ma
the
ma
tisch

cen
trum

AFDELING NUMERIEKE WISKUNDE NW 19/75 JUNE

H.J.J. TE RIELE

ON THE REPRESENTATION OF THE POSITIVE INTEGERS AS THE SUM OF TWO UNITARY ABUNDANT NUMBERS

amsterdam 1975

stichting
mathematisch
centrum

MC

AFDELING NUMERIEKE WISKUNDE NW 19/75 JUNE

H.J.J. TE RIELE

ON THE REPRESENTATION OF THE POSITIVE INTEGERS AS THE SUM OF TWO UNITARY ABUNDANT NUMBERS

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 10A20

On the representation of the positive integers as the sum of two unitary abundant numbers

by

H.J.J. te Riele

ABSTRACT

Any even integer > 530086 and any odd integer > 2004452254833 can be expressed as the sum of two unitary abundant numbers. The largest *even* integer, *not* expressible in such a way, is 530086.

KEY WORDS & PHRASES: *Unitary abundant numbers.*

CONTENTS

1. PRELIMINARIES AND RESULTS

Let $\sigma(n)$ denote the sum of the divisors of the positive integer n. If $\sigma(n) > 2n$, then n is called *abundant*. A divisor d of n is called *unitary*, if $(d,n/d) = 1$. By $\sigma^*(n)$ we denote the sum of the unitary divisors of n. If $\sigma^*(n) > 2n$, then n is called *unitary abundant*.

It is easy to prove that every integer ≥ 83160 can be expressed as the sum of two abundant numbers. The largest *even* number, *not* expressible as the sum of two abundant numbers, is 46. The largest *odd* number, *not* expressible as the sum of two abundant numbers, is 20161[1].

In this note, we shall prove the following results concerning the expressibility of the positive integers as the sum of two *unitary* abundant numbers.

THEOREM 1. *Every integer greater than*

$$550673\ 5446542339\ 7733736100 = \Big(\prod_{p\le 37} p\Big)^2$$

can be expressed as the sum of two unitary abundant numbers.

THEOREM 2. *The largest even number, not expressible as the sum of two unitary abundant numbers, is* 530086.

THEOREM 3. *Every odd number* ≥ 200 4452254835 *can be expressed as the sum of two unitary abundant numbers.*

The largest *odd* number, *not* expressible as the sum of two unitary abundant numbers, can be computed in the same way, as the largest even such number (see §4,step 2). However, we have not carried out this computation, since it probably would consume too much computer time.

As usual, we denote the number of different prime factors of n by $\omega(n)$. Let $\frac{\sigma^*(n)}{n} = \alpha(n)$. In the sequel, we shall frequently use the following facts, without explicitly stating them:

1. σ^* and α are multiplicative functions.

---
1) THOMAS R. PARKIN & LEON J. LANDER, Abundant Numbers, Aerospace Corp., Calif., July 1964.

2. $\sigma^*(n) \geq n$, with equality, if and only if $n = 1$.
3. n is unitary abundant, if and only if $\alpha(n) > 2$.
4. If n is unitary abundant, and if m is a positive integer, relatively prime to n, then also nm is unitary abundant.
5. If p is a prime and e is a positive integer, then $\alpha(p^e) = 1 + p^{-e}$ is monotonically decreasing in p and in e.

2. PROOF OF THEOREM 1

We first prove the

LEMMA. *Let* a *and* b *be relatively prime positive integers* > 1. *Then for every integer* $n > (ab)^2$ *there are positive integers* x *and* y, *with* $(a,x) = (b,y) = 1$, *such that* $n = ax + by$.

PROOF. Since $(a,b) = 1$, there are *positive* integers u and v, such that $au - bv = 1$. Multiplying by n yields

$$nau - nbv = n > (ab)^2,$$

so that

$$\frac{nu}{b} - \frac{nv}{a} > ab.$$

Hence, between $\frac{nv}{a}$ and $\frac{nu}{b}$ there are at least ab consecutive positive integers. If t is one of them, then it follows that $x = nu - bt$ and $y = at - nv$ are *positive* and also that

$$ax + by = anu - abt + bat - bnv = anu - bnv = n.$$

Moreover, from the Chinese remainder theorem, it follows that among any ab consecutive integers, there is (precisely) one integer t satisfying

$$x = nu - bt \equiv 1 \pmod{a}, \text{ and}$$
$$y = at - nv \equiv 1 \pmod{b},$$

so that $(x,a) = (y,b) = 1$. □

F OF THEOREM 1. Let $a = 2.3.37$ and $b = 5.7.11.13.17.19.23.29.31$. By the a, it follows that any integer $n > (ab)^2 = (\prod_{p\le 37} p)^2$ can be written in form $n = ax + by$, for some positive integers x and y, with $(a,x) = (b,y) = 1$. Since a and b are unitary abundant, and since $(a,x) = (b,y) = 1$, ax and by are unitary abundant. □

IE SMALLEST UNITARY ABUNDANT NUMBER ≡ 2ℓ(mod 36)

In order to compute the largest even number, not expressible as the f two unitary abundant numbers, it will appear to be useful to know mallest unitary abundant number ≡ 2ℓ(mod 36), for $\ell = 0,1,\dots,17$. These rs will be denoted by $s_{2\ell}$, and are listed in TABLE 1.

TABLE 1.

The smallest unitary abundant number ≡ 2ℓ(mod 36)

2ℓ	$s_{2\ell}$	factorization of $s_{2\ell}$
0	13860	2(2)3(2)5.7.11
2	1190	2.5.7.17
4	20020	2(2)5.7.11.13
6	42	2.3.7
8	497420	2(2)5.7.11.17.19
10	910	2.5.7.13
12	660	2(2)3.5.11
14	770	2.5.7.11
16	587860	2(2)5.7.13.17.19
18	630	2.3(2)5.7
20	460460	2(2)5.7.11.13.23
22	2002	2.7.11.13
24	420	2(2)3.5.7
26	1430	2.5.11.13
28	795340	2(2)5.7.13.19.23
30	30	2.3.5
32	340340	2(2)5.7.11.13.17
34	70	2.5.7

A short account follows of how TABLE 1 has been computed. Firstly, we notice that the numbers $6(6k\pm1)(k=1,2,\ldots)$ are unitary abundant, and since 6 is not, we have $s_6 = 42$ and $s_{30} = 30$. Secondly, systematic computation of all unitary abundant numbers $\not\equiv \pm6 \pmod{36}$ and ≤ 2002 (listed in TABLE 2), yielded $s_{2\ell}$, for $\ell = 17,12,9,6,7,5,1,13$ and 11, successively.

TABLE 2

The unitary abundant numbers $\not\equiv \pm6 \pmod{36}$ and ≤ 2002

n	factorization	residue(mod 36)	
70	2.5.7	34	(= s_{34})
420	2(2)3.5.7	24	(= s_{24})
630	2.3(2)5.7	18	(= s_{18})
660	2(2)3.5.11	12	(= s_{12})
770	2.5.7.11	14	(= s_{14})
780	2(2)3.5.13	24	
840	2(3)3.5.7	12	
910	2.5.7.13	10	(= s_{10})
924	2(2)3.7.11	24	
990	2.3(2)5.11	18	
1020	2(2)3.5.17	12	
1092	2(2)3.7.13	12	
1140	2(2)3.5.19	24	
1170	2.3(2)5.13	18	
1190	2.5.7.17	2	(= s_2)
1330	2.5.7.19	34	
1380	2(2)3.5.23	12	
1386	2.3(2)7.11	18	
1428	2(2)3.7.17	24	
1430	2.5.11.13	26	(= s_{26})
1530	2.3(2)5.17	18	
1596	2(2)3.7.19	12	
1610	2.5.7.23	26	
1638	2.3(2)7.13	18	
1710	2.3(2)5.19	18	

TABLE 2 (con'd)

n	factorization	residue(mod 36)
1740	2(2)3.5.29	12
1860	2(2)3.5.31	24
1870	2.5.11.17	34
1890	2.3(3)5.7	18
2002	2.7.11.13	22 (= s_{22})

The remaining values of $s_{2\ell}$ were computed as follows:

l = 0. For the unitary abundant numbers $n \equiv 0 \pmod{36}$ we have $2^2|n$ and $3^2|n$. If $\omega(n) \le 4$, then we have

$$\alpha(n) \le \frac{5}{4} \cdot \frac{10}{9} \cdot \frac{6}{5} \cdot \frac{8}{7} = \frac{40}{21} < 2.$$

If $\omega(n) \ge 5$, then we have $n \ge 2^2 3^2 5.7.11$, and since

$$\alpha(2^2 3^2 5.7.11) = \frac{40}{21} \cdot \frac{12}{11} = \frac{160}{77} > 2,$$

if follows that $s_0 = 2^2 3^2 5.7.11 = 13860$.

l = 2,4,8,10,14 and 16. For the unitary abundant numbers $n \equiv 2\ell \pmod{36}$ (ℓ=2,4,8,10,14 and 16) we have $2^2|n$ and $3 \nmid n$. If $\omega(n) \le 4$, then we have

$$\alpha(n) \le \frac{5}{4} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} = \frac{144}{77} < 2.$$

If $\omega(n) = 5$, then we have $n \ge 2^2 5.7.11.13$, and since

$$\alpha(2^2 5.7.11.13) = \frac{144}{77} \cdot \frac{14}{13} = \frac{288}{143} > 2, \text{ and } 2^2 5.7.11.13 \equiv 4 \pmod{36},$$

it follows that $s_4 = 2^2 5.7.11.13 = 20020$. Moreover, since

$$\alpha(2^2 5.7.11.17) = \frac{144}{77} \cdot \frac{18}{17} = \frac{2592}{1309} < 2,$$

20020 is the only unitary abundant number n for which $2^2|n$, $3 \nmid n$ and $\omega(n) = 5$. Now suppose $\omega(n) = 6$. Since

$$\alpha(2^2 11.13.17.19.23) = \frac{1814400}{1062347} < 2,$$

$$\alpha(2^2 7.11.13.17.19) = \frac{86400}{46189} < 2, \text{ and}$$

$$\alpha(2^2 5.11.13.17.19) = \frac{90720}{46189} < 2,$$

it follows that $5.7|n$. If $2^3|n$, then

$$\alpha(n) \le \frac{9}{8} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} = \frac{23328}{12155} < 2,$$

so that $2^2 \| n$. In TABLE 3 are listed the residues (modulo 36) of the numbers $2^2 5.7.11.13.p$, $17 \le p \le 43$, and of all other numbers of the form $2^2 5.7.p_1 p_2 p_3$, where p_1, p_2 and p_3 are different primes > 7 and $p_1 p_2 p_3 <$ $< 11.13.43$. Since all numbers n in this table are unitary abundant, and since $\omega(n) \ge 7$ implies $n \ge 2^2 5.7.11.13.17.19 = 6466460$, it follows that $s_{32} = 340340 = 2^2 5.7.11.13.17$, $s_{20} = 460460 = 2^2 5.7.11.13.23$, $s_8 =$ $= 497420 = 2^2 5.7.11.17.19$, $s_{16} = 587860 = 2^2 5.7.13.17.19$ and $s_{28} = 795340 =$ $= 2^2 5.7.13.19.23$.

TABLE 3

$n = 2^2 5.7.m$

m	residue of $2^2 5.7.m$ (mod 36)	
11.13.17 = 2431	32	($\Rightarrow s_{32}$)
11.13.19 = 2717	4	
11.13.23 = 3289	20	($\Rightarrow s_{20}$)
11.13.29 = 4147	8	
11.13.31 = 4433	16	
11.13.37 = 5291	4	
11.13.41 = 5863	20	
11.13.43 = 6149	28	
11.17.19 = 3553	8	($\Rightarrow s_8$)
11.17.23 = 4301	4	
11.17.29 = 5423	16	
11.17.31 = 5797	32	
11.19.23 = 4807	32	
11.19.29 = 6061	20	
13.17.19 = 4199	16	($\Rightarrow s_{16}$)
13.17.23 = 5083	8	
13.19.23 = 5681	28	($\Rightarrow s_{28}$)

4. PROOF OF THEOREM 2

Step 1. If $n \equiv a \pmod{36}$, and if a is even, then we have for $k = 1,2,\dots$

$$n \equiv 6(6k\mp 1) + a\pm 6 \pmod{36},$$

and since the numbers $6(6k\mp 1)$ are unitary abundant, it follows that all numbers $n \equiv a \pmod{36}$, satisfying

$$n \geq \min(42+s_{a-6},\ 30+s_{a+6}) \overset{=}{\text{def}}\ m_a,$$

are expressible as the sum of two unitary abundant numbers. In TABLE 4 are listed the numbers a and m_a, for $a = 0,2,\dots,34$, computed from the values of $s_{2\ell}$ in TABLE 1. It follows that all even numbers n satisfying $n \geq \max_{a=0,2,\dots,34}(m_a) = 587902$ are expressible as the sum of two unitary abundant numbers.

TABLE 4

$m_a = \min(42+s_{a-6},\ 30+s_{a+6})$

a	0	2	4	6	8	10
m_a	72	340382	112	690	800	20062
a	12	14	16	18	20	22
m_a	84	460490	952	450	812	587902
a	24	26	28	30	32	34
m_a	60	340370	100	462	1220	20050

Step 2. From step 1 it follows that, if the largest even number, which can *not* be expressed as the sum of two unitary abundant numbers, is ≥ 460490 and < 587902, then it must be congruent with 22(mod 36). Now the numbers $n \equiv 22 \pmod{36}$ and < 587902 can *not* be written as the sum of two *odd* unitary abundant numbers, since all odd unitary abundant numbers < 587902 are

divisible by 3. (Indeed, if n is odd, $3 \nmid n$ and $n < 587902$, then we clearly have $n < 5.7.11.13.17.19 = 1616615$, so that $\omega(n) \le 5$, and thus $\alpha(n) \le \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} = \frac{20736}{12155} < 2$). Hence, any even number < 587902, which is a sum of two odd unitary abundant numbers, is divisible by 3, and therefore incongruent with 22(mod 36). Furthermore, there are left seven possible ways to write the even numbers $n \equiv 22 \pmod{36}$, which are < 587902, as a sum of two unitary abundant numbers, namely as $n \equiv 0+22$, $2+20$, $4+18$, $8+14$, $10+12$, $24+34$ and $26+32 \pmod{36}$. In order to find such a splitting of n (if existent), we have computed and saved *all* unitary abundant numbers < 587902, which are congruent with 0,20,4,8,10,34 and 32(mod 36). Any of these seven congruence classes occur in one of the seven sums given above. There are 24,1, 2,2,466,473 and 1 unitary abundant numbers < 587902 in these seven congruence classes, respectively, thus a total of 969. It is clear that for any number $n \equiv 22 \pmod{36}$ and < 587902, which is representable as the sum of two unitary abundant numbers, one of the terms in that sum must belong to this set of 969 numbers. Using this fact, we have found such a representation for *any* even number $n \equiv 22 \pmod{36}$ in the interval $[530122, 587866]$, but no such representation for the number 530086. (A limited number of copies of the 30 pages computer output is available for the interested reader). □

REMARK. By extending the computer search, it was found that in the interval [300000,530086) there are seven even numbers, which are *not* the sum of two unitary abundant numbers, viz. 307442, 324886, 331334, 336242, 340334, 348286 and 382918.

5. PROOF OF THEOREM 3

We prove theorem 3 in the same way as theorem 2 (step 1). If $n \equiv a \pmod{36}$, and if a is odd, then we have for $k = 1,2,\ldots$

$$n \equiv 6(6k \mp 1) + a \pm 6 \pmod{36},$$

and since the numbers $6(6k \mp 1)$ are unitary abundant, all numbers $n \equiv a \pmod{36}$ satisfying

$$n \geq \min(42+s_{a-6},\ 30+s_{a+6}),$$

are expressible as the sum of two unitary abundant numbers (s_b denotes, as before, the smallest unitary abundant number, which is $\equiv b \pmod{36}$). In TABLE 5 the numbers $2\ell + 1$ and $s_{2\ell+1}$ are listed, for $\ell = 0,1,\ldots,17$. Hence, $\max_{a=1,3,\ldots,35}(\min(42+s_{a-6},30+s_{a+6})) = 2004452254835$, so that all odd numbers ≥ 2004452254835 are expressible as the sum of two unitary abundant numbers. □

The entries of TABLE 5 were computed as follows.

TABLE 5

The smallest unitary abundant number $\equiv 2\ell+1 \pmod{36}$

$2\ell+1$	$s_{2\ell+1}$	factorization of $s_{2\ell+1}$
1	157 10571 72685	5.7.11.13.17.19.23.29.31.47
3	15015	3.5.7.11.13
5	200 44522 54805	5.7.11.13.17.19.23.31.37.47
7	137 04966 82555	5.7.11.13.17.19.23.29.31.41
9	3346 39305	3(2)5.7.11.13.17.19.23
11	171 55469 88155	5.7.11.13.17.19.23.29.37.43
13	163 57541 04985	5.7.11.13.17.19.23.29.37.41
15	19635	3.5.7.11.17
17	143 73501 79265	5.7.11.13.17.19.23.29.31.43
19	190 10115 27415	5.7.11.13.17.19.23.29.41.43
21	21945	3.5.7.11.19
23	3 34267 48355	5.7.11.13.17.19.23.29.31
25	197 21781 52945	5.7.11.13.17.19.23.29.31.59
27	4219 36515	3(2)5.7.11.13.17.19.29
29	223 95921 39785	5.7.11.13.17.19.23.29.31.67
31	177 16176 62815	5.7.11.13.17.19.23.29.31.53
33	26565	3.5.7.11.23
35	174 85647 32915	5.7.11.13.17.19.23.31.37.41

l = 1,7,10,16. For the odd unitary abundant numbers $n \equiv 3,15,21$ and $33 \pmod{36}$, we have $3 \| n$. If $\omega(n) \le 4$, then we have

$$\alpha(n) \le \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} = \frac{768}{365} < 2.$$

If $\omega(n) = 5$, then we have $n \ge 3.5.7.11.13$. If $5|n$ and $7 \nmid n$, then we have

$$\alpha(n) \le \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} = \frac{24192}{12155} < 2,$$

so that $5.7|n$. The six smallest odd numbers n, satisfying $\omega(n) = 5$ and $5.7|n$ are given by $n = 105n_1$, for $n_1 = 11.13,\ 11.17,\ 11.19,\ 13.17,\ 13.19$ and 11.23, and their residues (mod 36) are 3,15,21,21,15 and 33, respectively. Since all these numbers are unitary abundant, and since $\omega(n) \ge 6$ implies $n \ge 3.5.7.11.13.17 = 255255$, it follows that $s_3 = 3.5.7.11.13 = 15015$, $s_{15} = 3.5.7.11.17 = 19635$, $s_{21} = 3.5.7.11.19 = 21945$ and $s_{33} = 3.5.7.11.23 =$ $= 26565$.

l = 4,13. For the odd unitary abundant numbers $n \equiv 9,27 \pmod{36}$, we have $3^2|n$. If $\omega(n) \le 7$, then we have

$$\alpha(n) \le \frac{10}{9} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} = \frac{92160}{46189} < 2.$$

If $\omega(n) \ge 8$, then we have $n \ge 3^2 5.7.11.13.17.19.23$. The two smallest numbers n, satisfying $\omega(n) = 8$ and $3^2|n$, are $3^2 5.7.11.13.17.19.23$ and $3^2 5.7.11.13.17.19.29$. Since they are both unitary abundant, and since their residues (mod 36) are 9 and 27, respectively, it follows that $s_9 = 3^2 5.7.11.13.17.19.23 =$ $= 334639305$ and $s_{27} = 3^2 5.7.11.13.17.19.29 = 421936515$.

l = 0,2,3,5,6,8,9,11,12,14,15 and 17. For these values of ℓ, the numbers $n \equiv 2\ell + 1 \pmod{36}$ satisfy $3 \nmid n$. If $\omega(n) \le 8$, then we have

$$\alpha(n) \le \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} \cdot \frac{24}{23} \cdot \frac{30}{29} = \frac{59719680}{30808063} < 2.$$

If $\omega(n) \ge 9$, then we have $n \ge 5.7.11.13.17.19.23.29.31$. Since $n = 5.7.11.13.17.19.23.29.31 \equiv 23 \pmod{36}$ is unitary abundant, whereas $n = 5.7.11.13.17.19.23.29.37$ is not, it follows that $s_{23} = 5.7.11.13.17.19.23.29.31 = 33426748355$, and that for the remaining values of $2\ell + 1$ we must have $\omega(s_{2\ell+1}) \ge 10$. The numbers $n = 5.7.11.13.17.19.23.29.31.p$, $37 \le p \le 157$, run through *all* residues $2\ell + 1 \pmod{36}$, for which the values of $s_{2\ell+1}$ remain to be computed. By computation of the residues (mod 36) of *all* numbers n, which satisfy $\omega(n) = 10$, and $n < 5.7.11.13.17.19.23.29.31.157$, the remaining values of $s_{2\ell+1}$ were easily derived.