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LINEAR MULTISTEP METHODS FOR A CLASS OF HYPERBOLIC DIFFERENTIAL EQUATIONS

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by

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ABSTRACT

Hyperbolic differential equations with a second order time derivative reduce to systems of second order, ordinary differential equations when the space variables are discretized. In this paper the special class is considered where the system does not contain first derivatives. Generally, the Jacobian matrix of the right hand side has a large negative eigenvalue spectrum, and therefore, integration methods with a large negative interval of stability are to be used. We construct several linear multistep formulas of first, second and third order, the implicit ones being unconditionally stable, the explicit ones having an extended interval of stability.

KEY WORDS & PHRASES: *Hyperbolic differential equations, second order equations, interval of stability.*

CONTENTS

1. INTRODUCTION

When hyperbolic differential equations of second order are semi-discretized with respect to the space variables, a (large) system of second order, ordinary differential equations of the form

$$(1.1)\qquad \frac{d^2\vec{y}}{dx^2} = \vec{f}\left(x,y,\frac{d\vec{y}}{dx}\right)$$

is obtained which are in general characterized by a *large negative* eigenvalue spectrum of the Jacobian matrix $\partial\vec{f}/\partial\vec{y}$. In order to solve the initial value problem for the general equation (1.1), the literature recommends (cf. HENRICI [1962, p.289]) to reduce (1.1) to a first order system and to apply integration methods for first order systems. Since these first order systems have Jacobian matrices of which the eigenvalues are located in a large strip along the imaginary axis, only methods with a relatively *large imaginary* stability interval are appropriate for numerical integration. Confining our considerations to linear multistep methods, we have at our disposal the first order backward Euler formula and the second order, two step backward differentiation formula of Curtiss and Hirschfelder. Both methods are *unconditionally, strongly stable* along the *imaginary* axis. As is well known, backward differentiation formulas of higher order are unstable in an imaginary interval around the orgin and therefore inappropriate for integrating hyperbolic systems.

When the first derivative $d\vec{y}/dx$ does not occur in the right hand side of equation (1.1), it is in many cases advantageous to omit the reduction to first order form and to construct algorithms starting from the second order form. In our case of negative eigenvalue spectra, these methods should have a large negative stability interval. Considering the linear multistep methods presented in textbooks such as those of HENRICI [1962] and LAMBERT [1973], it turns out that the stability intervals, even of the implicit ones, are relatively small. On the other hand, the order of accuracy is relatively high; the so-called optimal k-step methods which are usually presented in the literature, have orders k+1 and k+2 for odd and even values of k, respectively. In this paper,we shall derive first order two-step formulas which are unconditionally strongly stable along the whole imaginary axis. These formulas are slightly more economic with respect to their storage requirements than the corresponding formulas for first order systems (the calculation of the first order form requires additional storage!). In the near future, we intend to publish numerical results obtained by the formulas presented in this paper and by formulas for first order equations. We will also try to construct higher order formulas for first order equations having the complete imaginary axis in its stability region.

2. SUMMARY OF THE THEORY

A linear k-step method for the system of equations

$$\frac{d^2\vec{y}}{dx^2} = \vec{f}(x,\vec{y}) \tag{2.1}$$

is defined by

$$\vec{y}_{n+1} = \sum_{\ell=1}^{k} a_\ell \vec{y}_{n+1-\ell} + h^2 \sum_{\ell=0}^{k} b_\ell \vec{f}_{n+1-\ell}, \tag{2.2}$$

where $\vec{f}_{n+1-\ell} = \vec{f}(x_{n+1-\ell}, y_{n+1-\ell})$. Furthermore, $h = x_{n+1}-x_n$ and let $\vec{y}(x)$ be the solution of (2.1), then a_ℓ, b_ℓ are to be determined in such a way that $\vec{y}_{n+1}$ is a sufficiently accurate approximation to the solution $\vec{y}(x_{n+1-\ell})$

The conditions which make scheme (2.2) a *p-th order consistent* approximation to system (2.1) are given by (see e.g. LAMBERT [1973, p.253])

$$C_0 = C_1 = \dots = C_{p+1} = 0, C_{p+2} \neq 0$$

$$\begin{aligned} C_0 &= 1-(a_1+a_2+ \dots + a_k), \\ C_1 &= k-((k-1)a_1+(k-2)a_2+ \dots + a_{k-1}) \\ C_2 &= \tfrac{1}{2}[k^2-((k-1)^2 a_1 + \dots a_{k-1})] - [b_0 + b_1 + \dots + b_k], \\ C_j &= \frac{1}{j!}[k^j-((k-1)^j a_1 + (k-2)^j a_2 + \dots + a_{k-1})] \\ &\quad - \frac{1}{(j-2)!}[k^{j-2}b_0 + (k-1)^{j-2}b_1 + (k-2)^{j-2}b_2 + \dots + b_{k-1}], \\ &\qquad j = 2,3,\dots, \end{aligned} \tag{2.3}$$

where it is assumed that the integration steps do not depend on n.

Secondly, the condition which makes a p-th order consistent scheme *convergent* is given by (HENRICI [e.g. 62, p.25])

(2.4) $$|\zeta_j| \le 1, \qquad \mu(\zeta_j) \le 2 \quad \text{if} \quad |\zeta_j| = 1,$$

where the ζ_j are the zeroes of the function

$$C(\zeta,0) \equiv \zeta^k - a_1\zeta^{k-1} \ldots a_{k-1}\zeta - a_k$$

and

$\mu(\zeta)$ is the multiplicity of the root ζ_j.

Finally, we have the condition of stability (cf. LAMBERT [1973,p.258])

(2.5) $$h^2\Delta \subset \{z \mid |\zeta_j(z)| < 1\}, \qquad j = 1,2,\ldots,k,$$

where Δ is the eigenvalue spectrum of the Jacobian matrix $\partial\vec{f}/\partial\vec{y}$ and the $\zeta_j(z)$ are the roots of the function

(2.6) $$C(\zeta,z) = (1-b_0z)\zeta^k - (a_1+b_1z)\zeta^{k-1} - \ldots -(a_{k-1}+b_{k-1}z)\zeta - (a_k+b_k)$$

The function $C(\zeta,z)$ will be called the *characteristic function.*

In the following sections we construct integration formulas of first, second and third order accuracy with optimal negative stability intervals

(2.7) $$\{z \,|\, |\zeta_j(z)| < 1, \; z<0\}$$

3. TWO - STEP FORMULAS

From (2.3) it follows that a 2-step formula is consistent of first order when

(3.1) $a_1 = 2,\ a_2 = 1,\ b_2 = 1 - b_0 - b_1$,

second order consistent when, in addition,

(3.2) $2b_0 + b_1 = 1$,

and third order consists when, in addition to (3.1) and (3.2),

(3.3) $24b_0 + 6b_1 = 7$.

It is easily verified that in both cases the convergence condition (2.4) is satisfied. The main problem is the derivation of the stability interval (2.7). To that end we consider the characteristic function $C(\zeta,z)$ as defined in (2.6). We find

$$C(\zeta,z) = (1-b_0z)\zeta^2 - (2 + b_1z)\zeta + 1 - b_2z.$$

The solutions $\zeta_1(z)$ and $\zeta_2(z)$ of the equation $C(\zeta,z) \equiv 0$ assume values within the unit circle when

(3.5) $$\frac{1 - b_2z}{1 - b_0z} < 1, \qquad \left| \frac{2 + b_1z}{1 - b_0z} \right| < \frac{2 - (b_0 + b_2)z}{1 - b_0z}$$

provided that z is real. These conditions, therefore, determine the stability interval (2.7). A straightforward calculation yields the length of the stability intervals (denoted by β) as functions of the parameters b_0 and b_1 which are indicated in figure 3.1. It may be concluded that *unconditionally (strongly) stable* schemes arise for

(3.6) $b_1 \le \frac{1}{2},\ 2b_0 + b_1 > 1,\ b_2 = 1 - b_0 - b_1$

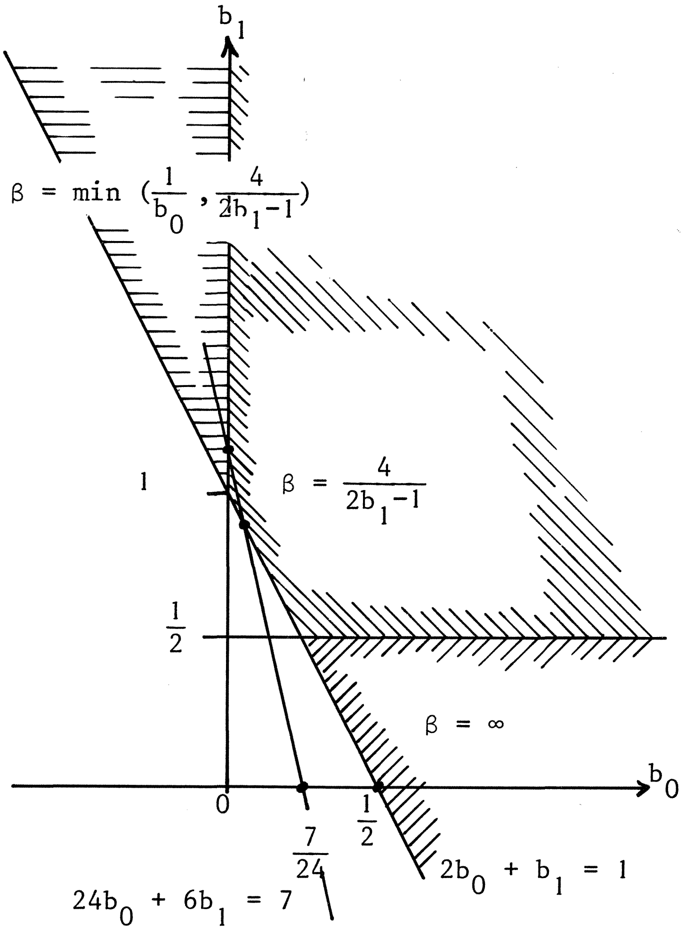

Fig. 3-1 Stability bounderies β in the (b_0, b_1) - plane

which are necessarily *implicit* and only of *first order*. In particular, we consider the case $b_1 = b_2 = 0$, i.e. the formula

$$(3.6) \qquad \vec{y}_{n+1} = 2\vec{y}_n - \vec{y}_{n-1} + h^2\vec{f}_{n+1}.$$

A simple calculation reveals that the roots of (3.4) satisfy the relation

$$|\zeta_j(z)| = \frac{\sqrt{1+|z|}}{1-z} \cong \frac{1}{\sqrt{|z|}}, \quad \text{as } |z| \to \infty.$$

Thus, formula (3.6) has a very strong damping effect on the higher harmonics.

For

$$(3.6'') \qquad b_1 \le \tfrac{1}{2}, \quad b_0 = \tfrac{1}{2}(1 - b_1), \quad b_2 = \tfrac{1}{2}(1 - b_1),$$

we obtain an *unconditionally weakly stable scheme of second order*, i.e. the roots $\zeta_{1,2}(z)$ of the characteristic function are for all z-values *on* the unit circle instead of *within* the unit circle. Finally when we impose condition (3.3), i.e.

$$(3.6''') \qquad b_0 = \frac{1}{12}, \quad b_1 = \frac{5}{6}, \quad b_2 = \frac{1}{12}$$

the scheme becomes fourth order exact, but has the stability boundary $\beta = 6$. In fact, this formula is the well-known formula of Numerov.

For the large systems which arise from two-dimensional hyperbolic differential equations it may be desirable to use *explicit* integration formulas, i.e. $b_0 = 0$. From figure 3.1 we see that the largest stability interval of explicit 2-step methods is obtained for

$$(3.7) \qquad b_0 = 0, \quad b_1 = 1, \quad b_2 = 0$$

yielding $\beta = 4$. The corresponding formula is second order accurate but only *weakly* stable. *Strongly* stable formulas are obtained by *increasing* b_1, *decreasing*, however, the value of β and *reducing* the order of accuracy to 1. As an illustration we consider the formula

$$(3.7') \qquad \vec{y}_{n+1} = 2\vec{y}_n - \vec{y}_{n-1} + h^2[(1 + \eta)\vec{f}_n - \eta\vec{f}_{n-1}], \quad 0 < \eta << 1$$

more closely. This formula is first order exact with the local truncation error (cf. LAMBERT [1973, p.253])

$$(3.8) \qquad -\eta h^3 \vec{y}'''(x_n) + \left(\frac{1}{12} - \frac{1}{2}\eta\right) h^4 \vec{y}''''(x_n) + O(h^5)$$

and amplification factors $\zeta_{1,2}(z)$ satisfying the relation

$$(3.9)\qquad |\zeta_{1,2}(z)| = \sqrt{1+\eta z} \cong 1 + \tfrac{1}{2}\eta z \qquad \text{as } \eta \to 0$$

provided that

$$(3.10)\qquad z > \frac{-4}{(1+\eta)^2} \cong -4(1-2\eta) \qquad \text{as } \eta \to 0$$

This formula has the advantage to damp the higher harmonics in the numerical solution at the cost of a small decrease of the stability boundary β and a "small" increase of the local truncation error.

4. THREE - STEP FORMULAS

For three - step formulas we have

$$(4.1)\qquad \begin{aligned} C_0 &= 1 - a_1 - a_2 - a_3, \\ C_1 &= 3 - 2a_1 - a_2, \\ C_2 &= \tfrac{9}{2} - 2a_1 - \tfrac{1}{2}a_2 - b_0 - b_1 - b_2 - b_3, \\ C_3 &= \tfrac{9}{2} - \tfrac{4}{3}a_1 - \tfrac{1}{6}a_2 - 3b_0 - 2b_1 - b_2, \\ C_4 &= \tfrac{27}{8} - \tfrac{2}{3}a_1 - \tfrac{1}{24}a_2 - \tfrac{9}{2}b_0 - 2b_1 - \tfrac{1}{2}b_2. \end{aligned}$$

Since we already have at our disposal strongly stable two-step formulas of first order, we now concentrate on second and third order formulas.

A simple calculation yields

$$(4.2)\qquad \begin{aligned} a_1 &= \tfrac{5}{2} - \tfrac{1}{2}(b_1 + 2b_2 + 3b_3), \\ a_2 &= -2 + (b_1 + 2b_2 + 3b_3), \\ a_3 &= \tfrac{1}{2} - \tfrac{1}{2}(b_1 + 2b_2 + 3b_3), \\ b_0 &= \tfrac{1}{2}(1 - b_1 + b_3) \end{aligned}$$

making the scheme second order consistent for all values of b_1, b_2 and b_3

by choosing

$$b_1 = \frac{81 - 16a_1 - a_2 - 108b_0 - 12b_2}{48} \tag{4.3}$$

we have third order accuracy.

In order to have convergence, equation (2.4) which is in the present case

$$\zeta^3 - \left(\frac{5}{2} - \frac{1}{2}b\right)\zeta^2 + (2 - b)\zeta - \frac{1}{2} + \frac{1}{2}b = 0, \tag{4.4}$$

should have roots satisfying conditions (2.4). Here, $b = b_1 + 2b_2 + 3b_3$. Since we may write for (4.4)

$$(\zeta - 1)^2\left(\zeta - \frac{1}{2} + \frac{1}{2}b\right) = 0 \tag{4.4'}$$

we satisfy condition (2.4) when

$$-1 < b \le 3. \tag{4.5}$$

Let us next consider the characteristic function

$$C(\zeta,z) \equiv (1 - b_0z)\zeta^3 - (a_1 + b_1z)\zeta^2 - (a_2 + b_2z)\zeta - (a_3 + b_3z) \tag{4.6}$$

Again we will assume that z only assumes negative values (negative spectrum) In order to force the zeros of $C(\zeta,z)$ within the unit circle we apply the Hurwitz criterion stating that the roots of the equation

$$\zeta^3 + c_1\zeta^2 + c_2\zeta + c_3 = 0$$

are within the unit circle when

$$1 + c_1 + c_2 + c_3 > 0,$$

$$1 - c_1 + c_2 - c_3 > 0,$$

$$3 - c_1 - c_2 + 3c_3 > 0,$$

$$1 - c_2 + c_1c_3 - c_3^2 > 0.$$

Applying this criterion to equation (4.6) and using that $b_0 > 0$, we arrive at the following inequalities for stable z - values:

$$
(4.7)\qquad
\begin{aligned}
&(1 - 3b_1 + 2b_2 - b_3)z < 4(3 - b_1 - 2b_2 - b_3) \\
&(1 + b_1 + 2b_2 + 3b_3)z < 0 \\
&(3 - 5b_1 - 2b_2 + 9b_3)z < 4(1 + b_1 + 2b_2 + b_3) \\
&[b_0(b_0 - b_2) + b_3(b_1 - b_3)]z < -\Big(-b_0(2 + a_2) + a_3b_1 + b_2 + \\
&\qquad\qquad + b_3(a_1 - 2a_3)\Big) .
\end{aligned}
$$

In order to obtain the whole negative z - axis in the stability region we have to require that

$$
(4.8)\qquad
\begin{aligned}
&b_0 = \tfrac{1}{2}(1 - b_1 + b_3) > 0 \\
&3 - b_1 - 2b_2 - b_3 > 0 \\
&1 - 3b_1 + 2b_2 - b_3 > 0 \\
&1 + b_1 + 2b_2 + 3b_3 > 0 \\
&1 + b_1 + 2b_2 + b_3 > 0 \\
&3 - 5b_1 - 2b_2 + 9b_3 > 0 \\
&b_0(b_0 - b_2) + b_3(b_1 - b_3) > 0
\end{aligned}
$$

Introducing the variable $b = b_1 + 2b_2 + 3b_3$ instead of b_2, we obtain the slightly more simple system

$$
(4.8')\qquad
\begin{aligned}
&-1 < b < 3 + 2b_3 \\
&1 - b_1 + b_3 > 0 \\
&1 - 4b_1 - 4b_3 > - b \\
&1 - 2b_3 > - b \\
&3 - 4b_1 + 12b_3 > b \\
&1 - (1 - b)b_1 + (5 - b)b_3 > b .
\end{aligned}
$$

In figure 4.1 the region of stable (b_1, b_3) points is shown. Furthermore, the line of points giving third order accuracy is drawn. This line is described

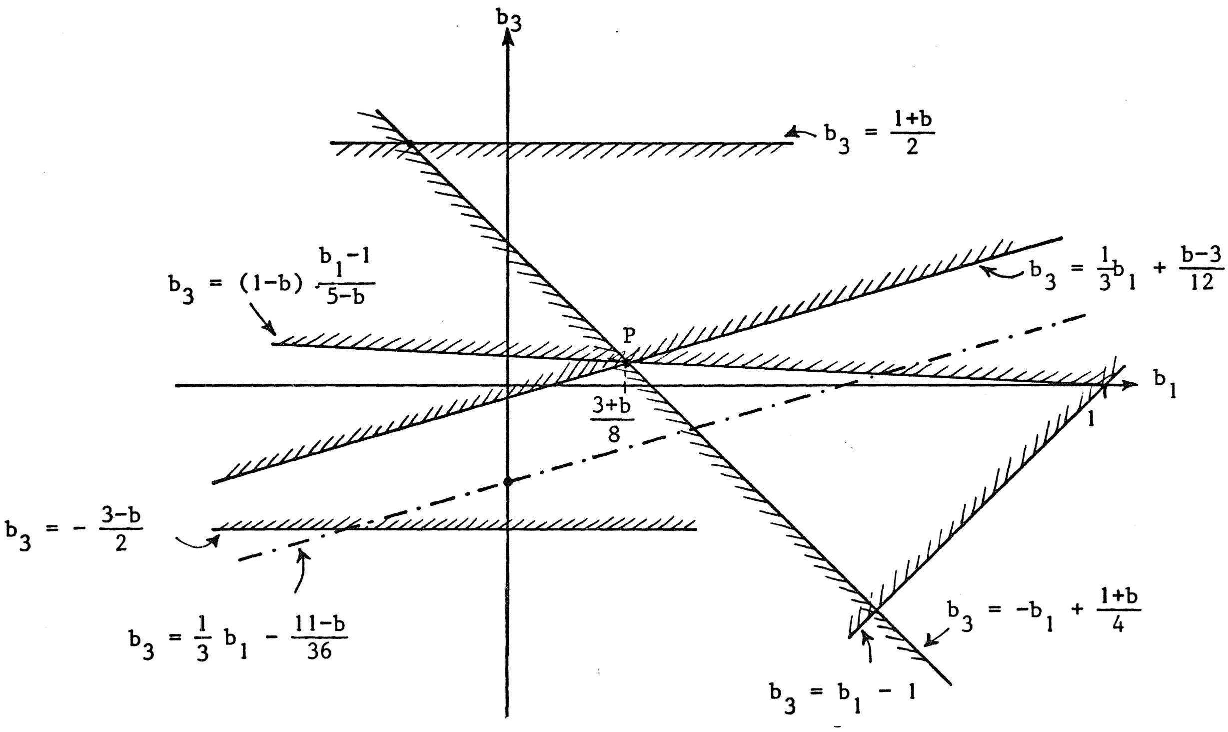

Fig. 4.1 Region of stable (b_1, b_3) - values

by

(4.3') $\quad 12b_1 - 36b_3 - 11 + b = 0,$

as may be derived from (4.3) by substituting $b_2 = \frac{1}{2}(b - b_1 - 3b_3)$. From this figure it may be concluded that *no third order formulas exist* *which are unconditionally stable.*

Let us consider more closely the class of *second order formulas*

(4.9)
$$\vec{y}_{n+1} = \frac{5-b}{2}\vec{y}_n + (b-2)\vec{y}_{n-1} + \frac{1-b}{2}\vec{y}_{n-2} + $$
$$+ \frac{h^2}{2}[(1-b_1+b_3)\vec{f}_{n+1} + 2b_1\vec{f}_n + (b-b_1-3b_3)\vec{f}_{n-1} + $$
$$+ 2b_3\vec{f}_{n-2}],$$

where b_1 and b_2 are assumed to lie in the stability region just established. The freedom left in the parameters b_1 and b_3 can be exploited to minimize the truncation error. According to Henrici [1962]one should consider the normalized error constant.

(4.10)
$$\tilde{C}_4 = \frac{C_4}{b_0+b_1+b_2+b_3} = \frac{\frac{1}{24}[-11+b+12b_1-36b_3]}{\frac{1}{2}(1+b)}.$$

The value of C_4 is minimized by choosing (b_1,b_3) in the neighbourhood of the point $P = (\frac{3+b}{8}, \frac{b-1}{8})$ to obtain

(4.10')
$$\tilde{C}_4 = \simeq \frac{1}{6}$$

for all values of b. Let us choose

$$b = 1-\varepsilon,\ b_1 = \frac{1-\varepsilon}{2},\ b_3 = 0,\ 0 < \varepsilon < 2.$$

It is readily verified that (b_1,b_3) lies within the stability region and coincides with P as $\varepsilon \to 0$. Formula (4.9) becomes

(4.11)
$$\vec{y}_{n+1} = \frac{4+\varepsilon}{2}\vec{y}_n - (1+\varepsilon)\vec{y}_{n-1} + \frac{1}{2}\varepsilon\vec{y}_{n-2} + \frac{1}{4}h^2(1+\varepsilon)\vec{f}_{n+1} + $$
$$+ \frac{1}{2}(1-\varepsilon)h^2\vec{f}_n + \frac{1}{4}(1-\varepsilon)h^2\vec{f}_{n-1},$$

with the error constant

(4.12) $$\tilde{C}_4 = -\frac{1}{6}\,\frac{1+\frac{7}{4}\varepsilon}{1-\frac{1}{2}\varepsilon}\,.$$

The special case $\varepsilon = 1$ is of interest because of its reduced storage requirements; the error constant $\tilde{C}_4$, however, is relatively large ($\tilde{C}_4 = -11/12$).

Having reduced the magnitude of the error constant, we investigate the roots of the characteristic equation of scheme (4.11) for large values of $|z|$. According to (4.6), the characteristic function then is approximately given by

(4.6') $$C(\zeta,z) \cong -\frac{1}{4}\left[(1+\varepsilon)z\zeta^3 + 2(1-\varepsilon)z\zeta^2 + (1-\varepsilon)z\zeta + 2\varepsilon\right].$$

For $z \to -\infty$ and $\varepsilon \to 0$ this function has the roots

(4.13) $$\zeta_{1,2}(z) = -1 + \frac{1}{3}\,\frac{z-1}{z}\varepsilon,\ \zeta_3(z) = -\frac{2\varepsilon}{(1-\varepsilon)z}$$

For $z \to -\infty$ and $\varepsilon = 1$, we have

(4.14) $$C(\zeta,z) \cong -\frac{1}{2}z\zeta^3 - \frac{5}{2}\zeta^2 + 2\zeta - \frac{1}{2}$$

Evidently, the roots $\zeta_j(z)$ satisfy the condition

(4.15) $$|\zeta_j(z)|^3 \le \frac{\frac{1}{2} + 2|\zeta_j| + \frac{5}{2}|\zeta_j^2|}{\frac{1}{2}|z|} \le \frac{10}{|z|}\,.$$

Thus, the formula

(4.11') $$\vec{y}_{n+1} = \frac{5}{2}\vec{y}_n - 2\vec{y}_{n-1} + \frac{1}{2}\vec{y}_{n-2} + \frac{1}{2}h^2\vec{f}_{n+1}$$

has a very strong damping effect on the higher harmonics at the cost of a larger error constant

(4.12') $$\tilde{C}_4 = -\frac{11}{12}\,.$$

We already observed that no third order formulas with an infinite interval of stability exist (for $k = 3$). Let us investigate how far the stability interval extends in the optimal case. To that end we substitute (4.3') into (4.7) to obtain in terms of b and b_3,

$$(4.7')\qquad \begin{aligned} &(32b-384b_3-64)z < 4(72+48b_3-24b),\\ &(1+b)z < 0,\\ &-16(1+b)z < 9b(1+b-2b_3),\\ &(1+b)(1-b+24b_3)z < 0\,. \end{aligned}$$

From these inequalities it follows that

$$(4.17)\qquad \beta = 6 \min \left\{\frac{1+b-2b_3}{1+6}, \frac{1}{2}\frac{3+2b_3-b}{2+12b_3-b}\right\},$$

provided that

$$(4.18)\qquad b_3 > \frac{b-1}{24}\,.$$

It is easily seen that β can never exceed the value 6, since the inequalities $1 + b - 2b_3 > 1 + b$, $(3+2b_3-b) > 2(2+12b_3-b)$ and $24b_3 > b - 1$ cannot be satisfied. Introducing the error constant

$$(4.19)\qquad \tilde{C}_5 = \frac{1-b+24b_3}{12(1+b)}$$

we may write

$$(4.17')\qquad \beta = 6 \text{ min. } \left\{\frac{13 - 12\tilde{C}_5 + (11-12\tilde{C}_5)b}{12(1+b)}, \frac{35 + 12\tilde{C}_5 -(11-12\tilde{C}_5)}{36 + 72\tilde{C}_5 - 12(1-12\tilde{C}_5)}\right\}$$

where (4.18) implies $\tilde{C}_5 > 0$.

Let us choose $b = 0$, then

$$(4.17'')\qquad \beta = 6 \min \left\{\frac{13 - 12C_5}{12}, \frac{35 + 12\tilde{C}_5}{36 + 72\tilde{C}_5}\right\},$$

which approximates the maximum attainable value as $\tilde{C}_5 \downarrow 0$. For example,

$C_5 = 1/12$ yields

(4.20) $$\beta = \frac{36}{7}$$

resulting in the formula

(4.21) $$\vec{y}_{n+1} = \frac{5}{2}\vec{y}_n - 2\vec{y}_{n-1} + \frac{1}{2}\vec{y}_{n-2} + \frac{1}{24}h^2\vec{f}_{n+1} + \frac{11}{12}h^2\vec{f}_n - \frac{11}{24}h^2\vec{f}_{n-1} .$$

Finally, we pay attention to *explicit* integration formulas, i.e.

(4.22) $$b_0 = 0,\ b_1 = 1 + b_3$$

Inequalities (4.7) yield the conditions (in terms of b and b_3)

$$(b-3-8b_3)z < 4(3-b+2b_3),$$

$$(1+b)z < 0,$$

$$(-b-1+8b_3)z < 4(1+b-2b_3),$$

$$b_3 z < 0.$$

Strong stability requires

$$b_3 > 0.$$

We distinguish two cases: firstly, let

$$0 < b_3 < \frac{b+1}{8}$$

then

(4.23) $$\beta = 4 \min \left\{\frac{3 - b + 2b_3}{3 - b + 8b_3}, \frac{1 + b - 2b_3}{1 + b - 8b_3}\right\}.$$

Secondly, let

$$\frac{b+1}{8} \leq b_3 < \frac{b+1}{2},$$

then

(4.23') $$\beta = 4\frac{3-b + 2b_3}{3-b + 8b_3} .$$

Clearly, β will never exceed the value 4. It turns out that this optimal value can be approximated within the class of third order formulas. To see this we express, by (4.3') and (4.22), b_3 in terms of b, i.e.

$$b_3 = \frac{1+b}{24}.$$

Thus,

(4.23") $$\beta = 4 \min \left\{\frac{37-11b}{40-8b}, \frac{11}{8}\right\} = 4\frac{37-11b}{40-8b}.$$

For $b \to -1$, β approximates the value of 4, the error constant $\tilde{C}_5$, however, becoming larger and larger. In fact we have

$$\tilde{C}_5 = \frac{\frac{1}{12}}{\frac{1+b}{2}} = \frac{1}{6(1+b)} .$$

An acceptable choice is b = 0, i.e.

(4.24) $$\beta = \frac{37}{10}, \tilde{C}_5 = \frac{1}{6}$$

and the integration formula becomes

(4.25) $$\vec{y}_{n+1} = \frac{5}{2} \vec{y}_n - 2\vec{y}_{n-1} + \frac{1}{2} \vec{y}_{n-2} + \frac{25}{24} h^2\vec{f}_n - \frac{7}{12} h^2\vec{f}_{n-1} + \frac{1}{24} h^2\vec{f}_{n-2}.$$

5. FOUR - STEP FORMULAS

In the preceding section we derived two formulas ((3.6) and (4.11')) which are respectively first and second order accurate possessing an infinite real stability interval. In this section we try to construct such a formula of third order.

The coefficients C_j are given by

$$(5.1)\qquad \begin{aligned} C_0 &= 1 - a_1 - a_2 - a_3 - a_4, \\ C_1 &= 4 - 3a_1 - 2a_2 - a_3, \\ C_2 &= 8 - \frac{9}{2}a_1 - 2a_2 - \frac{1}{2}a_3 - (b_0+b_1+b_3+b_4), \\ C_3 &= \frac{32}{3} - \frac{9}{2}a_1 - \frac{4}{3}a_2 - \frac{1}{6}a_3 - (4b_0+3b_1+2b_2+b_3), \\ C_4 &= \frac{32}{3} - \frac{27}{8}a_1 - \frac{2}{3}a_2 - \frac{1}{24}a_3 - (8b_0+\frac{9}{2}b_1+2b_2+b_3). \end{aligned}$$

The consistency conditions $C_j = 0$, $j = 0,1,\ldots,4$ are solved by

$$(5.2)\qquad \begin{aligned} a_1 &= 4 - 3b_0 - 2b_1 - b_2 + b_4, \\ a_2 &= -6 + 8b_0 + 5b_1 + 2b_2 - b_3 - 4b_4, \\ a_3 &= 4 - 7b_0 - 4b_1 - b_2 + 2b_3 + 5b_4, \\ a_4 &= -1 + 2b_0 + b_1 - b_3 - 2b_4, \end{aligned}$$

where

$$(5.3)\qquad b_0 = \frac{1}{35}(12-11b_1+b_2-5b_3-11b_4)$$

and where b_1, b_2, b_3, and b_4 are still free parameters.

In order to derive the stability region of this class of formulas we apply the Hurwitz criterion for fourth degree polynomials

$$\zeta^4 + c_1\zeta^3 + c_2\zeta^2 + c_3\zeta + c_4 = 0,$$

stating that all zeroes are within the unit circle if and only if

$$\begin{aligned} &\gamma_0 > 0,\ \gamma_1 > 0,\ \gamma_2 > 0,\ \gamma_3 > 0,\ \gamma_4 > 0, \\ &\gamma_1\gamma_2\gamma_3 - \gamma_1^2\gamma_4 - \gamma_0\gamma_3^2 > 0, \end{aligned}$$

where

$$\gamma_0 = 1 - c_1 + c_2 - c_3 + c_4 ,$$

$$\gamma_1 = 4 - 2c_1 + 2c_3 - 4c_4 ,$$

$$\gamma_2 = 6 - 2c_2 + 6c_4 ,$$

$$\gamma_3 = 4 + 2c_1 - 2c_3 -4c_4 ,$$

$$\gamma_4 = 1 + c_1 + c_2 + c_3 + c_4 .$$

In our case we find

$$\gamma_0 = (4p_1-p_2z)/(1-b_0z),$$

$$\gamma_1 = (8p_3-p_4z)/(1-b_0z),$$

$$\gamma_2 = (4p_5-p_6z)/(1-b_0z),$$

$$\gamma_3 = -p_3z/(1-b_0z),$$

$$\gamma_4 = -p_5z/(1-b_0z).$$

where

(5.5)

$$p_1 = 4 - 5b_0 - 3b_1 - b_2 + b_3 + 3b_4 ,$$

$$p_2 = b_0 - b_1 + b_2 - b_3 + b_4,$$

$$p_3 = 2b_0 + b_1 - b_3 - 2b_4,$$

$$p_4 = 2b_0 - b_1 + b_3 - 2b_4,$$

$$p_5 = b_0 + b_1 + b_2 + b_3 + b_4,$$

$$p_6 = 3b_0 - b_2 + 3b_4 .$$

Hence, we obtain for

$$(5.6)\qquad \begin{aligned} &b_0 > 0, \quad p_j > 0, \quad j = 1,\dots,6 \\ &p_2p_3^2 + (p_5p_4-2p_3p_6)p_4 < 0 \end{aligned}$$

the real stability interval

$$(5.7)\qquad \left(-\infty,\ \min\left\{0,\ 4p_3\,\frac{p_1p_3 + p_4p_5 - 2p_3p_6}{p_2p_3^2 + p_4(p_4p_5 - 2p_3p_6)}\right\}\right)$$

As an illustration we compute the stability interval of the formula generated by

$$(5.8)\qquad b_1 = b_2 = b_3 = b_4 = 0.$$

According to (5.3) and (5.5) this yields

$$b_0 = \frac{12}{35},\ p_1 = \frac{80}{35},\ p_2 = p_5 = \frac{12}{35},$$

$$p_3 = p_4 = \frac{24}{35},\ p_6 = \frac{36}{35}.$$

Evidently, inequalities (5.6) are satisfied, so that the real interval of stability becomes

$$(5.7')\qquad (-\infty, -\tfrac{5}{4}).$$

Unfortunately, this result drops out formula (5.8).

Next, we try

$$(5.9)\qquad b_1 = b_3 = b_4 = 0.$$

to obtain

$$b_0 = \frac{12+b_2}{35},\ p_1 = \frac{40(2-b_2)}{35},\ p_2 = p_5 = \frac{12(1+3b_2)}{35},$$

$$p_3 = p_4 = \frac{24}{35},\ p_6 = \frac{4(9-8b_2)}{35}.$$

Inequalities (5.6) are satisfied for

$$-\frac{1}{3} < b_2 < \frac{1}{3}.$$

The stability interval becomes

(5.7) $$\left(-\infty,\ \min\ \{0,\ \frac{48b_2 + 10}{17b_2 - 6}\}\right).$$

Thus, we have unconditional strong stability for

(5.10) $$-\frac{1}{3} < b_2 < -\frac{5}{24}\ .$$

The normalized error constant becomes

(5.11) $$\tilde{C}_5 = -\frac{5}{72}\,\frac{12+b_2}{1+3b_2}\ .$$

Finally, we consider the characteristic function of the formula generated by (5.9):

(5.12) $$C(\zeta,z) = (1-\frac{12+b_2}{35}\,z)\zeta^4 - \frac{104 - 38b_2}{35}\,\zeta^3 + $$

$$+ \frac{114-78b_2-35b_2 z}{35}\,\zeta^2 - \frac{56-42b_2}{35}\,\zeta + \frac{11-2b_2}{35}.$$

In figure 5.1 the behaviour of this function is illustrated for large values of $-z$. For $z \to -\infty$ the zeroes of $C(\zeta,z)$ converge to the points

$$\zeta = 0,\ \zeta = \pm\sqrt{\frac{-35b_2}{12+b_2}}\ .$$

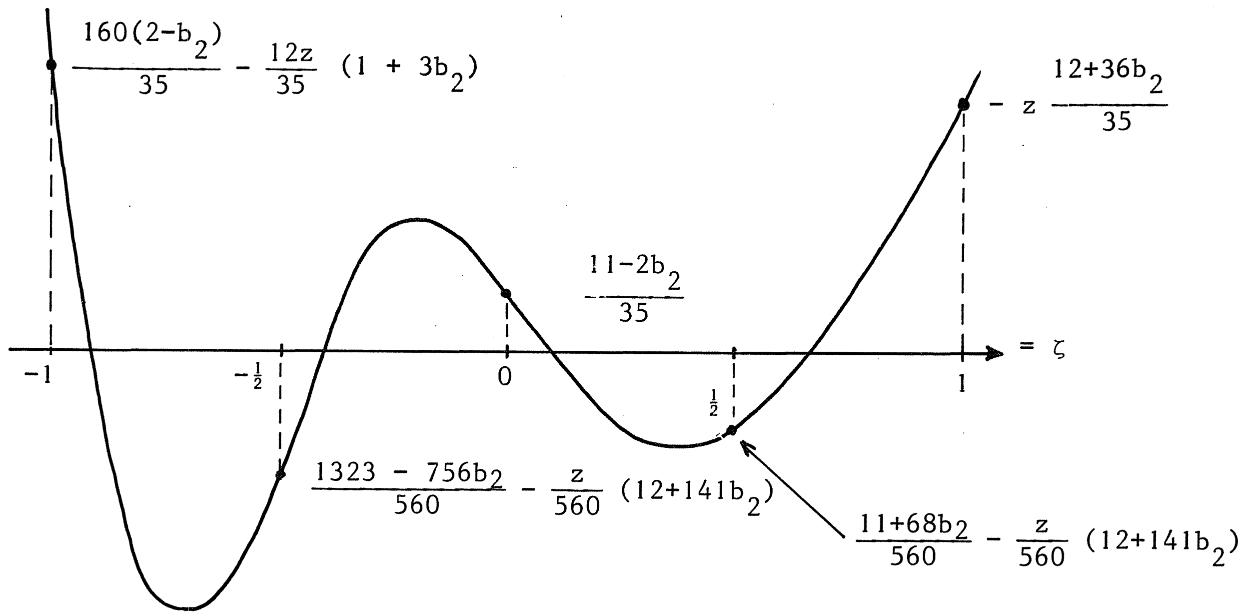

Fig.5.1. The function $C(\zeta,z)$ for large values of $-z$

Therefore, the strongest damping is obtained for $b_2 \cong -\frac{5}{24}$. Let us choose

(5.13) $$b_2 = -\frac{1}{4},$$

then

(5.14) $$\zeta_{1,2}(z) \cong 0, \quad \zeta_{3,4}(z) \cong \pm\sqrt{\frac{35}{47}} \cong \pm .86.$$

For $z \cong 0$ we may write

(5.12') $$C(\zeta,z) = (\zeta-1)^2(\zeta^2 - \frac{87}{70}\zeta + \frac{23}{70})$$

which has a double root at $\zeta = 1$ and simple roots at $\zeta \cong \frac{3}{8}$ and $\zeta \cong \frac{5}{8}$. Consequently, the condition of convergence is also satisfied. Furthermore, it may be concluded that the principal roots being located at $\zeta = 1$ for $z = 0$ are moving to $\zeta = + .86$ and $\zeta = 0$ as $z \to -\infty$, whereas the parasitic roots starting in $\zeta \cong \frac{3}{8}$ and $\zeta \cong \frac{5}{8}$ move to $\zeta = 0$ and $\zeta = \cdot 86$.

We conlcude our study of four-step methods by giving the complete integration formula for $b_2 = -1/4$:

$$\vec{y}_{n+1} = \frac{227}{70}\vec{y}_n - \frac{267}{70}\vec{y}_{n-1} + \frac{133}{70}\vec{y}_{n-2} - \frac{23}{70}\vec{y}_{n-3} + \frac{47}{140}h^2\vec{f}_{n+1} - \frac{1}{4}h^2\vec{f}_{n-1}. \tag{5.15}$$

6. SUMMARY OF RESULTS

In table 6.1 some characteristics are listed of the formulas derived in this paper.

Table 6.1 Survey of properties of some multistep methods for hyperbolic equations.

k	p	β	type	$\tilde{C}_{p+2}$	$\lvert\zeta_j\rvert$	formula
2	1	∞	implicit		$\lvert z\rvert^{-\frac{1}{2}}$ as $\lvert z\rvert\to\infty$	(3.6')
	2	∞	implicit		1	(3.6")
	4	6	implicit	$-\frac{1}{240}$	1	Numerov (3.6''')
	2	4	explicit		1	(3.7)
	1	$4(1-2\eta)$	explicit	$-\eta$	$(1+\eta z)^{\frac{1}{2}}$ as $\eta\to 0$	(3.7')
3	2	∞	implicit	$-\frac{1}{6}$	$1-\frac{1}{3}\varepsilon$ as $\lvert z\rvert\to\infty$	(4.11)
	2	∞	implicit	$-\frac{11}{12}$	$10\lvert z\rvert^{-1}$ as $\lvert z\rvert\to\infty$	(4.11')
	3	$\frac{36}{7}$	implicit	$\frac{1}{12}$		(4.21)
	3	$\frac{37}{10}$	explicit	$\frac{1}{6}$		(4.25)
4	3	∞	implicit	$-\frac{235}{72}$	$\cdot 86$ as $\lvert z\rvert\to\infty$	(5.15)

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