denoted by $z_{i,\ell}$. Then we are led to the condition that the roots ζ of

$$(3.20)\qquad \begin{aligned} &(1-\tfrac{1}{2}z_{n+1,n+1})\zeta^2-[2+\tfrac{1}{2}(z_{n,n}-z_{n+1,n+1})+\tfrac{1}{4}(z_{n+1,n}z_{n,n+1}-z_{n+1,n+1}z_{n,n})]\zeta \\ &\qquad +(1+\tfrac{1}{2}z_{n,n}) = 0 \end{aligned}$$

must be located within the unit circle.

Let the eigenvalues $z_{i,\ell}$ be negative (otherwise the integral equation itself is not stable), then by Hurwitz' criterion the roots of (3.20) are within the unit circle if

$$(3.21)\qquad \begin{aligned} &z_{n,n} < 0,\ z_{n+1,n+1} < 0,\ z_{n,n+1} < 0,\ z_{n+1,n} < 0 \\ &z_{n+1,n+1}\, z_{n,n} - z_{n+1,n}\, z_{n,n+1} > 0 \\ &z_{n,n} - z_{n+1,n+1} + \tfrac{1}{4}(z_{n+1,n}z_{n,n+1}-z_{n+1,n+1}z_{n,n}) > -4. \end{aligned}$$

For kernel functions of the form $K(x,y,f) = (-a+bx+cy)f$ the stability conditions (3.21) read

$$(3.21')\qquad \begin{aligned} &(b+c)x_n < a,\ (b+c)(x_n+h) < a,\ (b+c)x_n+ch < 0, \\ &(b+c)x_n+bh < a,\quad bch^4 < 0, \\ &(b+c)h^2-\tfrac{1}{4}bch^4 < 4. \end{aligned}$$

When we compare the conditions (3.21') with the conditions derived in [3] for the same class of kernel functions, we see that (3.21') yields more stringent conditions. In particular, the condition $bch^4 < 0$ was not met in [3]. This difference in stability conditions is explained by the difference in stability definitions. Recall that, in the present paper, the stability analysis can be carried out only when the kernel function is defined in the whole rectangle $x_0 \le x,t \le X$, whereas in [3] the kernel function has to be defined only on the triangle $x_0 \le t \le x \le X$. As a consequence the definition of stability given in [3] differs from the one given in this paper.

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