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MULTIPOINT MULTISTEP RUNGE-KUTTA METHODS II: THE CONSTRUCTION OF A CLASS OF STABILIZED THREE-STEP METHODS FOR PARABOLIC EQUATIONS

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Multipoint multistep runge-Kutta methods II: the construction of a class of stabilized three-step methods for parabolic equations

by

J.G. Verwer

ABSTRACT

A class of multipoint three-step Runge-Kutta methods is discussed for the numerical solution of initial value problems for systems of ordinary differential equations y' = f(y). These systems are supposed to originate from parabolic partial differential equations by applying the method of semi-discretization. In this report the discussion is concentrated on the construction of stabilized formulas of first and second order. For this construction a technique is discussed which makes use of the method of linear programming.

KEY WORDS & PHRASES: *Numerical analysis, Multipoint multistep Runge-Kutta method, Parabolic partial differential equations, Method of lines, Stability.*

CONTENTS

1. INTRODUCTION

Let

$$y' = f(y) \tag{1.1}$$

represent a system of ordinary differential equations of which the eigenvalues of the Jacobian matrix are situated in a long narrow strip around the negative axis. Systems of this type arise by applying the method of semi-discretization to parabolic partial differential equations.

In the present report we investigate the stability of a class of m-point three step Runge-Kutta methods when applied to such parabolic systems. This class of integration methods is defined by the scheme:

$$\begin{aligned}
y_{n+1}^{(0)} &= y_n, \\
y_{n+1}^{(1)} &= (1-b_1)y_n + b_1 y_{n-1} + c_1 hf(y_{n-1}) + \lambda_{1,0} hf(y_n), \\
y_{n+1}^{(j)} &= (1-b_j)y_n + b_j y_{n-1} + c_j hf(y_{n-1}) + \lambda_{j,0} hf(y_n) + \\
&\quad + \lambda_{j,j-1} hf(y_{n+1}^{(j-1)}), \qquad j = 2,\dots,m;\ m \geq 2, \\
y_{n+1} &= d y_{n+1}^{(m)} + (1-d) y_{n-2},
\end{aligned} \tag{1.2}$$

where y_{n+1} denotes a numerical approximation to the analytical solution $y(x)$ at $x = x_{n+1} = x_n + h$.

If d is set equal to one, method (1.2) is reduced to a two-step method. This two-step method is extensively discussed in VERWER [5], where a class of second order methods was constructed with a real boundary of absolute stability of $1.8\ m^2$. Unfortunately, those two-step methods are rather inaccurate when compared with stabilized one-step methods as discussed by VAN DER HOUWEN [4]. This inaccuracy is due to the fact that the normalized error-constants are chosen too large.

The purpose of the simple extension of the two-step to a three-step method is thus to develop formulas which have a similar accuracy behaviour as possessed by the one-step formulas, without loss of the large stability boundaries. To that end we shall require that the normalization factor for the error-constants is equal to one, i.e. we always assume (see VERWER [5])

(1.3) $$b_m = \frac{2(d-1)}{d}\ .$$

The stability of method (1.2) is investigated for real eigenvalues, say δ, of the Jacobian of (1.1). For these eigenvalues the following stability problem is discussed: determine the parameters of the scheme in such a way that

$$\max|\text{amplification factor}| \le \rho(h\delta),\ h\delta \in [-\beta,0],\ \beta \text{ maximal},$$

where ρ is a prescribed function with the properties $\rho(0) = 1$ and $0 < \rho(z) < 1$ for $z < 0$. The function ρ may be considered as a damping function for the higher harmonics. Moreover, if the ρ-values are not too close to one, the absolute stability region of a method will contain a long narrow strip along the negative axis.

The stability problem is stated in section 2 for first and second order methods. Higher order methods are less popular in the integration of partial differential equations. A technique to determine an almost optimal solution to the optimization problem stated above is discussed in section 3. This technique makes use of the method of linear programming. Results are given in the appendix for $m \le 12$.

The numerical calculations have been carried out on a CYBER 73-28 computer using 14 significant digits.

2. STATEMENT OF THE PROBLEM

Let us apply method (1.2) to the model-equation

(2.1) $$y' = \delta y,\ \delta < 0.$$

This yields the relation

(2.2) $$y_{n+1} = d[S(z)y_n + P(z)y_{n-1}] + (1-d)y_{n-2},$$

where S(z) and P(z) are polynomials of degree m in $z = h\delta$. We denote

(2.3) $$S(z) = \sum_{i=0}^{m} s_i z^i, \quad P(z) = \sum_{l=0}^{m} p_i z^i.$$

The coefficients s_i and p_i are dependent on the parameters of the scheme (see VERWER [5]); in particular

$$p_0 = b_m, \quad s_0 = 1 - b_m.$$

By substituting the second order Padé-approximation

(2.4) $$1 + z + \frac{1}{2} z^2$$

to exp(z) into the characteristic equation

(2.5) $$\alpha^3 - dS(z)\alpha^2 - dP(z)\alpha - 1 + d = 0,$$

we are able to express the consistency conditions for orders p = 1 and 2 into the parameter d and the coefficients s_i and p_i (see table 2.1).

p = 1	$s_0 + p_0 = 1,$ $s_1 - p_0 + p_1 = (3-2d)/d;$
p = 2	$s_2 + \frac{1}{2} p_0 - p_1 + p_2 = (-3/2 + 2d)/d;$

TABLE 2.1. Consistency conditions

Let $\alpha_i(z)$, i = 1,2,3, denote the roots of equation (2.5). The stability problem we intend to solve then reads:

<u>PROBLEM</u>. Let $\rho : (-\infty,0] \to [0,1]$, $\rho(0) = 1$ be given. Let $p_0 = 2(d-1)/d$. Determine the coefficients s_i and p_i, i = 0,...,m and the parameter d, in

such a way that

$$\max_{i=1,2,3} |\alpha_i(z)| \leq \rho(z), \quad z \in [-\beta,0], \quad \beta \text{ maximal},$$

where it is assumed that p = 1 or p = 2.

3. A SOLUTION TECHNIQUE

Define $\alpha = \rho\xi$ and substitute into equation (2.5). This yields a cubic equation in ξ:

$$\rho^3\xi^3 - dS\rho^2\xi^3 - dP\rho\xi - (1-d) = 0. \tag{3.1}$$

Let

$$\xi = \frac{1+\eta}{1-\eta}, \tag{3.2}$$

which maps the interior of the unit circle $|\xi| = 1$ into the half-plane $\mathrm{Re}(\eta) < 0$. Substitution of (3.2) into (3.1) yields a cubic equation in η:

$$a_0\eta^3 + a_1\eta^2 + a_2\eta + a_3 = 0, \tag{3.3}$$

where

$$\begin{aligned} a_0 &= \rho^3 + dS\rho^2 - dP\rho + 1 - d, \\ a_1 &= 3\rho^3 + dS\rho^2 + dP\rho - 3(1-d), \\ a_2 &= 3\rho^3 - dS\rho^2 + dP\rho + 3(1-d), \\ a_3 &= \rho^3 - dS\rho^2 - dP\rho - (1-d). \end{aligned} \tag{3.4}$$

Sufficient conditions for the roots of (3.1) to lie inside or on the unit circle can be obtained by applying the Routh-Hurwitz criterion to (3.3) (see LAMBERT [3]). These conditions read:

$$(3.5) \qquad a_i \geq 0, \quad i = 0,\ldots,3; \quad a_1 a_2 - a_0 a_3 \geq 0.$$

Observe that without the equality signs conditions (3.5) are necessary for the roots of (3.1) to lie inside the unit circle.

In terms of S, P, d and ρ conditions (3.5) give:

$$\rho S - P \geq \frac{d - 1 - \rho^3}{d\rho}$$

$$\rho S + P \geq \frac{3(1-d) - 3\rho^3}{d\rho}$$

$$(3.7) \qquad -\rho S + P \geq \frac{3(d-1) - 3\rho^3}{d\rho}$$

$$-\rho S - P \geq \frac{1 - d - \rho^3}{d\rho}$$

$$\frac{1-d}{\rho^2} S + P \geq \frac{(1-d)^2 - \rho^6}{d\rho^4} .$$

The problem stated in section 2 thus reads:

Let the function ρ be given and let $p_0 = 2(d-1)/d$. Determine the parameter d and the coefficients s_i and p_i, compatible to an imposed order of accuracy, in such a way that (3.7) is satisfied for $z \in [-\beta,0]$, β maximal.

The idea is now to discretize the variable z on an interval $[-\bar{\beta},0]$, i.e. we define the points $z_j = -j\Delta z$, $\Delta z = \bar{\beta}/N$, $j = 1,\ldots,N$, N prescribed, and to replace the five conditions (3.7) by a set of 5N inequalities which are linear provided that the parameter d is fixed beforehand. If $\bar{\beta} \leq \beta$, a feasible solution for this linear set of inequalities must exist. Such a feasible solution is easy to determine by using a linear programming method. If $\bar{\beta} > \beta$ and N large enough, no feasible solution will exist. Summarizing, once the optimal value for d is known, β can be approximated as accurate as possible by solving a sequence of linear programming problems.

Before describing the linear programming approach in detail we first expand the polynomials S and P in Chebyshev polynomials in order to prevent numerical difficulties for higher values of m. Let $\bar{\beta}$ be given. For $z \in [-\bar{\beta},0]$ we expand:

$$(3.8) \qquad P(z) = \sum_{k=0}^{m} \bar{p}_k T_k\left(1 + \frac{2z}{\bar{\beta}}\right), \qquad S(z) = \sum_{k=0}^{m} \bar{s}_k T_k\left(1 + \frac{2z}{\bar{\beta}}\right),$$

where $T_k(z) = \cos(m \arccos z)$. Define

$$t_{ij} = \frac{d^i}{dz^i} T_j(z)\Big|_{z=1}.$$

According to ABRAMOWITZ & STEGUN [1] (formulas 15.1.1, 15.2.2 and 15.4.3) we have

$$(3.9)\qquad t_{ij} = \begin{cases} 1, & i = 0 \\ \dfrac{j^2(j^2-1)\ldots(j^2-(i-1)^2)}{1.3\ldots(2i-1)}, & i = 1,\ldots,j. \end{cases}$$

By means of relation (3.9) the coefficients s_i and p_i are expressed as

$$(3.10)\qquad p_i = \frac{2^i \sum_{k=i}^{m} \bar{p}_k t_{ik}}{i!\bar{\beta}^i}, \qquad s_i = \frac{2^i \sum_{k=i}^{m} \bar{s}_k t_{ik}}{i!\bar{\beta}^i}, \qquad i = 0,\ldots,m.$$

Let

$$(3.11)\qquad T_{\ell k} = T_k\left(1 + \frac{2z_\ell}{\bar{\beta}}\right).$$

Then for $z = z_\ell$ and $\rho_\ell = \rho(z_\ell)$ conditions (3.7) read:

$$(3.12)\qquad \begin{aligned}
\sum_{k=0}^{m} -T_{\ell k}\bar{p}_k + \rho_\ell T_{\ell k}\bar{s}_k &\geq \frac{d-1-\rho_\ell^3}{d\rho_\ell}, \\
\sum_{k-0}^{m} T_{\ell k}\bar{p}_k + \rho_\ell T_{\ell k}\bar{s}_k &\geq \frac{3(1-d)-3\rho_\ell^3}{d\rho_\ell}, \\
\sum_{k=0}^{m} T_{\ell k}\bar{p}_k - \rho_\ell T_{\ell k}\bar{s}_k &\geq \frac{3(d-1)-3\rho_\ell^3}{d\rho_\ell}, \\
\sum_{k=0}^{m} -T_{\ell k}\bar{p}_k - \rho_\ell T_{\ell k}\bar{s}_k &\geq \frac{1-d-\rho_\ell^3}{d\rho_\ell}, \\
\sum_{k=0}^{m} T_{\ell k}\bar{p}_k + \frac{1-d}{\rho_\ell^2} T_{\ell k}\bar{s}_k &\geq \frac{(1-d)^2-\rho_\ell^6}{d\rho_\ell^4}.
\end{aligned}$$

Because the actual calculations are performed with the coefficients $\bar{s}_i$ and $\bar{p}_i$, we also have to transform the relation $p_0 = 2(d-1)/d$ as well as the consistency relations. By means of (3.10) the relation $p_0 = 2(d-1)/d$ is transformed into

(3.13) $$\sum_{k=0}^{m} \bar{p}_k = 2(d-1)/d.$$

The transformed conditions of consistency are given below:

(3.14)
$$p = 1 \quad \left| \begin{array}{l} \sum_{k=0}^{m} \bar{p}_k + \bar{s}_k = 1, \\ \sum_{k=0}^{m} \left(\frac{2}{\bar{\beta}} k^2 - 1\right) \bar{p}_k + \frac{2k^2}{\bar{\beta}} \bar{s}_k = \frac{3-2d}{d}\,; \end{array} \right.$$
$$p = 2 \quad \left| \; \sum_{k=0}^{m} \left(\frac{1}{2} - \frac{2k^2}{\bar{\beta}} + \frac{2k^2(k^2-1)}{3\bar{\beta}^2}\right) \bar{p}_k + \left(\frac{2k^2(k^2-1)}{3\bar{\beta}^2}\right) \bar{s}_k = \frac{2d - \frac{3}{2}}{d}\,. \right.$$

Next we define the vector

(3.15) $$X = [\bar{p}_0,\ldots,\bar{p}_m,\ \bar{s}_0,\ldots,\bar{s}_m]^T,$$

and we assume that $\bar{\beta}$, d and N are fixed. Then, inequalities (3.12) for $\ell = 1,\ldots,N$, relations (3.13) and (3.14) constitute a linear system of inequalities of the type

(3.16) $$AX \geq C,$$

where A represents a $(5N + 4 + 2p) * (2m + 2)$ matrix, and where C is a $5N + 4 + 2p$-vector of righthand sides.

As already observed we are only interested in the existence or non-existence of a feasible solution to (3.16). Thus, it is allowed to consider the following linear optimization problem:

(3.17) $$\min B^T X$$

subject to

(3.18)
$$AX \geq C,$$
$$-\infty \leq X_i \leq \infty,$$

where X_i denotes the i-th component of X and B represents an arbitrary $2m+2$-vector. The solution to (3.17) - (3.18) is of course a feasible solution to (3.17), provided such a solution exists. The reason for stating problem (3.17) - (3.18) will be clear from the following argument. In order to satisfy (3.7) for arguments z between points z_j it will be necessary to choose large values for N. As a consequence, the number of constraints is much greater than the number of variables. From a practical point of view it is therefore more convenient to consider problem (3.17) - (3.18) as the dual problem of the primal problem

$$\max C^T Y \tag{3.19}$$

subject to

$$\begin{aligned} &A^T Y = B, \\ &Y_i \geq 0, \end{aligned} \tag{3.20}$$

and to solve problem (3.19) - (3.20). The dual solution of this problem is the primal solution of problem (3.17) - (3.18), provided it exists. The time needed to solve (3.19) - (3.20) numerically is much smaller than the time needed for solving (3.17) - (3.18). The relation between both problems is given by the following theorem (see GASS [2]):

<u>THE DUALITY THEOREM</u>: *If either the primal or the dual problem has a finite optimum solution, then the other problem has a finite optimum solution and the extremes of the linear functions are equal. If either problem has an unbounded optimum solution, then the other problem has no feasible solution.*

The next step is to describe the determination of the parameter d and the boundary β. For one-step and two-step methods we have the relation

$$\beta \simeq Km^2, \tag{3.21}$$

where K is a constant, We shall assume that for our problems such a relation also holds. This assumption has been corroborated by practical experiments. Thus the idea is to determine a constant K by a numerical search program for low values of m, m = 2, 3 and 4 say, and after that to solve one linear programming problem for each m with a prescribed $\bar{\beta} = Km^2$.

The numerical search, performed for m = 2, 3 and 4, is very simple and can be described as follows: the components of the vector B has been set equal to 1. For N, the number of gridpoints, we have chosen 30 * m. Next, for a sequence of d-values, where $0 < d < 2$, a sequence of linear problems (3.19) - (3.20) has been solved by performing bisection on $\bar{\beta}$ in such a way that β is estimated within a relative accuracy of 0.01. For the linear programming problems we used the IMSL-routine ZX3LP. This routine is very easy to use and is suited for our purpose as it delivers the corresponding dual solution. Moreover, it yields error messages in case no feasible or bounded solution exists. We need these messages for the bisection process.

The damping function $\rho(z)$, used in the actual calculations, is defined by

$$\rho(z) = \begin{cases} 1, & -1.5 \le z \le 0, \\ 0.85, & z \le -1.5. \end{cases} \tag{3.22}$$

The results of the numerical search are:

$$\begin{aligned} &p = 1, \quad d = 1.375, \quad \beta = 5.15 * m^2, \\ &p = 2, \quad d = 0.775, \quad \beta = 2.29 * m^2. \end{aligned} \tag{3.23}$$

With these values for d and β the coefficients s_i and p_i were determined for $m \le 12$. We restricted the m-values to $m \le 12$, for we have to take into account the internal stability behaviour of the integration method. Emanating from a machine precision of 14 or 15 digits, higher values of m cannot be accounted for (see VERWER [5]). The corresponding s_i and p_i are listed in the appendix. In the appendix we have also given a figure of the absolute stability region $\{z \mid z \in \mathbb{C}, \max|\alpha_i(z)| < 1, \ i = 1,2,3\}$ for some values of m and p.

The problem stated in section 2 was solved by discretizing the continuous variable z. As a consequence, the condition

$$|\alpha_i(z)| \le \rho(z), \qquad i = 1,2,3,$$

is not necessarily satisfied for all $z \in [-\beta,0]$. The true damping factor

$$(3.24) \qquad \rho_{max} = \max_{i=1,2,3} |\alpha_i(z)|, \qquad z \in [-\beta,-1.5],$$

is approximately given in table 3.1. It is clear that $\rho_{max} \to 0.85$ if $N \to \infty$.

m	p = 1	p = 2
2	0.85	0.85
3	0.85	0.87
4	0.86	0.88
5	0.86	0.85
6	0.86	0.89
7	0.87	0.86
8	0.86	0.90
9	0.86	0.87
10	0.88	0.88
11	0.88	0.92
12	0.87	0.91

TABLE 3.1. ρ_{max}

From table 3.1 we see that ρ_{max} becomes larger with m. Thus, for higher values of m very large values of N are necessary if one applies the technique described above with an equidistant z-grid. In order to circumvent this problem we suggest for high values of m the following approach. Firstly, apply the technique using an equidistant grid with N not too large, $N = 10 * m$ say. This yields a number of points where ρ_{max} is too large. Secondly, apply the technique using a finer grid near these points. By using a finer grid in a neighborhood of these critical points, it is possible to apply the linear programming approach for relatively high values of m.

In the near future we intend to publish numerical results of three-step methods, possessing error and stepsize control, of which the parameters are chosen as suggested in VERWER [5], section 4.

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APPENDIX

```
ORDER = 1 DEGREE =  2 D = 1,375 RO =    ,85 BETA =  ,2060E+02

P 0 =  ,54545454545454E+00  S 0 =  ,45454545454545E+00
P 1 =  ,32974222139503E+00  S 1 =  ,39753050587769E+00
P 2 =  ,15874540963720E-01  S 2 =  ,19106034236683E-01
```

```
ORDER = 1 DEGREE =  3 D = 1,375 RO =    ,85 BETA =  ,4635E+02

P 0 =  ,54545454545454E+00  S 0 =  ,45454545454545E+00
P 1 =  ,32957779372404E+00  S 1 =  ,39769493354868E+00
P 2 =  ,18807742712872E-01  S 2 =  ,22675100922608E-01
P 3 =  ,26931406295668E-03  S 3 =  ,32438614977749E-03
```

```
ORDER = 1 DEGREE =  4 D = 1,375 RO =    ,85 BETA =  ,8240E+02

P 0 =  ,54545454545454E+00  S 0 =  ,45454545454545E+00
P 1 =  ,32959633626966E+00  S 1 =  ,39767639100306E+00
P 2 =  ,19843848141588E-01  S 2 =  ,23921246939481E-01
P 3 =  ,38370516974646E-03  S 3 =  ,46221334545758E-03
P 4 =  ,23214008199836E-05  S 4 =  ,27945492539796E-05
```

```
ORDER = 1 DEGREE =  5 D = 1,375 RO =    ,85 BETA =  ,1288E+03

P 0 =  ,54545454545454E+00  S 0 =  ,45454545454545E+00
P 1 =  ,32940480780399E+00  S 1 =  ,39786791946874E+00
P 2 =  ,20289622974884E-01  S 2 =  ,24509754044867E-01
P 3 =  ,43922728369494E-03  S 3 =  ,53071848010798E-03
P 4 =  ,38881393659393E-05  S 4 =  ,47002589400275E-05
P 5 =  ,12058633806519E-07  S 5 =  ,14585303356181E-07
```

```
ORDER = 1 DEGREE =  6 D = 1,375 RO =    ,85 BETA =  ,1854E+03

P 0 =  ,54545454545454E+00  S 0 =  ,45454545454545E+00
P 1 =  ,32915730747582E+00  S 1 =  ,39811541979691E+00
P 2 =  ,20529934599533E-01  S 2 =  ,24864589588956E-01
P 3 =  ,47014565070180E-03  S 3 =  ,56998860293388E-03
P 4 =  ,48729959290595E-05  S 4 =  ,59138340603510E-05
P 5 =  ,23293337843875E-07  S 5 =  ,28300608926316E-07
P 6 =  ,41768893290961E-10  S 6 =  ,50812598486162E-10
```

```
ORDER = 1 DEGREE =  7 D = 1,375 RO =    ,85 BETA =   ,2524E+03

P 0 =   ,54545454545454E+00   S 0 =   ,45454545454545E+00
P 1 =   ,32940290556199E+00   S 1 =   ,39786982171073E+00
P 2 =   ,20695785946957E-01   S 2 =   ,25000002282553E-01
P 3 =   ,48959305208277E-03   S 3 =   ,59148858437864E-03
P 4 =   ,55250561672575E-05   S 4 =   ,66757562996758E-05
P 5 =   ,32042572866540E-07   S 5 =   ,38720468964770E-07
P 6 =   ,92217187087773E-10   S 6 =   ,11144761163396E-09
P 7 =   ,10431596770258E-12   S 7 =   ,12608153728000E-12
```

```
ORDER = 1 DEGREE =  8 D = 1,375 RO =    ,85 BETA =   ,3296E+03

P 0 =   ,54545454545454E+00   S 0 =   ,45454545454545E+00
P 1 =   ,32904622047855E+00   S 1 =   ,39822650679417E+00
P 2 =   ,20769127247467E-01   S 2 =   ,25183525013803E-01
P 3 =   ,50144873108237E-03   S 3 =   ,60880314013113E-03
P 4 =   ,59538312498493E-05   S 4 =   ,72358433014328E-05
P 5 =   ,38413639404624E-07   S 5 =   ,46725317100106E-07
P 6 =   ,13735161731261E-09   S 6 =   ,16719594503501E-09
P 7 =   ,25577038177072E-12   S 7 =   ,31157527742621E-12
P 8 =   ,19355325549260E-15   S 8 =   ,23590370482110E-15
```

```
ORDER = 1 DEGPEE =  9 D = 1,375 RO =    ,85 BETA =   ,4172E+03

P 0 =   ,54545454545454E+00   S 0 =   ,45454545454545E+00
P 1 =   ,32902403825404E+00   S 1 =   ,39824868901869E+00
P 2 =   ,20835887364795E-01   S 2 =   ,25265819897767E-01
P 3 =   ,51019371265378E-03   S 3 =   ,61910411544859E-03
P 4 =   ,62671263848834E-05   S 4 =   ,76074711148850E-05
P 5 =   ,43273294211670E-07   S 5 =   ,52536344101559E-07
P 6 =   ,17557956622083E-09   S 6 =   ,21317642667569E-09
P 7 =   ,41529099936815E-12   S 7 =   ,50421345500406E-12
P 8 =   ,52977760638010E-15   S 8 =   ,64317688539028E-15
P 9 =   ,28162396623902E-18   S 9 =   ,34187163161474E-18
```

ORDER = 1 DEGREE = 10 D = 1,375 RO = ,85 BETA = ,5150E+03

P 0 =	,54545454545454E+00	S 0 =	,45454545454545E+00
P 1 =	,32900701770523E+00	S 1 =	,39826570956750E+00
P 2 =	,20847709891773E-01	S 2 =	,25287844091655E-01
P 3 =	,51474022625718E-03	S 3 =	,62494425734623E-03
P 4 =	,64655444450446E-05	S 4 =	,78540296938450E-05
P 5 =	,46694681553560E-07	S 5 =	,56743151456408E-07
P 6 =	,20545752812528E-09	S 6 =	,24973869169769E-09
P 7 =	,55985683650846E-12	S 7 =	,68066567167935E-12
P 8 =	,92238940881933E-15	S 8 =	,11216260688042E-14
P 9 =	,84171480799272E-18	S 9 =	,10236802978533E-17
P10 =	,32655765072494E-21	S10 =	,39720657964596E-21

ORDER = 1 DEGREE = 11 D = 1,375 RO = ,85 BETA = ,6232E+03

P 0 =	,54545454545454E+00	S 0 =	,45454545454545E+00
P 1 =	,32923047518706E+00	S 1 =	,39804225208567E+00
P 2 =	,20942105514450E-01	S 2 =	,25341708058167E-01
P 3 =	,52155114179973E-03	S 3 =	,63151501325496E-03
P 4 =	,66698403229501E-05	S 4 =	,80807554649258E-05
P 5 =	,49787290971999E-07	S 5 =	,60354664402269E-07
P 6 =	,23175109073032E-09	S 6 =	,28111642092611E-09
P 7 =	,69290153772379E-12	S 7 =	,84106334638657E-12
P 8 =	,13309584971259E-14	S 8 =	,16167280397671E-14
P 9 =	,15876152476718E-17	S 9 =	,19299941789907E-17
P10 =	,10702794409632E-20	S10 =	,13021732956413E-20
P11 =	,31159779644428E-24	S11 =	,37944453664700E-24

ORDER = 1 DEGREE = 12 D = 1,375 RO = ,85 BETA = ,7416E+03

P 0 =	,54545454545454E+00	S 0 =	,45454545454545E+00
P 1 =	,32888246227955E+00	S 1 =	,39838448104318E+00
P 2 =	,20930564082556E-01	S 2 =	,25416762495946E-01
P 3 =	,52399048369937E-03	S 3 =	,63697712076076E-03
P 4 =	,67865318951772E-05	S 4 =	,82551983508228E-05
P 5 =	,51892263386503E-07	S 5 =	,63149729888770E-07
P 6 =	,25160762750795E-09	S 6 =	,30629858708888E-09
P 7 =	,80319349035922E-12	S 7 =	,97808681916440E-12
P 8 =	,17105594522426E-14	S 8 =	,20836563130596E-14
P 9 =	,24063258323529E-17	S 9 =	,29320679574331E-17
P10 =	,21467888957759E-20	S10 =	,26166497017204E-20
P11 =	,11002739569540E-23	S11 =	,13415328284784E-23
P12 =	,24672233557657E-27	S12 =	,30092796196223E-27

ORDER = 2 DEGREE = 2 D = ,775 RO = ,85 BETA = ,916000E+01

P 0 = -,58064516129032E+00 S 0 = ,15806451612903E+01
P 1 = -,21559395174672E+00 S 1 = ,15059165323919E+01
P 2 = -,30544696579492E-01 S 2 = ,16978945451019E+00

ORDER = 2 DEGREE = 3 D = ,775 RO = ,85 BETA = ,206100E+02

P 0 = -,58064516129032E+00 S 0 = ,15806451612903E+01
P 1 = -,19115223031554E+00 S 1 = ,14814748109607E+01
P 2 = -,29473834786102E-01 S 2 = ,19316031414798E+00
P 3 = -,10066347258135E-02 S 3 = ,62334163398843E-02

ORDER = 2 DEGREE = 4 D = ,775 RO = ,85 BETA = ,366400E+02

P 0 = -,58064516129032E+00 S 0 = ,15806451612903E+01
P 1 = -,18233307297138E+00 S 1 = ,14726556536165E+01
P 2 = -,28236544473632E-01 S 2 = ,20074218117966E+00
P 3 = -,12030039601232E-02 S 3 = ,86336976630508E-02
P 4 = -,15208008334815E-04 S 4 = ,11558084587197E-03

ORDER = 2 DEGREE = 5 D = ,775 RO = ,85 BETA = ,572500E+02

P 0 = -,58064516129032E+00 S 0 = ,15806451612903E+01
P 1 = -,17833069941496E+00 S 1 = ,14686532800601E+01
P 2 = -,28475246894827E-01 S 2 = ,20498325715728E+00
P 3 = -,14606302030482E-02 S 3 = ,99938118863291E-02
P 4 = -,29964111364600E-04 S 4 = ,19922348036296E-03
P 5 = -,21343804115613E-06 S 5 = ,13919201448922E-05

ORDER = 2 DEGREE = 6 D = ,775 RO = ,85 BETA = ,824400E+02

P 0 = -,58064516129032E+00 S 0 = ,15806451612903E+01
P 1 = -,17610367098315E+00 S 1 = ,14664262516283E+01
P 2 = -,28260676645090E-01 S 2 = ,20699571533935E+00
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ORDER = 2 DEGREE =  7 D =  ,775 RO =   ,85 BETA =  ,112210E+03

P 0 = -,58064516129032E+00  S 0 =  ,15806451612903E+01
P 1 = -,17483390114911E+00  S 1 =  ,14651564817943E+01
P 2 = -,27741186226246E-01  S 2 =  ,20774599475456E+00
P 3 = -,14815956989918E-02  S 3 =  ,10971861052267E-01
P 4 = -,36301045393729E-04  S 4 =  ,27524368830002E-03
P 5 = -,44685007550778E-06  S 5 =  ,35403656934423E-05
P 6 = -,26875584507281E-08  S 6 =  ,22575321236761E-07
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ORDER = 2 DEGREE =  8 D =  ,775 RO =   ,85 BETA =  ,146560E+03

P 0 = -,58064516129032E+00  S 0 =  ,15806451612903E+01
P 1 = -,17396071288671E+00  S 1 =  ,14642832935319E+01
P 2 = -,28684134226593E-01  S 2 =  ,20956213101730E+00
P 3 = -,17432495919146E-02  S 3 =  ,11509566107375E-01
P 4 = -,50845860092993E-04  S 4 =  ,31097988163828E-03
P 5 = -,79137773126678E-06  S 5 =  ,45630986133302E-05
P 6 = -,67374753547729E-08  S 6 =  ,37079641130883E-07
P 7 = -,29589252318632E-10  S 7 =  ,15682590950194E-09
P 8 = -,52416294063507E-13  S 8 =  ,26933430035588E-12


ORDER = 2 DEGREE =  9 D =  ,775 RO =   ,85 BETA =  ,185490E+03

P 0 = -,58064516129032E+00  S 0 =  ,15806451612903E+01
P 1 = -,17339878373842E+00  S 1 =  ,14637213643836E+01
P 2 = -,28044727272007E-01  S 2 =  ,20948465321101E+00
P 3 = -,16253143532453E-02  S 3 =  ,11536416747709E-01
P 4 = -,45810065724231E-04  S 4 =  ,31784614870548E-03
P 5 = -,71155788003856E-06  S 5 =  ,49116720047009E-05
P 6 = -,64049072748140E-08  S 6 =  ,44528196080481E-07
P 7 = -,33259355915937E-10  S 7 =  ,23505328331393E-09
P 8 = -,92392278190522E-13  S 8 =  ,66870118493809E-12
P 9 = -,10625147968957E-15  S 9 =  ,79239438466385E-15
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ORDER = 2 DEGREE = 10 D =  ,775 RO =   ,85 BETA =  ,229000E+03

P 0 = -,58064516129032E+00  S 0 =  ,15806451612903E+01
P 1 = -,17144173377009E+00  S 1 =  ,14617643144152E+01
P 2 = -,28015092938518E-01  S 2 =  ,21141206884584E+00
P 3 = -,16157222373633E-02  S 3 =  ,11805056288024E-01
P 4 = -,46227724829057E-04  S 4 =  ,33440743994479E-03
P 5 = -,75479222345544E-06  S 5 =  ,54414235293880E-05
P 6 = -,74894835017511E-08  S 6 =  ,53918086434111E-07
P 7 = -,45979053535837E-10  S 7 =  ,33075450191903E-09
P 8 = -,17060137796770E-12  S 8 =  ,12264057386754E-11
P 9 = -,35056663086597E-15  S 9 =  ,25181064036724E-14
P10 = -,30628886618191E-18  S10 =  ,21977616072810E-17

ORDER = 2 DEGREE = 11 D =  ,775 RO =   ,85 BETA =  ,277090E+03

P 0 = -,58064516129032E+00  S 0 =  ,15806451612903E+01
P 1 = -,17300284166294E+00  S 1 =  ,14633254223081E+01
P 2 = -,27328009024214E-01  S 2 =  ,20916387703869E+00
P 3 = -,15056021545051E-02  S 3 =  ,11579454550239E-01
P 4 = -,41367453292114E-04  S 4 =  ,32857305529008E-03
P 5 = -,65884289780074E-06  S 5 =  ,54458388015784E-05
P 6 = -,65536146317341E-08  S 6 =  ,56371977519294E-07
P 7 = -,42091025159364E-10  S 7 =  ,37545083692440E-09
P 8 = -,17479977056317E-12  S 8 =  ,16091005950325E-11
P 9 = -,45368406820998E-15  S 9 =  ,42884398776166E-14
P10 = -,66928703208412E-18  S10 =  ,64665832685414E-17
P11 = -,42844311155752E-21  S11 =  ,42148457924105E-20

ORDER = 2 DEGREE = 12 D =  ,775 RO =   ,85 BETA =  ,329760E+03

P 0 = -,58064516129032E+00  S 0 =  ,15806451612903E+01
P 1 = -,17229366960984E+00  S 1 =  ,14626162502550E+01
P 2 = -,28800272381574E-01  S 2 =  ,21134531244915E+00
P 3 = -,18596174430420E-02  S 3 =  ,12108121568554E-01
P 4 = -,59976978730906E-04  S 4 =  ,35894153745450E-03
P 5 = -,11090566945359E-05  S 5 =  ,62664422624671E-05
P 6 = -,12745935692238E-07  S 6 =  ,69193362616297E-07
P 7 = -,95158694657869E-10  S 7 =  ,50195179261835E-09
P 8 = -,46969406476975E-12  S 8 =  ,24253292036317E-11
P 9 = -,15218064136528E-14  S 9 =  ,77310365461803E-14
P10 = -,31130952108508E-17  S10 =  ,15613921810526E-16
P11 = -,36466961532119E-20  S11 =  ,18102819599155E-19
P12 = -,18644934368193E-23  S12 =  ,91776107476727E-23
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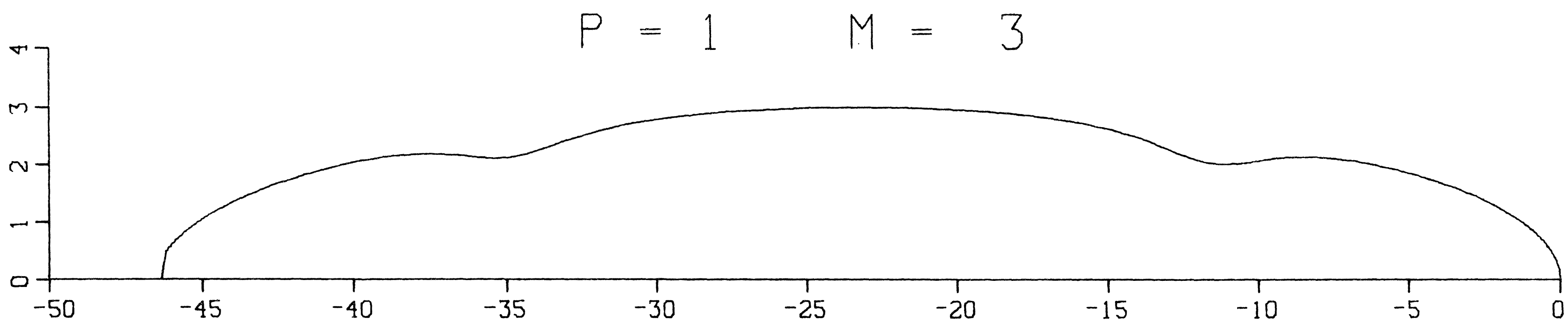
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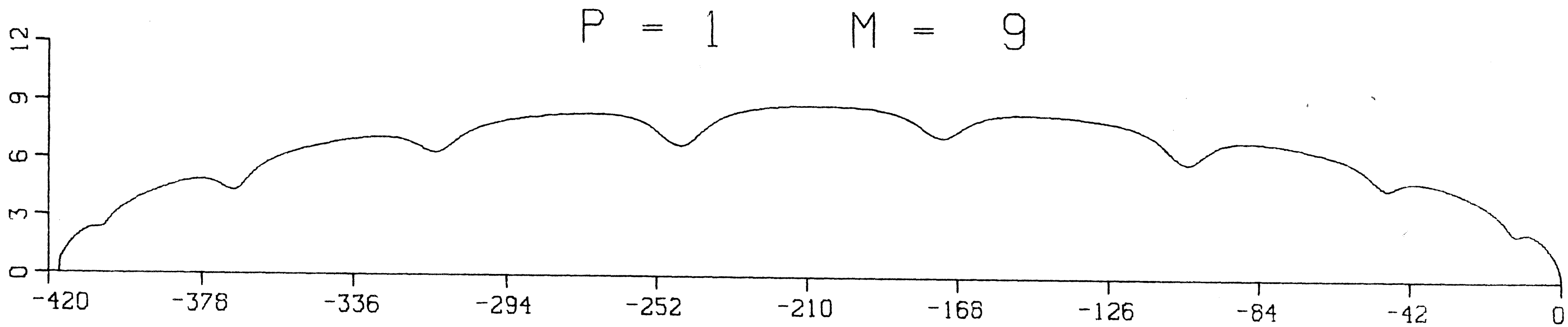
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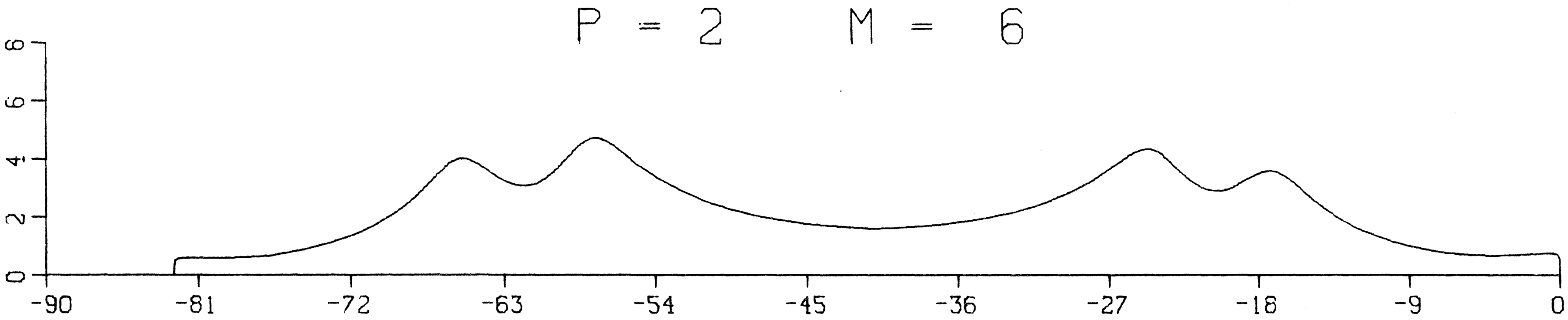
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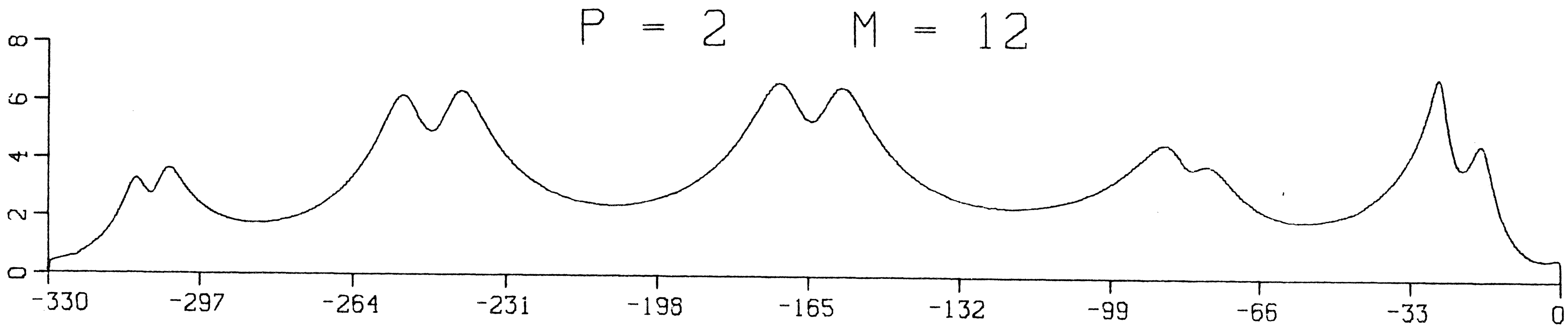
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-165
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-99
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-33
0