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ON THE LOCAL CONVERGENCE OF NEWTON'S METHOD

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On the local convergence of Newton's method *)

by

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ABSTRACT

In a Banachspace X, let F be an operator with Lipschitz continuous derivative F', and $x^* \in X$ such that $F(x^*) = 0$ and $F'(x^*)$ is invertible. In a recent paper, Rall showed that an open ball B_1 with centre x^* and a specified radius exists such that the Newton-Kantorovich theorem guarantees rapid convergence to x^* starting from any $x_0 \in B_1$. In this note we focus attention on the existence and convergence of the Newton sequence $\{x_k\}$ leaving out the question of whether the hypotheses of the Newton-Kantorovich theorem are satisfied. In this way we are able to prove that a ball $B_2 \supset B_1$ exists with centre x^* and a specified radius such that under the same hypotheses as Rall assumed, the Newton sequence converges quadratically to x^*, starting from any $x_0 \in B_2$. The radius of B_2 is shown to be the best possible.

KEY WORDS & PHRASES: *Newton's method, local convergence.*

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*) This report will be submitted for publication elsewhere

Let F be a nonlinear operator from a Banach space X into itself. Suppose that the Fréchet-derivative $F'(x)$ of F exists for all $x \in X$. In order to find a solution $x = x^*$ of the equation

(1) $$F(x) = 0$$

one might use *Newton's method*. It consists in constructing a so called *Newton sequence* $\{x_k\}$ defined by

(2) $$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \qquad k = 0,1,\dots,$$

where x_0 is some suitably chosen *starting point* of the sequence. Assume that there is a region $\Omega \subset X$ and a constant $\gamma > 0$, such that

(3) $$\|F'(x) - F'(y)\| \leq \gamma\|x - y\|, \qquad \text{for all } x,y \in \Omega.$$

Assume further that x^* is a solution of (1), and that $F'(x^*)$ has an inverse whose norm $\|F'(x^*)^{-1}\|$ satisfies

(4) $$\|F'(x\ )^{-1}\| \leq \beta.$$

Let $B(x,r) = \{y \mid \|y - x\| < r\}$ and let $r_1 = (2-\sqrt{2})/(2\beta\gamma)$ and $r_2 = 2/(3\beta\gamma)$. We note that $r_1 < r_2$.

In a recent paper, RALL [4] showed that, under conditions (3), (4) and

(5) $$B(x^*,1/(\beta\gamma)) \subset \Omega$$

the hypotheses of the famous Newton-Kantorovich theorem (cf. [1] and [3]) are satisfied at any starting point x_0 belonging to the ball $B(x^*,r_1)$. This implies that for any x_0 in this ball the Newton sequence $\{x_k\}$ exists and converges to x^*. Moreover, for any $x_0 \in B(x^*,r_1)$, the Newton-Kantorovich theorem was shown to imply that the sequence $\{x_k\}$ converges more rapidly than a geometric progression. Rall proved, by means of a counterexample, that no radius $r > r_1$ exists such that the Newton-Kantorovich theorem still ensures rapid convergence for all $x_0 \in B(x^*,r)$.

In this note we focus attention on the existence and convergence of the Newton-sequence $\{x_k\}$, leaving out the question of whether the hypotheses of the Newton-Kantorovich theorem are satisfied. In this way we are able to prove that, again under the conditions (3), (4) and (5), the Newton sequence $\{x_k\}$ exists and converges to x^* whenever x_0 belongs to the ball $B(x^*,r_2) \supset B(x^*,r_1)$. For all of these x_0 the convergence will be shown to be quadratic, and the value r_2 will be shown to be the best possible.

THEOREM. *Let* x^* *be a solution of* (1) *and let* F *satisfy* (3). *Assume further that* $F'(x^*)$ *has an inverse that satisfies* (4). *Finally let*

(6) $$B(x^*,2/(3\beta\gamma)) \subset \Omega.$$

Then for any starting point $x_0 \in B(x^*,r_2)$ *the Newton sequence* $\{x_k\}$ *exists and converges to* x^*. *Moreover a constant* $C > 0$ *exists such that*

$$\|x_{k+1}-x^*\| \le C\|x_k-x^*\|^2, \qquad k = 0,1,\ldots .$$

PROOF. (a) For any ε, where $0 < \varepsilon \le 2$, let $\alpha(\varepsilon) = (2-\varepsilon)/(2+2\varepsilon)$. We note that $0 \le \alpha(\varepsilon) < 1$.

Let $x \in B(x\ ,(2-\varepsilon)/(3\beta\gamma))$. Following the same argument as Rall, we may conclude that

(i) $F'(x)$ has an inverse whose norm $\|F'(x)^{-1}\|$ satisfies

$$\|F'(x)^{-1}\| \le \beta/(1-\beta\gamma\|x - x^*\|).$$

(ii) $$F(x^*)-F(x) = F'(x)(x^*-x)+\int_0^1 [F'(x+t(x^*-x))-F'(x)](x^*-x)dt.$$

Thus

$$x-F'(x)^{-1}F(x)-x^* = F'(x)^{-1}\int_0^1 [F'(x+t(x^*-x))-F'(x)](x^*-x)dt.$$

Therefore

$$\|x-F'(x)^{-1}F(x)-x^*\| = \|F'(x)^{-1}\int_0^1 [F'(x+t(x^*-x))-F'(x)](x^*-x)dt\| \le$$

$$\le \|F'(x)^{-1}\| \int_0^1 \gamma t dt \|x-x^*\|^2 \le \frac{\beta\gamma\|x-x^*\|^2}{2(1-\beta\gamma\|x-x^*\|)} .$$

So

$$(7) \qquad \|x-F'(x)^{-1}F(x)-x^*\| \le \frac{\beta\gamma\|x-x^*\|^2}{2(1-\beta\gamma\|x-x^*\|)} .$$

Thus for any $x \in B(x^*,(2-\varepsilon)/(3\beta\gamma))$,

$$(8) \quad \|x-F'(x)^{-1}F(x)-x^*\| \le \frac{\beta\gamma(2-\varepsilon)/(3\beta\gamma)}{2(1-\beta\gamma(2-\varepsilon)/(3\beta\gamma))}\|x-x^*\| = \alpha(\varepsilon)\|x-x^*\| .$$

(b) Let $x_0 \in B(x^*,r_2)$. Then an $\varepsilon > 0$ exists such that $x_0 \in B(x^*,(2-\varepsilon)/(3\beta\gamma))$. From (8) it follows that the Newton sequence $\{x_k\}$ with starting point x_0 exists and remains in $B(x^*,(2-\varepsilon)/(3\beta\gamma))$. Furthermore

$$\|x_k-x^*\| \le [\alpha(\varepsilon)]^k\|x_0-x^*\| \to 0, \qquad \text{for } k \to \infty.$$

From (7) it follows that

$$\|x_{k+1}-x^*\| \le \frac{3\beta\gamma}{2}\|x-x^*\|^2, \qquad k = 0,1,\ldots . \quad \square$$

The value $2/(3\beta\gamma)$ for r_2 is as large as possible, even if we strengthen condition (6) of the theorem by requiring that $\Omega = X$. This is shown by the following example.

Let $X = \mathbb{R}^1$, and $F: \mathbb{R}^1 \to \mathbb{R}^1$ be defined by

$$F(x) = \begin{cases} 1/(2\beta^2\gamma) , & \text{for } x > 1/(\beta\gamma) \\ -\frac{1}{2}\gamma x(x-2/(\beta\gamma)) , & \text{for } 0 \le x \le 1/(\beta\gamma) \\ \frac{1}{2}\gamma x(x+2/(\beta\gamma)) , & \text{for } -1/(\beta\gamma) \le x < 0 \\ -1/(2\beta^2\gamma) , & \text{for } x < -1/(\beta\gamma). \end{cases}$$

It is easily verified that $F(0) = 0$, $\|F'(0)^{-1}\| = \beta$ and $\|F'(x)-F'(y)\| \le$ $\le \gamma\|x-y\|$, for all $x,y \in X$. So for any starting point $x_0 \in (-2/(3\beta\gamma),2/(3\beta\gamma))$ the Newton sequence $\{x_k\}$ converges to $x^* = 0$. However, if we take

$x_0 = 2/(3\beta\gamma)$, the Newton-sequence $\{x_k\}$ satisfies: $x_k = (-1)^k 2/(3\beta\gamma)$ $(k = 1,2,3,\ldots)$. Consequently $\{x_k\}$ does not converge to $x^* = 0$.

If X is an arbitrary Hilbertspace, then it can also be shown by means of a counterexample that the value $2/(3\beta\gamma)$ of r_2 is as large as possible (cf. [2]).

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