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H.J.J. TE RIELE

ON INTEGER ARITHMETIC PROGRESSIONS OF LENGTH FOUR

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On integer arithmetic progressions of length four *)

by

H.J.J. te Riele

ABSTRACT

Let $t_i(n)$ be the number of pairs (a,b) $(a,b \in \mathbb{N},\ 1 \le a,b \le n,\ a \ne b)$ which belong to i integer arithmetic progressions of length four with positive terms $\le n$.

In this note it is shown that $t_i(n) \sim \frac{C_i}{1260} n^2$ $(n \to \infty)$, where $C_0 = 280$, $C_1 = 324$, $C_2 = 214$, $C_3 = 189$, $C_4 = 106$, $C_5 = 105$, $C_6 = 42$ and $C_{>6} = 0$.

KEY WORDS & PHRASES: *Arithmetic progressions*

*) This paper is not for review; it is meant for publication elsewhere

1. RESULTS

Let n, a and b be positive integers such that $1 \le a,b \le n$, $a \ne b$. Define $f_n(a,b)$ as the number of integer arithmetic progressions of length *four* with positive terms $\le n$ to which a and b belong (in this order). The possible values of $f_n(a,b)$ are $0,1,\dots,6$. We denote by $t_i(n)$ the number of pairs (a,b) for which $f_n(a,b)$ takes the value i.

In this note it is proved that for $n \to \infty$(*) $t_i(n) \sim \frac{C_i}{1260} n^2$, where $C_0 = 280$, $C_1 = 324$, $C_2 = 214$, $C_3 = 189$, $C_4 = 106$, $C_5 = 105$ and $C_6 = 42$. Moreover, it is proved that the limit of the average value of $f_n(a,b)$ is 2(as $n \to \infty$).

Analogous results for arithmetic progressions of length *three* were obtained by DRESSLER [1]. In principle, our method is a formalization of DRESSLER's method. It can be extended to arithmetic progressions of length greater than four, but the amount of work will be a very rapidly increasing "function" of the length.

2. PREPARATORY CALCULATIONS

Let Ω_n be the set of pairs (a,b) of integers a,b such that $1 \le a < b \le n$. Choose $(a,b) \in \Omega_n$. A necessary and sufficient condition for a and b to be the *first* and the *second* term, respectively, of an integer arithmetic progression of length four between 1 and n, inclusive, is that $b + 2(b-a) \le n$. A necessary and sufficient condition for a and b to be the *first* and the *third* term, respectively, is that $(2|b-a) \wedge (b+(b-a)/2 \le n)$, and so on. The *six* essentially different possibilities and the corresponding conditions are given in Table 1. The conditions are denoted by $c_1, c_2, \dots, c_6$, in the order indicated in the table.

(*) If g(n) and h(n) are defined and positive for all $n \in \mathbb{N}$, then by $g(n) \sim h(n)$ we mean $\lim_{n\to\infty} g(n)/h(n) = 1$, and we read: g(n) is asymptotic to h(n).

TABLE 1

The six necessary and sufficient conditions $c_1,\dots,c_6$

first term	second term	third term	fourth term	necessary and sufficient condition
a	b	$2b - a$	$3b - 2a$	$3b - 2a \le n$ (c_1)
a	$a + \frac{b-a}{2}$	b	$b + \frac{b-a}{2}$	$(2\mid b-a) \wedge (b+ \frac{b-a}{2} \le n)$ (c_3)
a	$a + \frac{b-a}{3}$	$a + 2\frac{b-a}{3}$	b	$3\mid b-a$ (c_6)
$2a - b$	a	b	$2b - a$	$(2a-b\ge 1) \wedge (2b-a\le n)$ (c_5)
$a - \frac{b-a}{2}$	a	$a + \frac{b-a}{2}$	b	$(2\mid b-a) \wedge (a- \frac{b-a}{2} \ge 1)$ (c_4)
$3a - 2b$	$2a - b$	a	b	$3a - 2b \ge 1$ (c_2)

Let $V_1,V_2,\dots,V_8$ be subsets of Ω_n, the elements of which satisfy, respectively, the following conditions

$$(2.1)\quad \begin{cases} 3b - a \le 2n \ (V_1), \quad 2b - a \le n \ (V_2), \quad 3b - 2a \le n \ (V_3), \\ 3a - b \ge 2 \ (V_4), \quad 2a - b \ge 1 \ (V_5), \quad 3a - 2b \ge 1 \ (V_6), \\ 2\mid b - a \ (V_7), \quad 3\mid b - a \ (V_8). \end{cases}$$

Then the pairs $(a,b) \in \Omega_n$ satisfying c_1 belong to V_3, the pairs satisfying c_2 belong to V_6, and so on:

(2.2)

condition on (a,b)	c_1	c_2	c_3	c_4	c_5	c_6
(a,b) belongs to	V_3	V_6	$V_1 \cap V_7$	$V_4 \cap V_7$	$V_2 \cap V_5$	V_8

In the sequel, the *negation* of c_i will be denoted by $\bar{c}_i$: for instance, $\bar{c}_3$ is the condition $(2 \nmid b-a) \vee (b+\frac{b-a}{2} > n)$. The *complement* of a set V_i (with respect to Ω_n) is denoted by $\bar{V}_i$ or $V_{\bar{i}}$. The intersection $\cap_{j=1}^{k} V_{i_j}$ of k sets $V_{i_1}, V_{i_2}, \ldots, V_{i_k}$ will be denoted by $V_{i_1,i_2,\ldots,i_k}$: for instance, $V_{1,\bar{3},7} = V_1 \cap \bar{V}_3 \cap V_7$.

In order to find an asymptotic estimate for $t_i(n)$, we shall determine all *disjoint* sets of pairs (a,b) which can be formed by the conjunction of i conditions out of $c_1, c_2, \ldots, c_6$ *set true* with the remaining 6-i conditions *set false*. This yields $2^6 = 64$ different sets. Since c_6 (resp. $\bar{c}_6$) contributes a factor 1/3 (resp. 2/3) to the estimates, we need only determine the 32 sets which remain after dropping the conditions c_6 and $\bar{c}_6$. These sets will be denoted by $W_0, W_1, \ldots, W_{31}$. The index k of W_k corresponds in the following way to the conditions to be satisfied by the elements of W_k: the j-th binary digit of k, counted from the left, is 0 or 1 according to whether the elements of W_k *do* or *do not* satisfy c_j (j=1,2,3,4,5). For example: $22_{10} = 10110_2$, so that

$$W_{22} = \{(a,b) \in \Omega_n \mid \bar{c}_1 \wedge c_2 \wedge \bar{c}_3 \wedge \bar{c}_4 \wedge c_5\}.$$

From (2.1) it follows that

$$V_3 \subset V_2 \subset V_1 \text{ and } V_6 \subset V_5 \subset V_4, \tag{2.3}$$

so that, using this and (2.2), we obtain

$$\begin{aligned} W_{22} &= \bar{V}_3 \cap V_6 \cap (\overline{V_1 \cap V_7}) \cap (\overline{V_4 \cap V_7}) \cap V_2 \cap V_5 \\ &= V_2 \cap \bar{V}_3 \cap (\bar{V}_1 \cup \bar{V}_7) \cap (\bar{V}_4 \cup \bar{V}_7) \cap V_6 \\ &= \left(V_{\bar{1},2,\bar{3}} \cup V_{2,\bar{3},\bar{7}}\right) \cap \left(V_{\bar{4},6} \cup V_{6,\bar{7}}\right) \\ &= V_{2,\bar{3},\bar{7}} \cap V_{6,\bar{7}} = V_{2,\bar{3},6,\bar{7}}. \end{aligned}$$

All sets W_k ($k=0,1,\ldots,31$) were determined in this way, and tabulated in Table 2 ($\emptyset$ denotes the empty set).

TABLE 2

The sets W_0, $W_1,\ldots,W_{31}$

k(decimal, binary)		W_k	k(decimal, binary)		W_k
0,	00000	$V_{3,6,7}$	16,	10000	$V_{2,\bar{3},6,7}$
1,	00001	$\emptyset$	17,	10001	$V_{1,\bar{2},6,7}$
2,	00010	$\emptyset$	18,	10010	$\emptyset$
3,	00011	$\emptyset$	19,	10011	$\emptyset$
4,	00100	$\emptyset$	20,	10100	$\emptyset$
5,	00101	$\emptyset$	21,	10101	$V_{\bar{1},6,7}$
6,	00110	$V_{3,6,\bar{7}}$	22,	10110	$V_{2,\bar{3},6,\bar{7}}$
7,	00111	$\emptyset$	23,	10111	$V_{\bar{2},6,\bar{7}}$
8,	01000	$V_{3,5,\bar{6},7}$	24,	11000	$V_{2,\bar{3},5,\bar{6},7}$
9,	01001	$V_{3,4,\bar{5},7}$	25,	11001	$V_{1,\bar{2},4,\bar{6},\bar{7}} \cup V_{1,\bar{3},4,\bar{5},\bar{7}}$
10,	01010	$\emptyset$	26,	11010	$\emptyset$
11,	01011	$V_{3,\bar{4},7}$	27,	11011	$V_{1,\bar{3},\bar{4},7}$
12,	01100	$\emptyset$	28,	11100	$\emptyset$
13,	01101	$\emptyset$	29,	11101	$V_{\bar{1},4,\bar{6},7}$
14,	01110	$V_{3,5,\bar{6},\bar{7}}$	30,	11110	$V_{2,\bar{3},5,\bar{6},\bar{7}}$
15,	01111	$V_{3,\bar{5},\bar{7}}$	31,	11111	$V_{\bar{1},\bar{4}} \cup V_{\bar{2},\bar{6},\bar{7}} \cup V_{\bar{3},\bar{5},\bar{7}}$

Now we shall determine asymptotic estimates for the *number of elements* in the sets $W_0, W_1, \ldots, W_{31}$. Let the number of elements in a set S be denoted by $|S|$. We first notice that both V_7 and $\bar{V}_7$ contribute a factor $\frac{1}{2}$ to the estimates. For instance, $|W_0| = |V_{3,6,7}| \sim \frac{1}{2}|V_{3,6}|$. Furthermore, from (2.1) one may derive the following permutation property: The number of elements in a set S_1, which is the intersection of some sets from the collection $\{V_1,V_2,V_3,V_4,V_5,V_6,\bar{V}_1,\bar{V}_2,\bar{V}_3,\bar{V}_4,\bar{V}_5,\bar{V}_6\}$, *equals* the number of elements in the set S_2 which is obtained from S_1 after replacing

$$V_1,V_2,V_3,V_4,V_5,V_6,\bar{V}_1,\bar{V}_2,\bar{V}_3,\bar{V}_4,\bar{V}_5,\bar{V}_6$$

by

$$V_4,V_5,V_6,V_1,V_2,V_3,\bar{V}_4,\bar{V}_5,\bar{V}_6,\bar{V}_1,\bar{V}_2,\bar{V}_3,$$

respectively. For instance, $|W_8| = |V_{3,5,\bar{6},7}| = |V_{6,2,\bar{3},7}| = |W_{16}|$. Finally, we observe that

$$|W_{25}| = |V_{1,\bar{2},4,\bar{6},\bar{7}} \cup V_{1,\bar{3},4,\bar{5},\bar{7}}|$$

$$= |V_{1,\bar{2},4,\bar{6},\bar{7}}| + |V_{1,\bar{3},4,\bar{5},\bar{7}}| - |V_{1,\bar{2},\bar{3},4,\bar{5},\bar{6},\bar{7}}|,$$

so that

$$(2.4) \qquad |W_{25}| = |V_{1,\bar{2},4,\bar{6},\bar{7}}| + |V_{1,\bar{3},4,\bar{5},\bar{7}}| - |V_{1,\bar{2},4,\bar{5},\bar{7}}| \qquad \text{(by (2.3))},$$

and

$$|W_{31}| = |V_{\bar{1},\bar{4}} \cup V_{\bar{2},\bar{6},\bar{7}} \cup V_{\bar{3},\bar{5},\bar{7}}|$$

$$= |V_{\bar{1},\bar{4}}| + |V_{\bar{2},\bar{6},\bar{7}}| + |V_{\bar{3},\bar{5},\bar{7}}|$$

$$- |V_{\bar{1},\bar{2},\bar{4},\bar{6},\bar{7}}| - |V_{\bar{1},\bar{3},\bar{4},\bar{5},\bar{7}}| - |V_{\bar{2},\bar{3},\bar{5},\bar{6},\bar{7}}| + |V_{\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7}}|$$

$$= |V_{\bar{1},\bar{4}}| + |V_{\bar{2},\bar{6},\bar{7}}| + |V_{\bar{3},\bar{5},\bar{7}}| - |V_{\bar{1},\bar{4},\bar{7}}| - |V_{\bar{1},\bar{4},\bar{7}}|$$

$$- |V_{\bar{2},\bar{5},\bar{7}}| + |V_{\bar{1},\bar{4},\bar{7}}| \qquad \text{(by (2.3))},$$

so that

$$(2.5) \qquad |W_{31}| = |V_{\bar{1},\bar{4}}| + |V_{\bar{2},\bar{6},\bar{7}}| + |V_{\bar{3},\bar{5},\bar{7}}| - |V_{\bar{1},\bar{4},\bar{7}}| - |V_{\bar{2},\bar{5},\bar{7}}|.$$

From these three observations one can easily deduce that, in order to compute asymptotic estimates for the number of pairs (a,b) in the sets $W_0, W_1, \ldots, W_{31}$, it is sufficient to determine these estimates only for the following twelve sets:

$$(2.6) \quad \begin{cases} V_{3,6},\ V_{3,\bar{4}},\ V_{3,\bar{5}},\ V_{\bar{1},\bar{4}},\ V_{\bar{2},\bar{6}},\ V_{\bar{2},\bar{5}}, \\ V_{3,5,\bar{6}},\ V_{3,4,\bar{5}},\ V_{1,\bar{3},\bar{4}}, \\ V_{2,\bar{3},5,\bar{6}},\ V_{1,\bar{2},4,\bar{6}},\ V_{1,\bar{2},4,\bar{5}}. \end{cases}$$

In order to save space we only give detailed computations for the three sets $V_{3,6}$, $V_{3,5,\bar{6}}$ and $V_{2,\bar{3},5,\bar{6}}$. The examples are fully illustrative for the other nine sets. The results are given in Table 3. This table also gives *all* sets which have, by the permutation property, the same number of elements as one of the twelve sets in (2.6).

TABLE 3

Asymptotic estimates of the number of elements in certain sets

set	estimate ($n\to\infty$)	set	estimate ($n\to\infty$)
$V_{3,6}$	$\sim \frac{1}{10} n^2$	$V_{3,5,\bar{6}},\ V_{2,\bar{3},6}$	$\sim \frac{1}{40} n^2$
$V_{3,\bar{4}},\ V_{\bar{1},6}$	$\sim \frac{1}{42} n^2$	$V_{3,4,\bar{5}},\ V_{1,\bar{2},6}$	$\sim \frac{1}{56} n^2$
$V_{3,\bar{5}},\ V_{\bar{2},6}$	$\sim \frac{1}{24} n^2$	$V_{1,\bar{3},\bar{4}},\ V_{\bar{1},4,\bar{6}}$	$\sim \frac{5}{84} n^2$
$V_{\bar{1},\bar{4}}$	$\sim \frac{1}{12} n^2$	$V_{2,\bar{3},5,\bar{6}}$	$\sim \frac{1}{60} n^2$
$V_{\bar{2},\bar{6}},\ V_{\bar{3},\bar{5}}$	$\sim \frac{5}{24} n^2$	$V_{1,\bar{2},4,\bar{6}},\ V_{1,\bar{3},4,\bar{5}}$	$\sim \frac{9}{280} n^2$
$V_{\bar{2},\bar{5}}$	$\sim \frac{1}{6} n^2$	$V_{1,\bar{2},4,\bar{5}}$	$\sim \frac{1}{60} n^2$

$\underline{V_{3,6}}$. By (2.1), any element $(a,b) \in V_{3,6}$ satisfies $3b - 2a \le n$ and $3a - 2b \ge 1$, so that

$$a \ge \max\left(\frac{3b-n}{2}, \frac{2b+1}{3}\right) = \begin{cases} \frac{3b-n}{2}, & \text{if } b > \frac{3n+2}{5}, \\ \frac{2b+1}{3}, & \text{if } b \le \frac{3n+2}{5}. \end{cases}$$

It follows that if $b \le (3n+2)/5$, then $(2b+1)/3 \le a < b$, and if $b > (3n+2)/5$, then $(3b-n)/2 \le a < b$.

Hence,

$$|V_{3,6}| \sim \sum_{b=1}^{(3n+2)/5} (b-(2b+1)/3) + \sum_{b=(3n+2)/5}^{n} (b-(3b-n)/2)$$

$$\sim \sum_{b=1}^{3n/5} b/3 + \sum_{b=3n/5}^{n} (n-b)/2 \sim n^2/10.$$

$\underline{V_{3,5,\bar{6}}}$. By (2.1), any element $(a,b) \in V_{3,5,\bar{6}}$ satisfies $3b - 2a \le n$, $2a - b \ge 1$ and $3a - 2b < 1$, so that

$$\begin{cases} a \ge \max\left(\frac{3b-n}{2}, \frac{b+1}{2}\right) = \begin{cases} \frac{3b-n}{2}, & \text{if } b > \frac{n+1}{2}, \\ \frac{b+1}{2}, & \text{if } b \le \frac{n+1}{2}, \end{cases} \quad \text{and} \\ a < \frac{2b+1}{3}. \end{cases}$$

If $b > (n+1)/2$, then the conditions $a \ge (3b-n)/2$ and $a < (2b+1)/3$ make sense only if $(3b-n)/2 < (2b+1)/3$, so that $b < (3n+2)/5$. Furthermore, if $b \le (n+1)/2$, then $(b+1)/2 \le a < (2b+1)/3$.

Hence,

$$|V_{3,5,\bar{6}}| \sim \sum_{b=1}^{(n+1)/2} ((2b+1)/3-(b+1)/2) + \sum_{b=(n+1)/2}^{(3n+2)/5} ((2b+1)/3-(3b-n)/2)$$

$$\sim \sum_{b=1}^{n/2} b/6 + \sum_{b=n/2}^{3n/5} (n/2-5b/6) \sim n^2/40.$$

$\underline{V_{2,\bar{3},5,\bar{6}}}$. By (2.1), any element $(a,b) \in V_{2,\bar{3},5,\bar{6}}$ satisfies $(2b-a) \le n$, $3b - 2a > n$, $2a - b \ge 1$ and $3a - 2b < 1$, so that

$$\begin{cases} a \ge \max(2b-n,\ (b+1)/2) = \begin{cases} 2b-n, & \text{if } b > \frac{2n+1}{3}, \\ \frac{b+1}{2}, & \text{if } b \le \frac{2n+1}{3}, \end{cases} \quad \text{and} \\ a < \min\left(\frac{3b-n}{2}, \frac{2b+1}{3}\right) = \begin{cases} \frac{3b-n}{2}, & \text{if } b \le \frac{3n+2}{5}, \\ \frac{2b+1}{3}, & \text{if } b > \frac{3n+2}{5}. \end{cases} \end{cases}$$

The condition $b \le (3n+2)/5$ implies that $b < (2n+1)/3$, so that if $b \le (3n+2)/5$ we have $a < (3b-n)/2$ and $a \ge (b+1)/2$. This makes sense only if $(b+1)/2 < (3b-n)/2$, so that $b > (n+1)/2$. Furthermore, if $(3n+2)/5 < b \le (2n+1)/3$, then $(b+1)/2 \le a < (2b+1)/3$. Finally, if $b > (2n+1)/3$, then $2b - n \le a < (2b+1)/3$. This makes sense only if $2b - n < (2b+1)/3$, so that $b < (3n+1)/4$.

Hence,

$$|V_{2,\bar{3},5,\bar{6}}| \sim \sum_{b=(n+1)/2}^{(3n+2)/5} ((3b-n)/2-(b+1)/2) + \sum_{b=(3n+2)/5}^{(2n+1)/3} ((2b+1)/3-(b+1)/2) + \sum_{b=(2n+1)/3}^{(3n+1)/4} ((2b+1)/3-(2b-n))$$

$$\sim \sum_{b=n/2}^{3n/5} (b-n/2) + \sum_{b=3n/5}^{2n/3} b/6 + \sum_{b=2n/3}^{3n/4} (n-4b/3) \sim n^2/60.$$

3. THE ASYMPTOTIC BEHAVIOUR OF $t_i(n)$

It is clear that $t_i(n)$, the number of pairs (a,b) with $1 \le a,b \le n (a \ne b)$, which belong to i integer arithmetic progressions of length four between 1 and n, inclusive, is twice the number of pairs for which $a < b$.

For $(a,b) \in \Omega_n$ we have $f_n(a,b) = 0$ if and only if (a,b) satisfies the condition $c_1 \wedge c_2 \wedge c_3 \wedge c_4 \wedge c_5 \wedge c_6$. Hence, $f_n(a,b) = 0$ if and only if $(a,b) \in W_{31} \cap \bar{V}_8$. From (2.5) and Table 3 it follows that

$$t_0(n) \sim 2\cdot\frac{2}{3}\cdot\left(\frac{1}{12}+\frac{5}{48}+\frac{5}{48}-\frac{1}{24}-\frac{1}{12}\right)\cdot n^2 = \frac{2}{9}n^2.$$

For $(a,b) \in \Omega_n$ we have $f_n(a,b) = 1$ if and only if (a,b) satisfies the condition

$$(\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge c_6) \vee (\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge c_5\wedge\bar{c}_6) \vee (\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge c_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee$$

$$(\bar{c}_1\wedge\bar{c}_2\wedge c_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (\bar{c}_1\wedge c_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (c_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6).$$

Hence, $f_n(a,b) = 1$ if and only if

$$(a,b) \in (W_{31}\cap V_8) \cup (W_{30}\cup W_{29}\cup W_{27}\cup W_{23}\cup W_{15})\cap\bar{V}_8),$$

so that

$$t_1(n) \sim 2\cdot\left(\frac{1}{6}\cdot\frac{1}{3}+\frac{2}{3}\cdot\left(\frac{1}{120}+\frac{5}{168}+\frac{5}{168}+\frac{1}{48}+\frac{1}{48}\right)\right)\cdot n^2 = \frac{9}{35}n^2.$$

For $(a,b) \in \Omega_n$, we have $f_n(a,b) = 2$ if and only if (a,b) satisfies the condition

$$(\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge c_5\wedge c_6) \vee (\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge c_4\wedge\bar{c}_5\wedge c_6) \vee (\bar{c}_1\wedge\bar{c}_2\wedge c_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge c_6) \vee$$

$$(\bar{c}_1\wedge c_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge c_6) \vee (c_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge c_6) \vee (\bar{c}_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge c_4\wedge c_5\wedge\bar{c}_6) \vee$$

$$(\bar{c}_1\wedge\bar{c}_2\wedge c_3\wedge\bar{c}_4\wedge c_5\wedge\bar{c}_6) \vee (\bar{c}_1\wedge c_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge c_5\wedge\bar{c}_6) \vee (c_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge c_5\wedge\bar{c}_6) \vee$$

$$(\bar{c}_1\wedge\bar{c}_2\wedge c_3\wedge c_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (\bar{c}_1\wedge c_2\wedge\bar{c}_3\wedge c_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (c_1\wedge\bar{c}_2\wedge\bar{c}_3\wedge c_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee$$

$$(\bar{c}_1\wedge c_2\wedge c_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (c_1\wedge\bar{c}_2\wedge c_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6) \vee (c_1\wedge c_2\wedge\bar{c}_3\wedge\bar{c}_4\wedge\bar{c}_5\wedge\bar{c}_6).$$

Hence, $f_n(a,b) = 2$ if and only if

$$(a,b) \in ((W_{30} \cup W_{29} \cup W_{27} \cup W_{23} \cup W_{15}) \cap V_8) \cup$$

$$((W_{28} \cup W_{26} \cup W_{22} \cup W_{14} \cup W_{25} \cup W_{21} \cup W_{13} \cup W_{19} \cup W_{11} \cup W_7) \cap \bar{V}_8),$$

so that, using (2.4),

$$t_2(n) \sim 2((\frac{1}{120} + \frac{5}{168} + \frac{5}{168} + \frac{1}{48} + \frac{1}{48}) \cdot \frac{1}{3} +$$

$$(\frac{1}{80} + \frac{1}{80} + \frac{9}{560} + \frac{9}{560} - \frac{1}{120} + \frac{1}{84} + \frac{1}{84}) \cdot \frac{2}{3}) \cdot n^2$$

$$= \frac{107}{630} n^2.$$

For $(a,b) \in \Omega_n$, we have $f_n(a,b) = 3$ if and only if (a,b) satisfies the condition

$$(\bar{c}_1 \wedge \bar{c}_2 \wedge \bar{c}_3 \wedge c_4 \wedge c_5 \wedge c_6) \vee (\bar{c}_1 \wedge \bar{c}_2 \wedge c_3 \wedge \bar{c}_4 \wedge c_5 \wedge c_6) \vee (\bar{c}_1 \wedge c_2 \wedge \bar{c}_3 \wedge \bar{c}_4 \wedge c_5 \wedge c_6) \vee$$

$$(c_1 \wedge \bar{c}_2 \wedge \bar{c}_3 \wedge \bar{c}_4 \wedge c_5 \wedge c_6) \vee (\bar{c}_1 \wedge \bar{c}_2 \wedge c_3 \wedge c_4 \wedge \bar{c}_5 \wedge c_6) \vee (\bar{c}_1 \wedge c_2 \wedge \bar{c}_3 \wedge c_4 \wedge \bar{c}_5 \wedge c_6) \vee$$

$$(c_1 \wedge \bar{c}_2 \wedge \bar{c}_3 \wedge c_4 \wedge \bar{c}_5 \wedge c_6) \vee (\bar{c}_1 \wedge c_2 \wedge c_3 \wedge \bar{c}_4 \wedge \bar{c}_5 \wedge c_6) \vee (c_1 \wedge \bar{c}_2 \wedge c_3 \wedge \bar{c}_4 \wedge \bar{c}_5 \wedge c_6) \vee$$

$$(c_1 \wedge c_2 \wedge \bar{c}_3 \wedge \bar{c}_4 \wedge \bar{c}_5 \wedge c_6) \vee (\bar{c}_1 \wedge \bar{c}_2 \wedge c_3 \wedge c_4 \wedge c_5 \wedge \bar{c}_6) \vee (\bar{c}_1 \wedge c_2 \wedge \bar{c}_3 \wedge c_4 \wedge c_5 \wedge \bar{c}_6) \vee$$

$$(c_1 \wedge \bar{c}_2 \wedge \bar{c}_3 \wedge c_4 \wedge c_5 \wedge \bar{c}_6) \vee (\bar{c}_1 \wedge c_2 \wedge c_3 \wedge \bar{c}_4 \wedge c_5 \wedge \bar{c}_6) \vee (c_1 \wedge \bar{c}_2 \wedge c_3 \wedge \bar{c}_4 \wedge c_5 \wedge \bar{c}_6) \vee$$

$$(c_1 \wedge c_2 \wedge \bar{c}_3 \wedge \bar{c}_4 \wedge c_5 \wedge \bar{c}_6) \vee (\bar{c}_1 \wedge c_2 \wedge c_3 \wedge c_4 \wedge \bar{c}_5 \wedge \bar{c}_6) \vee (c_1 \wedge \bar{c}_2 \wedge c_3 \wedge c_4 \wedge \bar{c}_5 \wedge \bar{c}_6) \vee$$

$$(c_1 \wedge c_2 \wedge \bar{c}_3 \wedge c_4 \wedge \bar{c}_5 \wedge \bar{c}_6) \vee (c_1 \wedge c_2 \wedge c_3 \wedge \bar{c}_4 \wedge \bar{c}_5 \wedge \bar{c}_6).$$

Hence, $f_n(a,b) = 3$ if and only if

$$(a,b) \in ((W_{28} \cup W_{26} \cup W_{22} \cup W_{14} \cup W_{25} \cup W_{21} \cup W_{13} \cup W_{19} \cup W_{11} \cup W_7) \cap V_8) \cup$$

$$((W_{24} \cup W_{20} \cup W_{12} \cup W_{18} \cup W_{10} \cup W_6 \cup W_{17} \cup W_9 \cup W_5 \cup W_3) \cap \bar{V}_8),$$

so that, using (2.4),

$$t_3(n) \sim 2\Big[(\frac{1}{80}+\frac{1}{80}+\frac{9}{560}+\frac{9}{560}-\frac{1}{120}+\frac{1}{84}+\frac{1}{84})\ \frac{1}{3}+$$

$$(\frac{1}{120}+\frac{1}{20}+\frac{1}{112}+\frac{1}{112})\ \frac{2}{3}\Big]\cdot n^2$$

$$= \frac{3}{20}n^2.$$

Similarly, it follows that $f_n(a,b) = 4$ if and only if

$$(a,b) \in ((W_{24}\cup W_{20}\cup W_{12}\cup W_{18}\cup W_{10}\cup W_6\cup W_{17}\cup W_9\cup W_5\cup W_3)\cap V_8)\ \cup$$

$$(W_{16}\cup W_8\cup W_4\cup W_2\cup W_1)\cap\bar{V}_8),$$

so that

$$t_4(n) \sim 2\Big[(\frac{1}{120}+\frac{1}{20}+\frac{1}{112}+\frac{1}{112})\ \frac{1}{3}+(\frac{1}{80}+\frac{1}{80})\ \frac{2}{3}\Big]\cdot n^2$$

$$= \frac{53}{630}n^2.$$

Furthermore, $f_n(a,b) = 5$ if and only if

$$(a,b) \in ((W_{16}\cup W_8\cup W_4\cup W_2\cup W_1)\cap V_8)\ \cup\ (W_0\cap\bar{V}_8),$$

so that

$$t_5(n) \sim 2\Big[(\frac{1}{80}+\frac{1}{80})\cdot\frac{1}{3}+\frac{1}{20}\cdot\frac{2}{3}\Big]\cdot n^2 = \frac{1}{12}n^2.$$

Finally, $f_n(a,b) = 6$ if and only if $(a,b) \in W_0 \cap V_8$, so that

$$t_6(n) \sim 2\cdot\frac{1}{20}\cdot\frac{1}{3}\cdot n^2 = \frac{1}{30}n^2.$$

The limit of the average value a_n of f_n may be computed as follows. The number of integer arithmetic progressions of length four all of whose terms are between 1 and n, inclusive, is given by $\sum_{k=1}^{n} [(n-k)/3]$, which is asymptotic to $n^2/6$. Counting each such progression 12 times (once for each pair in it), we obtain

$$\sum_{\substack{1\le a,b\le n \\ a\ne b}} f_n(a,b) \sim 2n^2.$$

Hence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2-n} \sum f_n(a,b) = 2.$$

This result provides a check of the values of $t_i(n)$, computed before, since $\sum f_n(a,b) = \sum_{i=0}^{6} it_i(n)$, so that we must have

$$2 = \lim_{n\to\infty} \frac{1}{n^2-n} \sum f_n(a,b) = \lim_{n\to\infty} \frac{1}{n^2-n} \sum_{i=0}^{6} it_i(n)$$
$$= \sum_{i=0}^{6} i \lim_{n\to\infty} (t_i(n)/(n^2-n)).$$

REFERENCES

[1] DRESSLER, R.E., *A note on arithmetic progressions of length four*, Math. Mag. 47 (1974) 31-34.