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A NOTE ON DEFECT CORRECTION PROCESSES WITH AN APPROXIMATE INVERSE OF DEFICIENT RANK

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A NOTE ON DEFECT CORRECTION PROCESSES WITH AN APPROXIMATE INVERSE OF DEFICIENT RANK

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A note on defect correction processes with an approximate inverse of deficient rank

by

P.W. Hemker

ABSTRACT

In view of the analysis of multiple grid methods, in this note we consider Defect Correction Processes of deficient rank. Both for the error and for the residual, the convergence of the defect correction iterative process is studied in terms of the range and the kernel of the approximate inverse. Since the coarse grid correction in the multiple grid algorithm can be seen as a step in such an iterative process, the present study can be used in the convergence analysis of these algorithms. In this sense, the present paper is a generalization of previous work by S.F. McCormick.

KEY WORDS & PHRASES: *defect correction, multi-grid methods*

THE DEFECT CORRECTION PROCESS

In order to solve the operator equation

$$Fx = y, \tag{1}$$

$F: B_1 \to B_2$, B_1, B_2 Banach spaces, we consider the defect correction iterative process

$$\begin{cases} x_0 = \tilde{G}y, & y \in B_2, \\ x_{i+1} = x_i - \tilde{G}Fx_i + \tilde{G}y. \end{cases} \tag{2}$$

The process is determined by the operator $\tilde{G}: B_2 \to B_1$, which is called the *approximate inverse* of F.

In this paper we consider only linear operators F and $\tilde{G}$. We notice that the process (2) converges to the solution x^* of (1) if $\tilde{G}$ is injective and

$$\| I - \tilde{G}F \|_{B_1 \to B_1} < 1.$$

The value $e_i = x_i - x^*$ is called the error of x_i; the operator

$$M = I - \tilde{G}F$$

plays an important part in our considerations and is called the *amplification operator of the error*, since

$$e_{i+1} = Me_i.$$

We notice also that, due to the linearity of $\tilde{G}$, the process (2) is equivalent with

$$\begin{cases} \ell_0 = y, \\ \ell_{i+1} = \ell_i - F\tilde{G}\ell_i + y, \end{cases} \tag{3}$$

when x_i is identified with

$$x_i = \tilde{G}\ell_i.$$

The process (3) converges to the solution of (1) if

$$\| I - F\tilde{G} \|_{B_2 \to B_2} < 1;$$

the value $r_i = y - Fx_i$ is called the residual of x_i and the operator

$$\hat{M} = I - F\tilde{G}$$

is called the *amplification operator of the residual* since $r_{i+1} = \hat{M} r_i$.

In particular we shall here consider the processes (2) and (3) where F and $\tilde{G}$ are operators $\mathbb{R}^n \to \mathbb{R}^n$, where F is a full rank matrix, such that the original problem (1) has a unique solution, and $\tilde{G}$ is of deficient rank, i.e. $\tilde{G}$ is neither injective nor surjective.

Because $\text{rank}(\tilde{G}) = k < n$, we know that $N = \text{Range}(\tilde{G})$ is a k-dimensional subspace of $\mathbb{R}^n$ and $Z = \text{Kernel}(\tilde{G})$ is a (n-k)-dimensional subspace of $\mathbb{R}^n$.

In order to define orthonormal bases in N and Z, we can decompose the n*n matrix $\tilde{G}$ into its singular value decomposition (cf. Lawson & Hanson):

$$\tilde{G} = U \Sigma V^T,$$

where U, Σ and V are n*n matrices, U and V are orthonormal and Σ is a non-negative diagonal matrix. Except for the ordering of the elements of Σ, this decomposition is uniquely determined. The diagonal elements of Σ are the singular values and normally they are ordered such that

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0.$$

Because $\text{rank}(\tilde{G}) = k$, we know that $\sigma_1, \sigma_2, \ldots, \sigma_k$ are non-zero and $\sigma_j = 0$, $j = k+1, \ldots, n$.

Since $\tilde{G}$ is not surjective, possibly $x^* \notin N$; however, from (2) we see that all approximate solutions $\{x_i\}$ are in N. Hence, if $\{x_i\}_{i=0,1,\ldots}$ attains a stationary point $\hat{x}$, $\hat{x}$ is not necessarily the solution of (1). However, we know

$$\tilde{G}(y - F\hat{x}) = 0,$$

i.e. the residual $\hat{r} = y - F\hat{x} \in Z$. Thus, with $\Delta = V_1 V_1^T$ the projection $\mathbb{R}^n \to Z^\perp$, instead of the sequence $\{\ell_i\}$ in (3) we may consider the sequence $\{\lambda_i\}$, with $\lambda_i = \Delta\ell_i$:

$$\begin{cases} \lambda_0 = \Delta y, \\ \lambda_{i+1} = \lambda_i - \Delta F\tilde{G}\lambda_i + \Delta y, \end{cases}$$

which has a unique stationary point $\hat{\lambda}$, satisfying

$$\Delta F\tilde{G}\hat{\lambda} = \Delta y.$$

Clearly, $N = \mathrm{Span}(U_1)$, where U_1 are the first k column vectors of U and $N^\perp = \mathrm{Span}(U_2)$, the last n-k columns of U. Analogously, $Z = \mathrm{Span}(V_2)$ and $Z^\perp = \mathrm{Span}(V_1)$.

From the singular value decomposition we easily see that for an arbitrary $P: \mathbb{R}^k \to \mathbb{R}^n$ and $R: \mathbb{R}^n \to \mathbb{R}^k$, with range(P) = N and Kernel(R) = Z, we may write

$$\tilde{G} = PSR,$$

where $S: \mathbb{R}^k \to \mathbb{R}^k$ is the nonsingular k*k matrix for which

$$S^{-1} = (RV_1)\,\mathrm{diag}\left(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_k}\right)\,(U_1^T P).$$

The operators P and R are called *prolongation* and *restriction* respectively.

Because P and R are full rank matrices, rank(P) = rank(R) = k, P has the left-inverse $\hat{R} = (U_1^T P)^{-1} U_1^T$ and R has the right-inverse $\hat{P} = V_1 (RV_1)^{-1}$. Moreover, we know that

$$P\hat{R} = P(U_1^T P)^{-1} U_1^T : \mathbb{R}^n \to N,$$

and

$$\hat{P}R = V_1 (RV_1)^{-1} R : \mathbb{R}^n \to Z^\perp,$$

are projection operators.

Now we can consider what happens to the error to the solution or to the residual after one iteration step of the defect correction process.

I. To study the effect on the error of the solution, we consider (2), of which the transition matrix is

$$M = I - \tilde{G}F = I - PSRF.$$

We decompose the error e into two parts: $e = e_s + e_u$, with $e_s \in N$ and $e_u \in N^{\perp}$. Analogously, we write

$$Me = (Me)_s + (Me)_u,$$

with $(Me)_s \in N$ and $(Me)_u \in N^{\perp}$.

From the relation

$$Me_s = MP\hat{R}e_s = (P\hat{R} - PSRFP\hat{R})e_s = P(I - SRFP)\hat{R}e_s,$$

we see that $Me_s \in N$. Moreover, we notice that in the special case when $S^{-1} = RFP$, we have $Me_s = 0$.

In the general case, with $S^{-1} = RFP + E$ we have

$$Me_s = PS\hat{E}Re_s = \tilde{G}\hat{P}\hat{E}Re_s.$$

In practice, where (for good convergence) $\tilde{G} = PSR$ should approximate F^{-1}, it is often possible to choose S^{-1} equal or close to RFP, the Galerkin approximation of F.

The contribution from e_u to Me is given by

$$Me_u = e_u - \tilde{G}Fe_u,$$

with $\tilde{G}Fe_u \in N$ and $e_u \in N^{\perp}$.

We conclude that

$$(Me)_s = \tilde{G}\hat{P}\hat{E}Re_s - \tilde{G}Fe_u,$$

$$(Me)_u = e_u.$$

Note: In the context of multigrid methods, the components in N are those grid functions in the fine grid that can be obtained by prolongation from a coarse grid function; therefore they are called the *smooth components of the error*. Those in $N^{\perp}$ are the *unsmooth components of the error*.

II. For the residual, the transition matrix is

$$\hat{M} = I - F\tilde{G} = I - FPSR.$$

Now we decompose the residual r into two parts $r = r_s + r_u$, with $r_u \in Z$ and $r_s \in Z$. Analogously we write

$$\hat{M}r = (\hat{M}r)_s + (\hat{M}r)_u.$$

Again, a simple computation shows

$$(\hat{M}r)_s = \hat{P}E\hat{R}\tilde{G}r_s,$$

$$(\hat{M}r)_u = -(I-\hat{P}R)F\tilde{G}r_s + r_u.$$

Note: In the context of multi-grid methods, the components in Z are those grid functions on the fine grid that vanish by restriction to the coarse grid and therefore they are called the *unsmooth components of the residual*, those in $Z^{\perp}$ are the *smooth components of the residual*.

III. In the special case that $R = P^T$ we see that

$$N = \text{Range}(P) = \text{Span}(U_1) = \text{Span}(V_1) = Z^{\perp},$$

$$Z = \text{Kernel}(R) = \text{Span}(U_2) = \text{Span}(V_2) = N^{\perp}.$$

In this case the subspace of the smooth (resp. unsmooth) components of the residual is the same as the subspace of the smooth (unsmooth) components in the error.

SUMMARY

The effect of one iteration step in a defect correction process with an approximate inverse of deficient rank can be summarized as follows:

(1) For the error in the solution:

$$\begin{array}{lll} \text{Smooth components} & = \text{Range}(P) & = N \xrightarrow{\tilde{G}\,\hat{P}E\hat{R}} N \\ & & \quad \searrow^{\tilde{G}F} \\ \text{Unsmooth components} & = \text{Kernel}(\hat{R}) & = N^{\perp} \xrightarrow[I]{} N^{\perp}. \end{array}$$

(2) For the residual:

$$\begin{array}{lll} \text{Smooth components} & = \text{Range}(\hat{P}) & = Z^{\perp} \xrightarrow{\hat{P}E\hat{R}\,\tilde{G}} Z^{\perp} \\ & & \quad \searrow^{(\hat{P}R-I)FG} \\ \text{Unsmooth components} & = \text{Kernel}(R) & = Z \xrightarrow[I]{} Z. \end{array}$$

(3) In the special case $R = P^T$ we have

$$\text{Range}(P) = \text{Range}(\hat{P}) = N = Z^{\perp},$$

and

$$\text{Kernel}(R) = \text{Kernel}(\hat{R}) = Z = N^{\perp}.$$

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