ma
the
ma
tisch

cen
trum

AFDELING NUMERIEKE WISKUNDE NW 44/77 SEPTEMBER
(DEPARTMENT OF NUMERICAL MATHEMATICS)

J. VAN DE LUNE & H.J.J. TE RIELE

EXPLICIT COMPUTATION OF SPECIAL ZEROS OF PARTIAL SUMS OF RIEMANN'S ZETA FUNCTION

Preprint

amsterdam 1977

stichting mathematisch centrum

MC

AFDELING NUMERIEKE WISKUNDE
(DEPARTMENT OF NUMERICAL MATHEMATICS)

NW 44/77 SEPTEMBER

J. VAN DE LUNE & H.J.J. TE RIELE

EXPLICIT COMPUTATION OF SPECIAL ZEROS OF PARTIAL SUMS OF RIEMANN'S ZETA FUNCTION

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 10F10, 10H05, 65H05

Explicit computation of special zeros of partial sums of Riemann's zeta function *)

by

J. van de Lune & H.J.J. te Riele

ABSTRACT

In this report we present two different methods for the explicit computation of zeros of the entire functions

$$\zeta_N(s) := \sum_{n=1}^{N} n^{-s}$$

in the halfplane $\mathrm{Re}(s) > 1$.

Many such (special) zeros are listed here, as far as we know, for the first time.

KEY WORDS & PHRASES: *zeros, partial sums (sections) of Riemann's zeta function, simultaneous approximation of irrational numbers.*

---

*) This report will be submitted for publication elsewhere

0. INTRODUCTION

In 1948 TURÁN [6] showed that the Riemann hypothesis for $\zeta(s)$ is true if there are positive numbers N_0 and C such that for all $N>N_0$, $N\in\mathbb{N}$,

$$\zeta_N(s) := \sum_{n=1}^{N} n^{-s}, \qquad (s\in\mathbb{C},\ s=\sigma+it)$$

has no zeros in the halfplane $\sigma \geq 1 + C/\sqrt{N}$.

In 1958 HASELGROVE [2] showed that there exist infinitely many $N\in\mathbb{N}$ such that $\zeta_N(s) = 0$ for some s with $\sigma>1$.

In 1968 SPIRA [4] proved, using a computer, that $\zeta_N(s)$ has zeros with $\sigma>1$, for N = 19,22(1)27,29(1)50. In this report we shall call zeros of $\zeta_N(s)$ with $\sigma>1$ "special zeros".

As far as we know, up till now no special zero of any $\zeta_N(s)$ is explicitly known. In this report we present two different methods for the explicit computation of special zeros of ζ_N. The first method is exhaustive, since it produces all special zeros of ζ_N with imaginary part in a given interval (sections 1, 2, 3 and 4). In the second method we first compute several "almost-periods" of ζ_N and then find special zeros of ζ_N by adding the almost-periods to zeros of ζ_N with real part very close to $\sigma=1$, but not necessarily in $\sigma>1$ (section 5). Of course, this second method is not exhaustive, but it is much less time consuming than the first one.

Finally, we present a selection of the special zeros of ζ_N for N = 19,22(1)27,29(1)35,37(1)41,47, computed by the two methods.

1. PREPARATIONS

Let $N\geq 3$ be fixed. We consider the zero-set of

$$R_N(\sigma,t) := \operatorname{Re} \zeta_N(s) = \sum_{n=1}^{N} \frac{\cos(t \log n)}{n^\sigma}$$

in the halfplane $\sigma<0$. If $R_N(\sigma_0,t_0) = 0$ then

$$-\frac{1}{N^{\sigma_0}}\cos(t_0 \log N) = \sum_{n=1}^{N-1} \frac{1}{n^{\sigma_0}} \cos(t_0 \log n)$$

so that

$$|\cos(t_0 \log N)| \le \sum_{n=1}^{N-1} (\tfrac{n}{N})^{-\sigma_0} < N \int_0^1 x^{-\sigma_0}\, dx = \frac{N}{1-\sigma_0} .$$

Now choose a small $\varepsilon>0$ ($\varepsilon = \frac{1}{N}$ is sufficient) and take $\sigma_0 < 1-N/\varepsilon$. Then we have

$$|\cos(t_0 \log N)| < \varepsilon$$

so that we must have

$$t_0 \log N \sim \frac{\pi}{2} + k\pi, \qquad (k\in\mathbb{Z})$$

or equivalently

$$t_0 \sim \frac{(2k+1)\pi}{2 \log N}, \qquad (k\in\mathbb{Z}).$$

From this it follows that the zero set of $R_N(\sigma,t)$ in the halfplane $\sigma < 1-N/\varepsilon$ consists of simple zero curves having

$$-\infty + \frac{(2k+1)\pi i}{2 \log N}, \qquad (k\in\mathbb{Z})$$

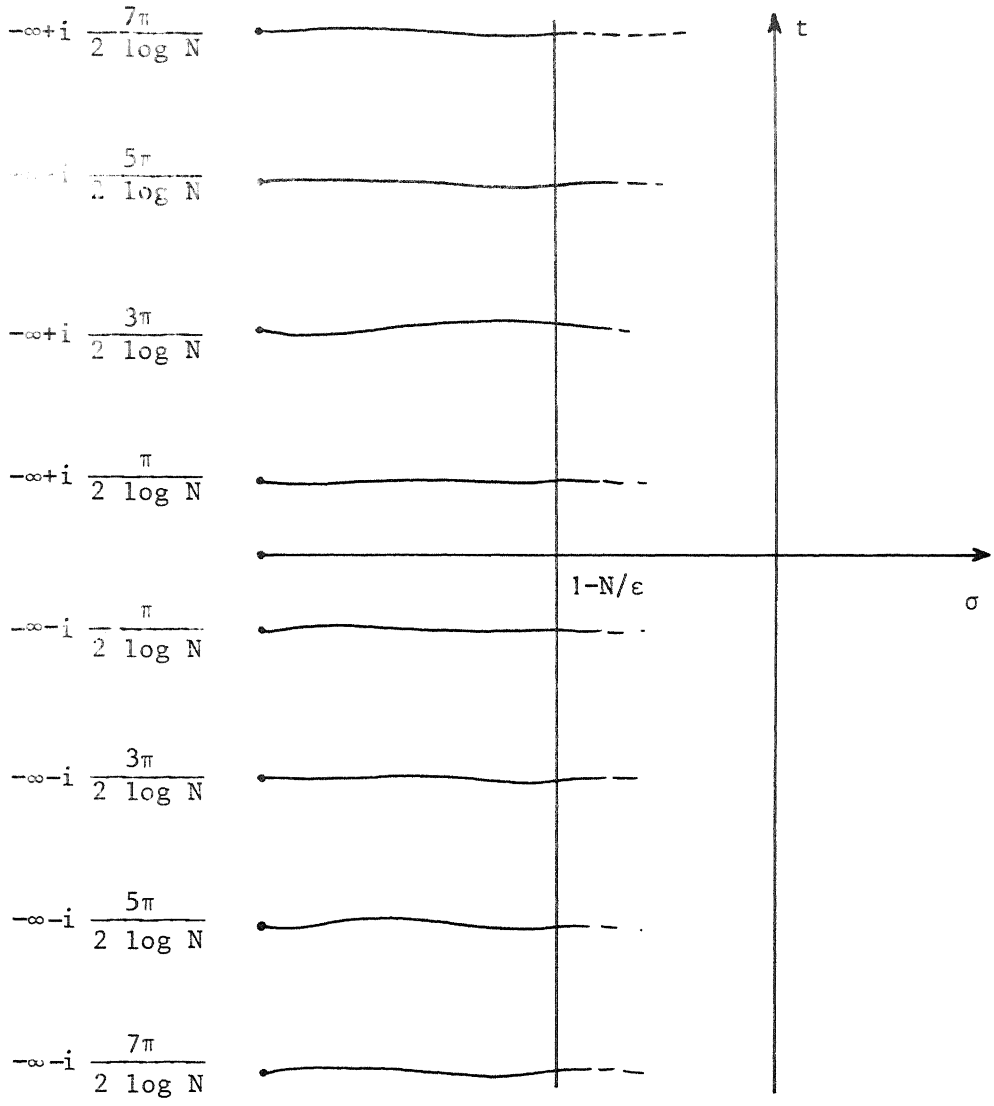
as asymptotical points. See Figure 1.

It is easy to see that

$$R_N(\sigma,t) > 0 \text{ for } \sigma \ge 2$$

so that the entire zero set of $R_N(\sigma,t)$ is contained in the halfplane $\sigma<2$. For $\sigma=1$ (or any other fixed $\sigma\in R$) we have that $R_N(1,t)$ is an almost periodic function of t and since

$$\max_{t\in\mathbb{R}} R_N(1,t) = R_N(1,0) = \sum_{n=1}^{N} \frac{1}{n}$$



$-\infty+i\,\frac{7\pi}{2\log N}$
$-\infty+i\,\frac{5\pi}{2\log N}$
$-\infty+i\,\frac{3\pi}{2\log N}$
$-\infty+i\,\frac{\pi}{2\log N}$
$-\infty-i\,\frac{\pi}{2\log N}$
$-\infty-i\,\frac{3\pi}{2\log N}$
$-\infty-i\,\frac{5\pi}{2\log N}$
$-\infty-i\,\frac{7\pi}{2\log N}$
t
$1-N/\varepsilon$
σ

Figure 1.

there exist arbitrarily large values of t for which

$$R_N(1,t) > -\varepsilon + \sum_{n=1}^{N} \frac{1}{n}$$

or equivalently

$$\sum_{n=1}^{N} \frac{1}{n} \cos(t \log n) > -\varepsilon + \sum_{n=1}^{N} \frac{1}{n} . \tag{1}$$

Choosing $\varepsilon>0$ small enough it follows that all cosines in (1) are close to 1 and hence positive so that for these particular values of t we have

$$R_N(\sigma,t) = \sum_{n=1}^{N} \frac{1}{n^\sigma} \cos(t \log n) > 0 \text{ for all } \sigma\in\mathbb{R} .$$

Since the zero lines of any harmonic function on the entire plane cannot have endpoints, it follows that a zero line of $R_N(\sigma,t)$ "starting" at a point

$$-\infty + \frac{(2k+1)\pi i}{2 \log N}$$

must return to some other asymptotical point of the same form (possibly not a neighboring one). See Figure 2.

Now we consider the zero lines of

$$I_N(\sigma,t) := \operatorname{Im} \zeta_N(s) = -\sum_{n=2}^{N} \frac{\sin(t \log n)}{n^\sigma} .$$

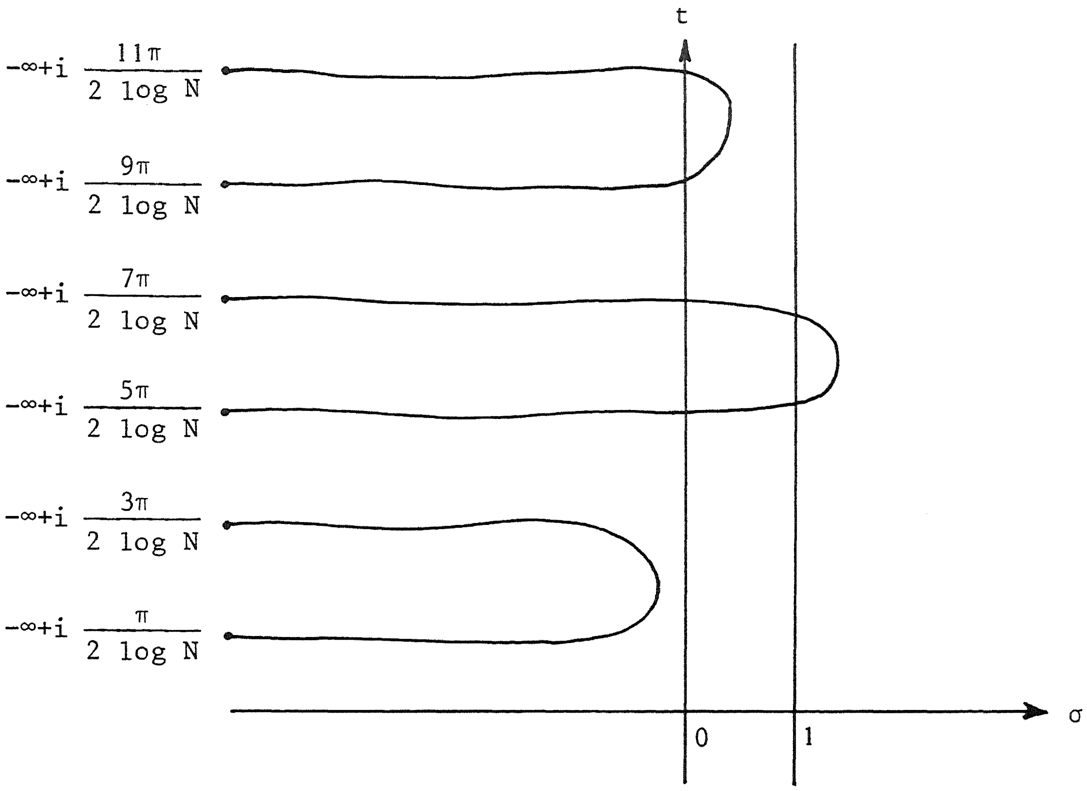
If $I_N(\sigma_0,t_0) = 0$ then

$$\frac{1}{N^{\sigma_0}} \sin(t_0 \log N) = -\sum_{n=2}^{N-1} \frac{1}{n^{\sigma_0}} \sin(t_0 \log n)$$

so that for $\sigma_0<0$

$$|\sin(t_0 \log N)| \le \sum_{n=1}^{N-1} \left(\frac{n}{N}\right)^{-\sigma_0} < \frac{N}{1-\sigma_0} .$$

Similarly as before, we choose a small $\varepsilon>0$ and take $\sigma_0 < 1-N/\varepsilon$ so that



$-\infty + i \frac{11\pi}{2 \log N}$
$-\infty + i \frac{9\pi}{2 \log N}$
$-\infty + i \frac{7\pi}{2 \log N}$
$-\infty + i \frac{5\pi}{2 \log N}$
$-\infty + i \frac{3\pi}{2 \log N}$
$-\infty + i \frac{\pi}{2 \log N}$
t
σ
0
1

<u>Figure 2.</u>

$$|\sin(t_0 \log N)| < \varepsilon.$$

Consequently

$$t_0 \log N \sim k\pi, \qquad (k\in\mathbb{Z})$$

or

$$t_0 \sim \frac{k\pi}{\log N}, \qquad (k\in\mathbb{Z}).$$

Hence, the zero set of $I_N(\sigma,t)$ in the halfplane $\sigma < 1-N/\varepsilon$ consists of a system of simple zero curves having the points

$$-\infty + \frac{k\pi i}{\log N}, \qquad (k\in\mathbb{Z})$$

as asymptotical points. See Figure 3.

For large positive σ we have in case of a zero of $I_N(\sigma,t)$

$$\frac{1}{2^{\sigma_0}} \sin(t_0 \log 2) = -\sum_{n=3}^{N} \frac{1}{n^{\sigma_0}} \sin(t_0 \log n)$$

and hence

$$|\sin(t_0 \log 2)| \leq \sum_{n=3}^{N} \left(\frac{2}{n}\right)^{\sigma_0} < N\left(\frac{2}{3}\right)^{\sigma_0}.$$

Chosing a small $\varepsilon>0$ and taking

$$\sigma_0 > \frac{\log(N/\varepsilon)}{\log(3/2)}$$

we thus have

$$|\sin(t_0 \log 2)| < \varepsilon$$

so that

$$t_0 \log 2 \sim k\pi, \qquad (k\in\mathbb{Z})$$

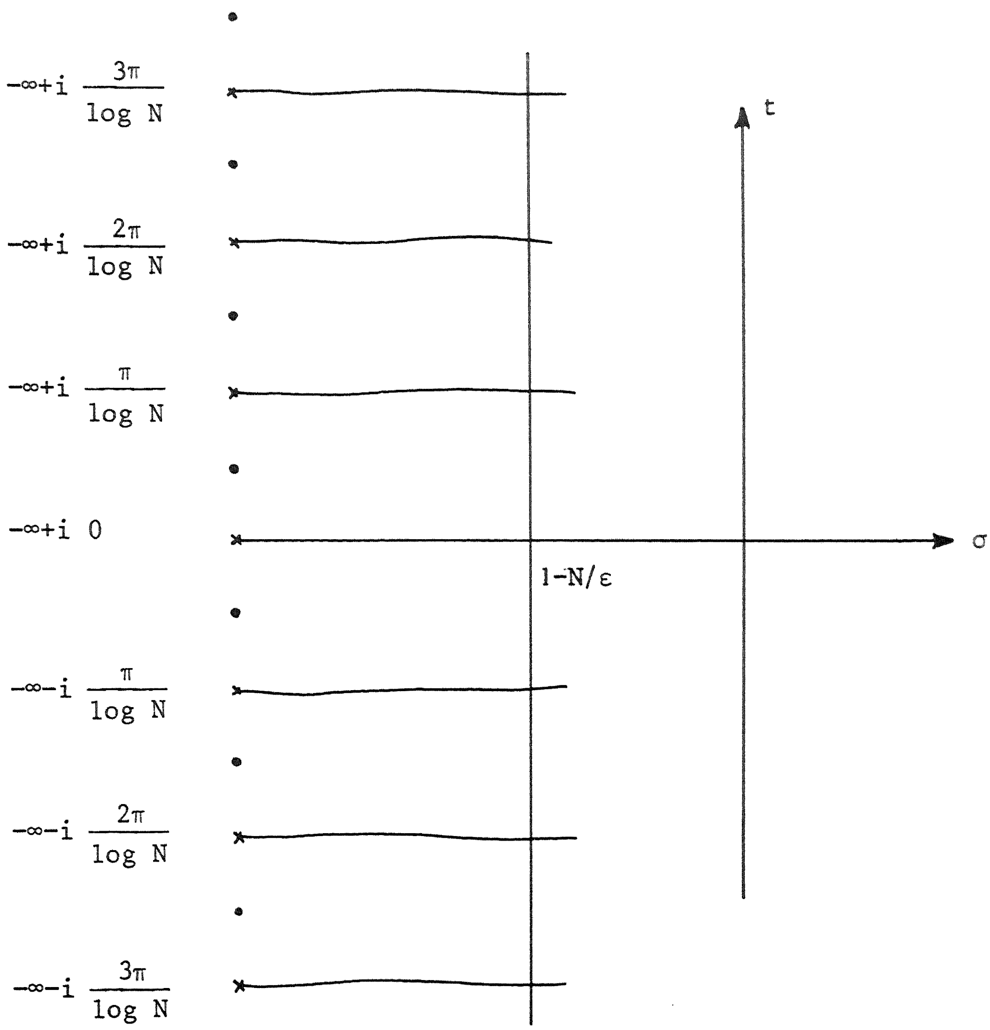

$-\infty + i \frac{3\pi}{\log N}$
$-\infty + i \frac{2\pi}{\log N}$
$-\infty + i \frac{\pi}{\log N}$
$-\infty + i\ 0$
$-\infty - i \frac{\pi}{\log N}$
$-\infty - i \frac{2\pi}{\log N}$
$-\infty - i \frac{3\pi}{\log N}$
$1-N/\varepsilon$
t
σ

Figure 3.

or equivalently

$$t_0 \sim \frac{k\pi}{\log 2}, \qquad (k \in \mathbb{Z}).$$

It follows that the zero set of $I_N(\sigma,t)$ in the halfplane $\sigma > \frac{\log(N/\varepsilon)}{\log(3/2)}$ consists of simple zero curves having

$$+\infty + \frac{k\pi i}{\log 2}, \qquad (k \in \mathbb{Z})$$

as asymptotical points. See Figure 4.

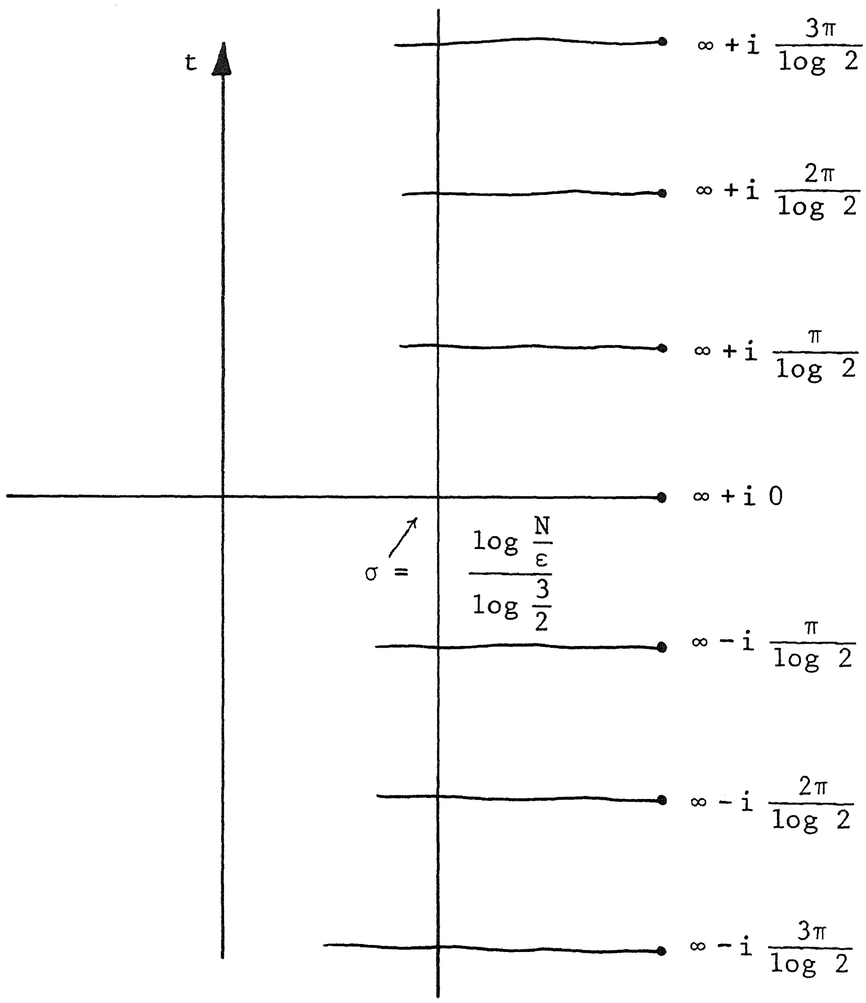


Figure 4.

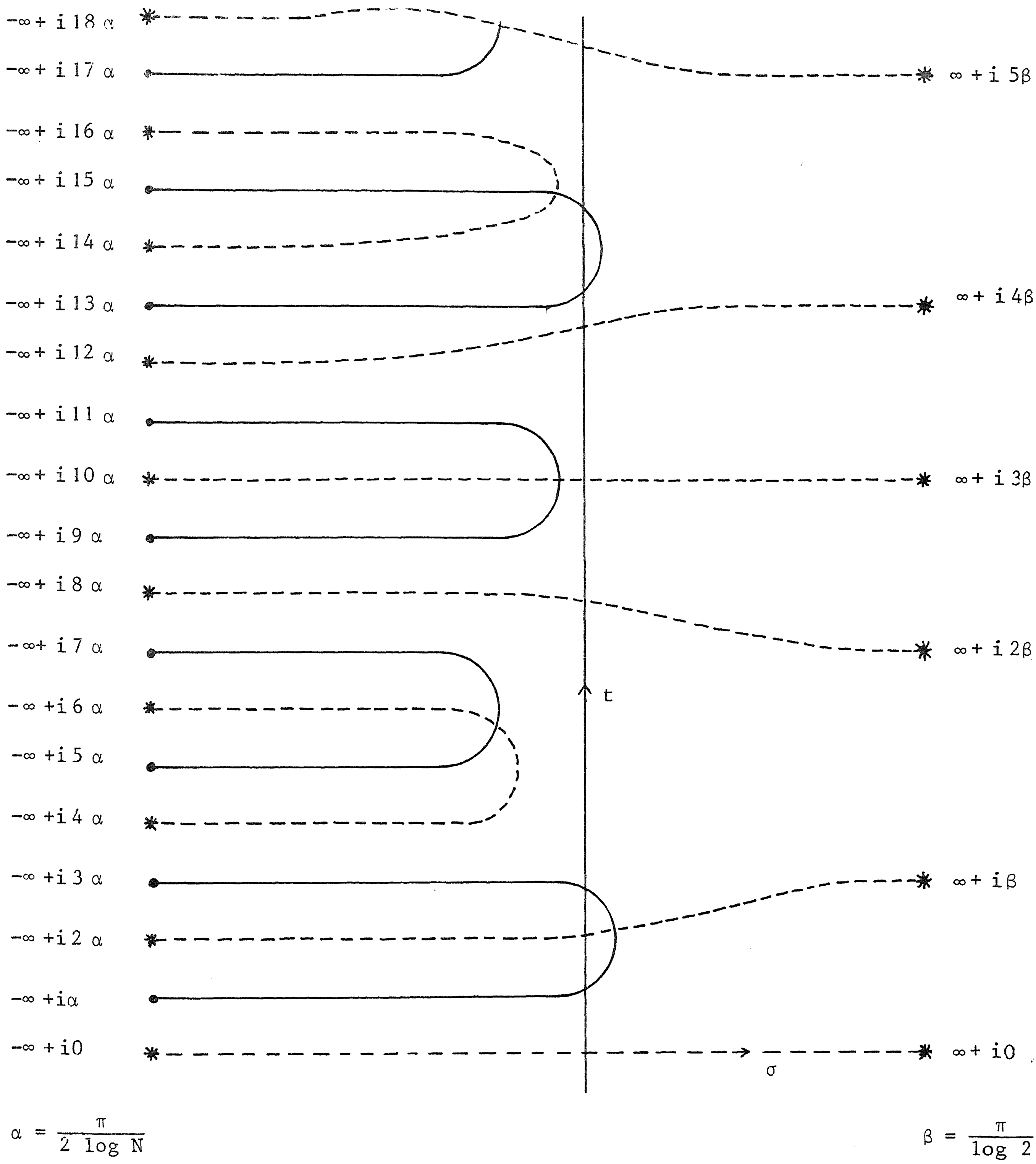

−∞ + i 18 α
−∞ + i 17 α
−∞ + i 16 α
−∞ + i 15 α
−∞ + i 14 α
−∞ + i 13 α
−∞ + i 12 α
−∞ + i 11 α
−∞ + i 10 α
−∞ + i 9 α
−∞ + i 8 α
−∞ + i 7 α
−∞ + i 6 α
−∞ + i 5 α
−∞ + i 4 α
−∞ + i 3 α
−∞ + i 2 α
−∞ + iα
−∞ + i0
∞ + i 5β
∞ + i 4β
∞ + i 3β
∞ + i 2β
∞ + iβ
∞ + i0
t
σ
α = π / (2 log N)
β = π / log 2

Figure 5.

It can be shown that every zero curve of $I_N(\sigma,t)$ starting at some asymptotical point $+\infty + k\pi i(\log 2)^{-1}$ is somehow connected with some asymptotical point $-\infty + l\pi i(\log N)^{-1}$. In other words: such a zero curve crosses over the s-plane "horizontally".

Moreover, every zero curve of $I_N(\sigma,t)$ starting at $-\infty + k_0\pi i(\log N)^{-1}$ is either connected with an asymptotical point $+\infty + l\pi i(\log 2)^{-1}$ or with an asymptotical point of the form $-\infty + m\pi i(\log N)^{-1}$.

Drawing the zero curves of $I_N(\sigma,t)$ as dotted lines, the zero curves of $I_N(\sigma,t)$ and $R_N(\sigma,t)$ have a pattern as pictured in Figure 5.

2. THE HEURISTIC PRINCIPLE

Again we denote zero curves of $I_N(\sigma,t)$ by dotted lines.

In case of a zero of $\zeta_N(s)$, we expect to have a pattern either as plotted in Figure 6a or as in Figure 6b.

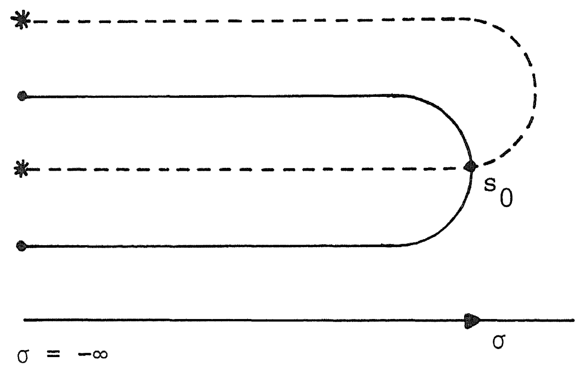


Figure 6a.

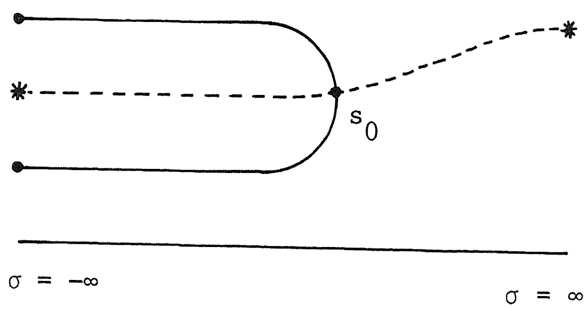


Figure 6b.

This heuristical argument is also based on the empirical observation that any zero curve of $R_N(\sigma,t)$ starting at $-\infty + \frac{(4k+1)\pi i}{2\log N}$ $(k>0)$ is connected with the "next" asymptotical point
$-\infty + \frac{(4k+3)\pi i}{2\log N}$. Hence, in order to have a *special* zero $s_0 = \sigma_0 + it_0$ of ζ_N, we expect to have a situation as plotted in Figure 7.

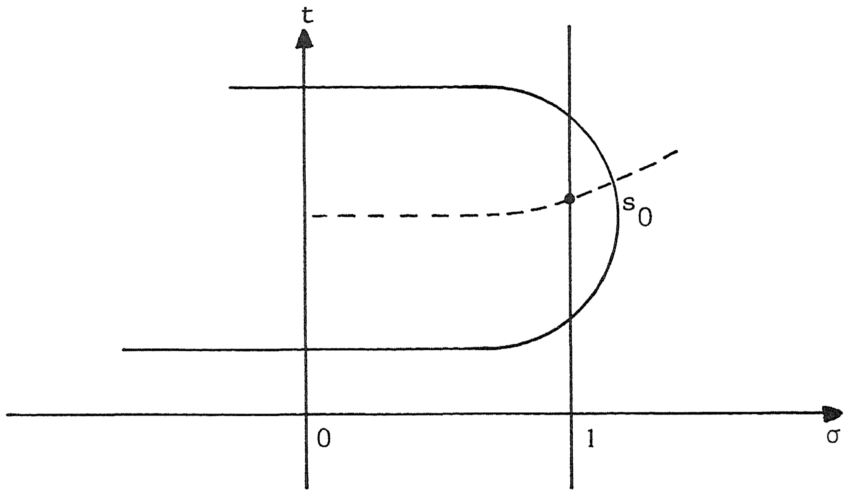


Figure 7.

In order to detect such a pattern of the zero curves of R_N and I_N one has to compute the zeros of $R_N(1,t)$ for $t>0$, yielding the increasing sequence $\{t_k\}_{k=1}^{\infty}$ of zeros $R_N(1,t)$. Once the zeros $t_{2\ell-1}$ and $t_{2\ell}$ have been located one checks whether $I_N(1,t)$ has a zero between $t_{2\ell-1}$ and $t_{2\ell}$. If so, it is a simple matter to locate the corresponding zero of $\zeta_N(s)$.

A slight modification of this procedure may be used in order to obtain zeros of ζ_N with real part just less than 1.

3. FIRST METHOD: THE SYSTEMATIC SEARCH

In this section we describe our first implementation (in FORTRAN) of the heuristical ideas for locating a special zero of $\zeta_N(s)$.

Since

$$R_N(1,t) = \sum_{n=1}^{N} \frac{1}{n} \cos(t \log n)$$

we have

$$\frac{\partial}{\partial t} R_N(1,t) = - \sum_{n=2}^{N} \frac{\log n}{n} \sin(t \log n)$$

and

$$\sup_{t \in \mathbb{R}} \left| \sum_{n=2}^{N} \frac{\log n}{n} \sin(t \log n) \right| \le \sum_{n=2}^{N} \frac{\log n}{n} =: M'_N.$$

In order to find a zero of $R_N(1,t)$ one may proceed as follows: Since $R_N(1,0) = \Sigma_{n=1}^{N} \frac{1}{n}$, we have by the maximal slope principle that $R_N(1,t)$ has no zeros on the interval $0 \le t \le R_N(1,0)/M'_N =: p_1$.

Since $R_N(1,p_1) > 0$ the same technique yields that $R_N(1,t)$ has no zeros in the interval $p_1 \le t \le p_1 + R_N(1,p_1)/M'_N =: p_2$, etc. As soon as $R_N(1,p_k) < \varepsilon$ we compute $R_N(1,p_k+\delta)$ and investigate whether $R_N(1,p_k+\delta) < 0$. In fact we took $\varepsilon=10^{-5}$ and $\delta=10^{-2}$. As soon as the first zero of $R_N(1,t)$ has been located in this way one proceeds in a similar manner starting from the point $t = p_k+\delta$. As soon as the second zero of $R_N(1,t)$ has been located one starts investigating whether $I_N(1,t)$ has a zero between these two zeros of $R_N(1,t)$. If this is the case one may draw the zero curves of R_N and I_N and find a special zero of $\zeta_N(s)$.

For N=23 this procedure leads very quickly to the special zero

$$\sigma = 1.008\ 496\ 93, \quad t = 8645.524\ 423\ 32.$$

For N=19, on a CDC 6600 computer, it took us about one hour computer time to find the special zero

$$\sigma = 1.001\ 095\ 51, \quad t = 600\ 884.203\ 427\ 78.$$

SPIRA's investigations [4] show that N=19, 22 and 23 are the first candidates for having special zeros. Clearly we wanted to see a special zero of $\zeta_{22}(s)$. Indeed, 19 and 23 are primes whereas 22 is the smallest composite N for which $\zeta_N(s)$ has special zeros.

However, neither the systematic search described above nor the acceleration of this procedure described in section 4 did produce any special zero of $\zeta_{22}(s)$ in the range $0 \le t \le 75\ 000\ 000$. Anticipating the results of section 5 we already remark here that by the method described there we have found the special zero

$$(N=22) \quad \sigma = 1.002\ 890\ 95,\ t = 558\ 159\ 406.148\ 225\ 57.$$

However, we do not know whether this special zero is the one with smallest positive imaginary part. We have given up our effort to "fill the gap" between t = 75,000,000 and t = 558,159,407 since it still might take several hundreds of hours of computer time to reach this goal.

4. ACCELERATION OF THE SYSTEMATIC SEARCH

The first thing to improve was to replace M'_N by a better (=smaller) estimate of

$$\sup_{t \in \mathbb{R}} \left| \sum_{n=2}^{N} \frac{\log n}{n} \sin(t \log n) \right| =: D_N.$$

Since

$$\sum_{n=2}^{22} \frac{\log n}{n} \sin(t \log n) = \frac{\log 2}{2} \sin(t \log 2) + \frac{\log 3}{3} \sin(t \log 3) + \frac{\log 4}{4} \sin(2t \log 2) + + \frac{\log 5}{5} \sin(t \log 5) + + \frac{\log 6}{6} \sin(t \log 2 + t \log 3) + \ldots\ldots + \frac{\log 22}{22} \sin(t \log 2 + t \log 11)$$

and since the logarithms of the primes are linearly independent over the rationals, it was possible to find the following numerical upper bound:

$$D_{22} \le 4.2725 \text{ (compare: } M'_{22} = 4.77\ldots).$$

However, it turned out that the replacement of M'_{22} by 4.2725 did not speed up the systematic search considerably.

The most time consuming thing in the systematic search is the evaluation of the transcendental functions sin(t log n) and cos(t log n).

We now describe how the systematic search can be speeded up considerabl (to about three times as fast as the original procedure). It is based on a generalization of the maximal slope principle to higher derivatives.

Observe that all derivatives of $R_N(1,t)$ are bounded:

$$|R_N^{(k)}(1,t)| \leq \sum_{n=2}^{N} \frac{(\log 2)^k}{n} =: R_N^{(k)}, \qquad k \in \mathbb{N},$$

so that by Taylor's expansion formula

$$\begin{aligned} R_N(1,t) = R_N(1,t_0) &+ \frac{(t-t_0)}{1!} R_N'(1,t_0) + \dots \\ &+ \frac{(t-t_0)^{k-1}}{(k-1)!} R_N^{(k-1)}(1,t_0) + \frac{(t-t_0)^k}{k!} R_N^{(k)}(1,\xi) \end{aligned}$$

for some $\xi \in (t_0,t)$. Hence

$$R_N(1,t) \geq \sum_{n=0}^{k-1} \frac{(t-t_0)^n}{n!} R_N^{(n)}(1,t_0) - \frac{(t-t_0)^k}{k!} R_N^{(k)}$$

and

$$R_N(1,t) \leq \sum_{n=0}^{k-1} \frac{(t-t_0)^n}{n!} R_N^{(n)}(1,t_0) + \frac{(t-t_0)^k}{k!} R_N^{(k)}$$

for all $t \geq t_0$. Writing

$$P_{1,k}(t_0,t) := \sum_{n=0}^{k-1} \frac{(t-t_0)^n}{n!} R_N^{(n)}(1,t_0) - \frac{(t-t_0)^k}{k!} R_N^{(k)}$$

and

$$P_{2,k}(t,t_0) := \sum_{n=0}^{k-1} \frac{(t-t_0)^n}{n!} R_N^{(n)}(1,t_0) + \frac{(t-t_0)^k}{k!} R_N^{(k)}$$

we clearly have that

$$P_{1,k}(t_0,t) \le R_N(1,t)$$

and

$$P_{2,k}(t_0,t) \ge R_N(1,t)$$

for all $t \ge t_0$.

From

$$P_{1,k}(t_0,t) \le R_N(1,t), \qquad (t \ge t_0)$$

and

$$D_N \ge \sup_{t \in \mathbb{R}} |R_N'(1,t)|$$

it follows that, if $R_N(1,t_0) > 0$ then $R_N(1,t)$ does not have a zero on the interval

$$t_0 \le t \le t_0 + \frac{P_{1,k}(t_0,t_0)}{D_N} =: t_1 .$$

See figure 8.

If $P_{1,k}(t_0,t_1) > \varepsilon > 0$ we can go a step further and say that $R_N(1,t)$ has no zeros on the interval

$$t_1 \le t \le t_1 + \frac{P_{1,k}(t_0,t_1)}{D_N} =: t_2$$

and so on, until one reaches a point t_r such that

$$P_{1,k}(t_0,t_r) \le \varepsilon, \qquad (\text{where } \varepsilon = 10^{-6}, \text{ say}).$$

At such an instance we compute a new polynomial $P_{1,k}(t_r,t)$. Noting that

$$P_{1,k}(t_r,t_r) = R_N(1,t_r)$$

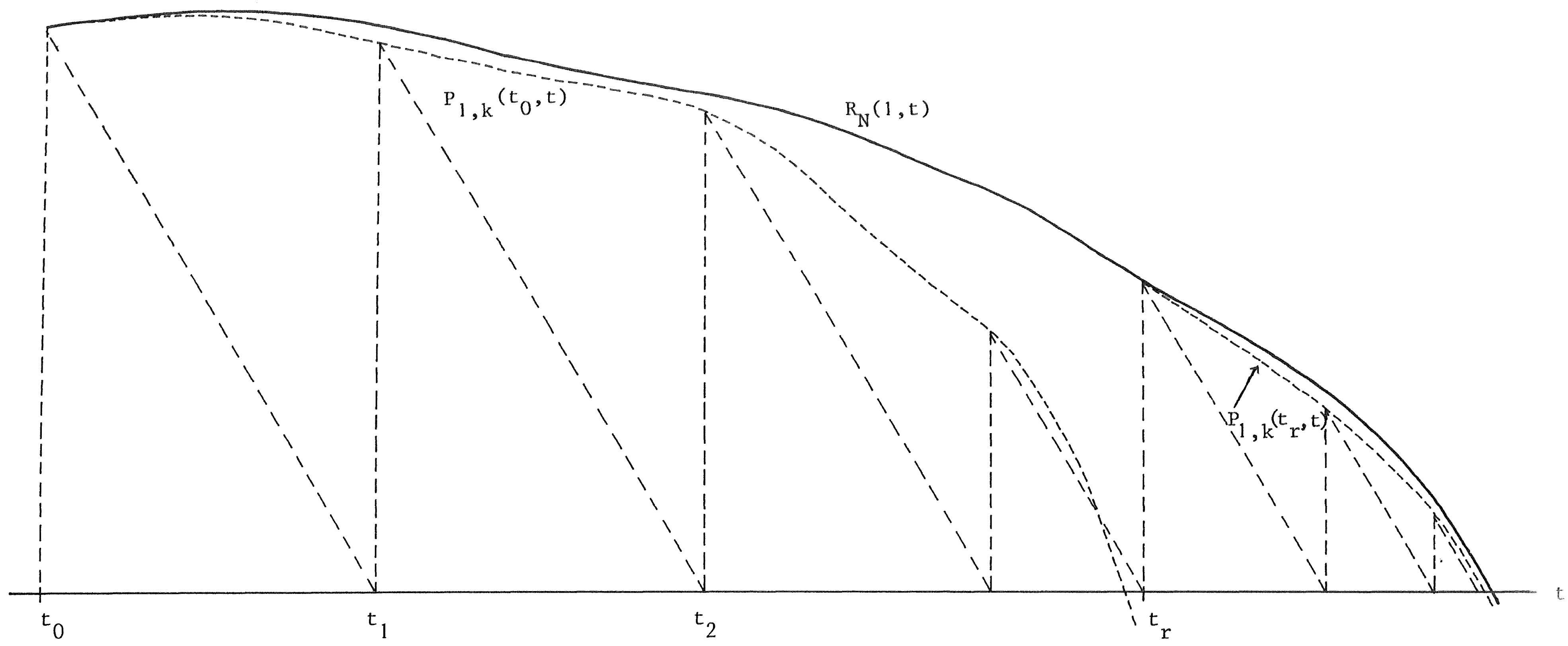


Figure 8.

we check whether $R_N(1,t_r) \le \varepsilon$. If not, we proceed with $P_{1,k}(t_r,t)$ in the same way as described above. If $R_N(1,t_r) \le \varepsilon$, we check whether $R_N(1,t_r+\delta) < 0$. If so, we compute the polynomial $P_{2,k}(t_r+\delta,t)$ and proceed similarly as above in order to determine the next zero of $R_N(1,t)$.

A similar procedure may be applied to compute the successive zeros of $I_N(1,t)$.

The advantage of the above procedure is that a considerable number of transcendental evaluations are replaced by polynomial calculations, which are performed considerably faster.

For N=22 we have tested out various values of k, resulting in the experimental observation that the total procedure was running fastest for k=14, and in fact about three times as fast as our original procedure.

5. SECOND METHOD: SEARCH BY USE OF ALMOST-PERIODS

In this section we describe a second method for the computation of special zeros of ζ_N. In fact, by this method we are able to construct (finite) sequences of zeros of ζ_N, all with real part close to one, some of them with real part *greater* than one.

The starting point is the supposition that already a zero s_0 of ζ_N is available, for which $|\mathrm{Re}\ s_0-1|$ is small. Such a zero may be found, for instance, by applying our first method to a line $\sigma = 1-\varepsilon$. Let $T_1 \in \mathbb{R}$ be such that $|\zeta_N(s) - \zeta_N(s+iT_1)|$ is small for all s on the line $\sigma=1$. Such a T_1 exists since $\zeta_N(1+it)$ is an almost-periodic function of t. Then one may expect that also $|\zeta_N(s) - \zeta_N(s_0 \pm iT_1)|$ is small, and there may be a zero, s_1 say, of ζ_N in the neighborhood of $s_0 + iT_1$. If $\mathrm{Re}\ s_1 > \mathrm{Re}\ s_0$, we look for another zero, s_2 say, of ζ_N in the neighborhood of $s_1 + iT_1$, and so on. In order to cross the line $\sigma=1$, we always demand that $\mathrm{Re}\ s_j > \mathrm{Re}\ s_{j-1}$. If $\mathrm{Re}\ s_j \le \mathrm{Re}\ s_{j-1}$ we continue with another almost-period T_2. After crossing the line $\sigma=1$ we may still continue this procedure in order to find more and more special zeros of ζ_N.

The crucial point in the above procedure is, of course, the availability of sufficiently many almost-periods of ζ_N on the line $\sigma=1$. We have

LEMMA 5.1. *Almost-periods of $\zeta_N(s)$ can be computed if one is able to find "sufficiently good" (to be specified later) approximations of the $\pi(N)(>1)$ numbers $\log p_j/\log p_{j_0}$, $(j=1,2,\ldots,\pi(N);\ j_0 \in \{1,2,\ldots,\pi(N)\})$ by rational numbers with the same denominator.*

PROOF. Let k be that common denominator, i.e., $k \log p_j/\log p_{j_0} \equiv \varepsilon_j \pmod 1$ where $\varepsilon_{j_0} = 0$ and the other ε_j's are small (but not zero, since the logarithms of the primes are independent over $\mathbb{Q}$). Let the canonical factorization of $n(\le N)$ be given by $n = \prod_{j=1}^{\pi(N)} p_j^{\alpha_j(n)}$. Then for $T := k\cdot 2\pi/\log p_{j_0}$ and for any fixed $s\in\mathbb{C}$ we have

$$\zeta_N(s+iT) = \sum_{n=1}^{N} n^{-s} \exp(-iT \log n) = \sum_{n=1}^{N} n^{-s} \exp(-i\theta_n),$$

where

$$\begin{aligned}\theta_n = T \log n &= (k\cdot 2\pi/\log p_{j_0}) \log \prod_{j=1}^{\pi(N)} p_j^{\alpha_j(n)} \\ &= 2\pi \prod_{j=1}^{\pi(N)} \alpha_j(n) k \log p_j/\log p_{j_0} \\ &\equiv \Big(\prod_{j=1}^{\pi(N)} \varepsilon_j \alpha_j(n)\Big) \pmod{2\pi}.\end{aligned}$$

If the ε_j's are small enough, we may expect the value of $\zeta_N(s+iT)$ to be close to the value of $\zeta_N(s)$, for any fixed $s\in\mathbb{C}$. Hence, T is an almost-period of ζ_N. The same argument holds, if one replaces T by $-T$. □

We have used the well-known modified Jacobi-Perron algorithm [1] and the less-known Szekeres algorithm [5] for the computation of the rational approximations of $\log p_j/\log p_{j_0}$ $(j=1,2,\ldots,\ (N);\ j\neq j_0)$. We first give a description of both algorithms in the style of KNUTH [3]. Both algorithms are simplified and put in a form suitable for our purpose.

ALGORITHM JP (Jacobi-Perron). Given $n\ge 1$ positive irrational numbers $\alpha_1,\alpha_2,\ldots,\alpha_n$. In step JP2 a positive integer k is computed such that $\{k\alpha_i\}$ is small, for $i=1,2,\ldots,n$ (where $\{x\}$ means the distance of x to the nearest integer). Auxiliary vectors $\vec{a} = (a_1,a_2,\ldots,a_n)$, $\vec{b} = (b_1,\ldots b_n)$ and $\vec{c} = (c_0,c_1,\ldots,c_n)$ are used. The algorithm terminates when k>kmax.

JP1. [Initialize]. Set $c_0 \leftarrow 0$ and set $a_i \leftarrow \alpha_i$ and $c_i \leftarrow 0$, for $i = 1,2,\ldots,n$.

JP2. [Take integer part of $\vec{a}$ and compute new k]. Set $b_i \leftarrow [a_i]$ for $i = 1,2,\ldots,n$ and set $k \leftarrow c_0 + \Sigma_{i=1}^{n} c_i b_i$. If k>kmax then stop.

JP3. [Compute new $\vec{c}$ and $\vec{a}$]. Set $c_0 \leftarrow c_1$, $c_i \leftarrow c_{i+1}$ and $a_i \leftarrow (a_{i+1}-b_{i+1})/(a_1-b_1)$, for $i = 1,2,\ldots,n-1$ and set $c_n \leftarrow k$ and $a_n \leftarrow 1/(a_1-b_1)$. Go to JP2.

Note that for n=1, this algorithm produces the denominators of the convergents of the regular continued fraction expansion of α_1.

The Szekeres algorithm is more complicated than JP, but it will appear to produce much better approximations than JP.

ALGORITHM SZ (Szekeres). Given n≥1 positive irrational numbers $\alpha_1,\alpha_2,\ldots,\alpha_n$, with $1 > \alpha_1 > \alpha_2 > \ldots > \alpha_n$. In step SZ6 a positive integer k is computed such that $\{k\alpha_i\}$ is small, for $i = 1,2,\ldots,n$. An auxiliary vector $\vec{\gamma} = (\gamma_0,\gamma_1,\ldots,\gamma_n)$, auxiliary arrays $A = (a_{ij})$, $i,j = 0,1,\ldots,n$ and $V = (v_{ij})$, $i,j = 1,2,\ldots,n$, and an auxiliary scalar h are used. The algorithm terminates, when k>kmax. In order to explain the notation in SZ3, we define a partial ordering of n-component vectors as follows: let $\vec{x} = (x_1,\ldots,x_n)$ and $\vec{y} = (y_1,\ldots,y_n)$ and let $i_1,i_2,\ldots,i_n$ be a permutation of $1,2,\ldots,n$ such that $|x_{i_1}| \geq |x_{i_2}| \geq \ldots \geq |x_{i_n}|$; similarly, let $|y_{j_1}| \geq |y_{j_2}| \geq \ldots \geq |y_{j_n}|$. We write $\vec{x} \simeq \vec{y}$ if $|x_{i_\mu}| = |y_{j_\mu}|$, for $\mu = 1,2,\ldots,n$ and $\vec{x} \prec \vec{y}$ if $\exists\nu$, $1 \leq \nu \leq n$ such that $|x_{j_\nu}| < |y_{j_\nu}|$, and $|x_{j_\mu}| = |y_{j_\mu}|$, for $1 \leq \mu < \nu$.

SZ1. [Initialize]. Set $\gamma_0 \leftarrow 1-\alpha_1$, $\gamma_i \leftarrow \alpha_i-\alpha_{i+1}$, $i = 1,2,\ldots,n-1$, $\gamma_n \leftarrow \alpha_n$. Set $a_{ij} \leftarrow 1$, $i = 0,1,\ldots,n$ and $j = 0,1,\ldots,i$ and $a_{ij} \leftarrow 0$, $i = 0,1,\ldots,n-1$ and $j = i+1, i+2,\ldots,n$.

SZ2. [Compute the differences v_{ij}]. Set $v_{ij} \leftarrow \left|\frac{a_{ij}}{a_{i0}} - \frac{a_{0j}}{a_{00}}\right|$, $i,j = 1,2,\ldots,n$.

SZ3. [Select index μ]. Let $\vec{v}_i$ be the i-th row of V, so $\vec{v}_i = (v_{i1},v_{i2},\ldots,v_{in})$. Find the largest index μ such that for every $1 \leq i \leq n$

$$\text{either} \quad \vec{v}_i \prec \vec{v}_\mu, \quad \text{or} \quad \vec{v}_i \simeq \vec{v}_\mu.$$

If $\gamma_0 < \gamma_\mu$, then go to SZ5.

SZ4. [$\gamma_0 \geq \gamma_\mu$]. Set $\gamma_0 \leftarrow \gamma_0-\gamma_\mu$ and $a_{\mu j} \leftarrow a_{\mu j}+a_{0j}$, $j = 0,1,\ldots n$. Go to SZ6.

SZ5 [$\gamma_0 < \gamma_\mu$]. Set $h \leftarrow \gamma_0$ and $\gamma_0 \leftarrow \gamma_\mu-\gamma_0$, $\gamma_\mu \leftarrow h$. Set $h \leftarrow a_{0j}$ and $a_{0j} \leftarrow a_{\mu j}$, $a_{\mu j} \leftarrow a_{\mu j}+h$, for $j = 0,1,\ldots,n$.

SZ6. [New k]. Set $k \leftarrow a_{\mu 0}$. If k≤kmax, then go to SZ2, else stop.

For n=1, this algorithm not only produces the denominators of the convergents of the regular continued fraction expansion of α_1, but also the denominators of the *intermediary* convergents.

Both algorithms were coded in FORTRAN, and run on a CDC 6600 computer, in double precision (28 significant digits) with kmax = 10^{20}, n=6 and for α_i the six irrationals log 3/log 2, log5/log 2, log7/log2, log11/log 2, log13/log 2, and log 17/log 2. Let $k_1, k_2, \ldots$ be the sequence of k's produced by one of the algorithms. Define $m_i := \max_{1 \le j \le 6} \{k_i \alpha_j\}$. In Table 1, for both algorithms we give the values of k_j and m_j, such that $m_j < m_i$, for $1 \le i \le j-1$. Clearly the results of SZ are much better than those of JP, so that we decided to choose the Szekeres algorithm for our further computations.

Table 1

Results of runs with the Jacobi-Perron Algorithm
and the Szekeres Algorithm

ALG.	j	k_j	m_j
JP	1	1	.460
	3	2	.401
	8	168	.365
	9	877	.331
	10	882	.219
	17	278575	.164
	25	1170241231	.158
	26	18158873714	.0675
	31	9176933208351	.0654
	35	259812674489863	.0349
SZ	1	2	.401
	8	4	.350
	19	9	.304
	30	31	.289
	49	311	.201
	57	764	.181
	71	2414	.139
	80	5855	.111
	83	14348	.0910
	113	88209	.0871
	116	119365	.0798
	125	272356	.0483
	149	2316275	.0276
	169	23993538	.0221
	218	890512495	.0184
	225	2039172447	.0178
	234	2929684942	.0167
	239	5312742147	.0115
	246	9640622028	.0106
	263	69123516771	.00715
	296	1903569470016	.00704
	297	2244797172219	.00615
	299	1740704456733	.00548
	300	2907809851158	.00522
	325	13059799506657	.00353
	339	61833456490027	.00344
	343	65818958118979	.00180
	392	7164194803257268	.00167
	407	38101473715080026	.00115
	419	102025501759257846	.00107
	447	1778599299350212805	.00053
	448	1485640231520813937	.00046

As indicated in section 3, we first applied our method to N=22. In order to find almost periods for N=22, we ran the SZ algorithm with N=19, i.e. $\pi(N) = 8$ and $i_0 = 1,2,3$ and 4. This yielded sufficiently many almost periods, and with the strategy described in the beginning of this section, we found many special zeros of $\zeta_{22}(s)$.

Although we already had found a few special zeros of ζ_{19} by the systematic method, we also applied the almost period method to ζ_{19}. As an illustration of the power of this method, we select the following result:

$$\zeta_{19}(s) = 0 \quad \text{for} \quad s = \sigma_0 + it_0, \text{ where}$$

$$\sigma_0 = 1.002\,793\,85, \; t_0 = 987\,047\,804\,990\,437\,138.210\,000\,67$$

and for $k = 1,2,\ldots,58$ the numbers $t_k = t_0 + kP$, where

$$P = 119\,473\,414\,699\,017\,719\,233.343\,2$$

are approximations, with absolute error of, at most, 0.1, of the imaginary parts of special zeros of ζ_{19}. These zeros are listed in Table 2 (σ rounded to 8, t to 5 decimals). We have also listed the first zero in this "almost-arithmetic progression" with real part < 1 (namely the zero with imaginary part $\approx t_0 + 59P$).

Table 2

59 special zeros of ζ_{19}, the imaginary parts of which form an "almost" arithmetic progression, and the first "non-special" zero in this progression.

σ	t
1.00279385	987047804990437138.21000
1.00287891	120460462504008156371.55227
1.00295917	239933877203025875604.89453
1.00303464	359407291902043594838.23680
1.00310532	478880706601061314071.57906
1.00317121	598354121300079033304.92133
1.00323237	717827535999096752538.26360
1.00328876	837300950698114471771.60587
1.00334038	956774365397132191004.94813
1.00338727	1076247780096149910238.29040

Table 2 (cont'd)

1.00342941	1195721194795167629471.63267
1.00346685	1315194609494185348704.97495
1.00349959	1434668024193203067938.31722
1.00352756	1554141438892220787171.65949
1.00355087	1673614853591238506405.00176
1.00356948	1793088268290256225638.34404
1.00358339	1912561682989273944871.68631
1.00359263	2032035097688291664105.02859
1.00359720	2151508512387309383338.37086
1.00359712	2270981927086327102571.71314
1.00359237	2390455341785344821805.05542
1.00358294	2509928756484362541038.39770
1.00356893	2629402171183380260271.73997
1.00355030	2748875585882397979505.08225
1.00352700	2868349000581415698738.42453
1.00349914	2987822415280433417971.76681
1.00346660	3107295829979451137205.10910
1.00342954	3226769244678468856438.45138
1.00338783	3346242659377486575671.79366
1.00334159	3465716074076504294905.13595
1.00329071	3585189488775522014138.47823
1.00323534	3704662903474539733371.82052
1.00317535	3824136318173557452605.16280
1.00311082	3943609732872575171838.50509
1.00304179	4063083147571592891071.84738
1.00296821	4182556562270610610305.18966
1.00289013	4302029976969628329538.53195
1.00280750	4421503391668646048771.87424
1.00272038	4540976806367663768005.21653
1.00262865	4660450221066681487238.55883
1.00253266	4779923635765699206471.90112
1.00243208	4899397050464716925705.24341
1.00232686	5018870465163734644938.58570
1.00221735	5138343879862752364171.92800
1.00210347	5257817294561770083405.27029
1.00198488	5377290709260787802638.61259
1.00186194	5496764123959805521871.95489
1.00173467	5616237538658823241105.29718
1.00160285	5735710953357840960338.63948
1.00146665	5855184368056858679571.98178
1.00132607	5974657782755876398805.32408
1.00118127	6094131197454894118038.66638
1.00103183	6213604612153911837272.00868
1.00087808	6333078026852929556505.35098
1.00071993	6452551441551947275738.69329
1.00055737	6572024856250964994972.03559
1.00039068	6691498270949982714205.37789
1.00021931	6810971685649000433438.72020
1.00004367	6930445100348018152672.06250
.99986388	7049918515047035871905.40481

In order to find almost periods for ζ_N, $23 \le N \le 28$, we ran the SZ algorithm with N=23, i.e. $\pi(N) = 9$, and $i_0 = 1,2,3$ and 4.

Unfortunately the SZ algorithm did not produce satisfactory results for $\pi(N) \ge 10$, unless we extended the precision of the calculations. Instead of doing this we decided to try to find zeros of ζ_N, $N \ge 29$ with the use of the almost periods found with the SZ algorithms, for the cases $\pi(N) = 8$ and $\pi(N) = 9$. This had to work, and in fact it did, by the independency of the logarithms of the primes over Q.

In Table 3 we give a selection of special zeros found with the two methods described above. σ and t are rounded to 8 decimals. All zeros with imaginary part greater than 5.10^8 were found by the method of almost periods described in this section.

REFERENCES

[1] BERNSTEIN, L., *The modified algorithm of Jacobi-Perron*, Memoirs of the Amer. Math. Soc., Number 67, 1966.

[2] HASELGROVE, C.B., *A disproof of a conjecture of Pólya*, Matematika, 5 (1958), 141-145.

[3] KNUTH, D.E., *The art of computer programming*, Vol. 1, Fundamental Algorithms, Addison-Wesley, 1968.

[4] SPIRA, R., *Zeros of sections of the zeta function, II*, Math. Comp., 22 (1968), 163-173.

[5] SZEKERES, G., *Multidimensional continued fractions*, Ann. Univ. Sc. Budapest. de Rolando Eötvös nom., 13 (1970), 113-140.

[6] TURAN, P., *On some approximative Dirichlet polynomials in the theory of the zeta function of Riemann*, Danske Vid. Selsk. Mat. Fys. Medd., 24 (1948), 3-36.

Table 3

A selection of special zeros of ζ_N, N = 19,22(1)27,29(1)35,37(1)41,47, computed with the systematic and with the almost period method

N	σ	t
19	1.00109551	600884.20342778
19	1.00235653	11771253.22839263
22	1.00289095	558159406.14822557
22	1.00159434	46892766540.42816696
23	1.00849693	8645.52442332
23	1.00519091	938296.18122556
23	1.00010041	2330124.70064096
23	1.00006953	3202110.39681165
23	1.00103136	3277066.40576762
23	1.00721589	3946708.69254419
23	1.01126485	4547478.18108028
23	1.00571315	4893650.03988065
23	1.00019718	5629488.54597714
23	1.00113166	6164062.17543663
23	1.00256708	7815899.06171757
23	1.00165130	8007793.91903903
23	1.01044335	8502832.39912066
23	1.01168877	9432483.05547926
23	1.00193093	9584842.76629013
23	1.00829376	11771253.27977385
23	1.00913875	13387837.27431388
23	1.00408121	16794145.94826183
23	1.00288075	18540790.53294455
23	1.00152197	19811202.31452277
23	1.00141400	20749500.16765432
23	1.00076491	22343785.04497516
23	1.00859454	23079623.19611120
23	1.00376614	26882617.70286760
23	1.01267753	27034977.40765425
23	1.00069855	27981919.11520594
23	1.00483371	29252330.88830235
23	1.00348478	29750694.85030826
23	1.00604019	30837971.91770344
23	1.00396132	31096062.63391930
23	1.00378926	31591101.11935353
23	1.01338428	32520751.77163493
23	1.00033024	33055325.40544247
23	1.00216134	33207685.11072094
23	1.00219355	33553689.36071613
23	1.00068643	34859521.99944206
23	1.01064524	34899057.14427724
23	1.00808078	36323746.28414194
23	1.01861685	36476105.99181750
23	1.00544284	38244881.72222851
23	1.00517487	39590249.50342533
23	1.00246299	39744526.87338768
23	1.00905381	40279100.50245581
23	1.00933119	41014938.64597675
23	1.00355840	41498998.65643527
23	1.00069483	42047876.23596304

Table 3 (cont'd)

23	1.00636263	43393244.01834429
23	1.01243966	44970292.86562675
23	1.00180913	45301993.16371658
23	1.00069052	45454352.87687934
23	1.00355256	47686010.30077727
23	1.00159992	48238713.74929047
23	1.00050265	53926240.63493770
23	1.00248352	57987325.85676374
23	1.00604546	59564374.70585854
23	1.00175842	61333150.43688606
23	1.01226660	66481512.73165926
23	1.00528559	75922641.31586465
23	1.00809725	198275746.89594875
23	1.00997921	221364015.61065165
23	1.00257039	307680947.42369788
23	1.01718912	558159406.13576644
23	1.00015407	1206746410567.01135674
23	1.00325336	1206750949399.66901277
24	1.00404187	32520751.78599510
24	1.00356213	36476106.00198972
24	1.00266176	558159406.14677888
25	1.00044920	32520751.80223907
25	1.00281451	1948209609528.90258422
25	1.00290925	2417014270341.99476594
25	1.00042574	19875494142569090677.75149100
26	1.00147172	3202110.43537085
26	1.00172491	9432483.09742690
26	1.00014747	27034977.36446369
26	1.00121135	31096062.59302785
26	1.00515827	32520751.81725186
26	1.00105189	34899057.10041968
26	1.00635285	36323746.32695248
26	1.00260254	39590249.46365969
26	1.00042865	41014938.68527968
26	1.00246238	66481512.68792064
26	1.00008033	198275746.84905529
26	1.00080101	221364015.56587153
27	1.00041028	6124205416040893 8.59965064
27	1.00014698	61876989689005520.81033424
27	1.00003079	364399200006758011.70965177
29	1.00370506	25891589773524 18.11781520
29	1.00263365	3162664354156986 8.61843369
29	1.00285421	206325152546206301.92606158
29	1.00516811	5478708916576279669.14757267
29	1.00247602	168005639371162389355.3563667
30	1.00035753	2589158977352418.10546556

Table 3 (cont'd)

30	1.00134674	3161854562023728.21687853
30	1.00091143	1236709808364235551.51367576
31	1.00710369	52331955.65876128
31	1.01237852	2589158977352418.10678941
31	1.01173696	3161854562023728.20489056
31	1.01213846	31626643541569868.60340243
31	1.00697816	206325152546206301.91152722
31	1.00938716	5478708916576279669.13509577
31	1.00654906	168005639371162389355.3484815
32	1.00165867	2589158977352418.10218851
32	1.00092022	3161854562023728.20489056
32	1.00064974	31626643541569868.59995286
33	1.00311308	2589158977352418.09084140
33	1.00006912	31626643541569868.58813015
33	1.00006291	5478708916576279669.12056897
34	1.00224271	2589158977352418.07991295
34	1.00244777	3161854562023728.18212929
34	1.00231563	31626643541569868.57704514
34	1.00429911	206325152546206301.88684313
34	1.00568156	5478708916576279669.10985066
35	1.00271904	2589158977352418.06938499
35	1.00546689	3161854562023728.17185247
35	1.00632459	31626643541569868.56710359
35	1.00306822	206325152546206301.87536194
35	1.00382418	5478708916576279669.09848015
37	1.00386526	2589158977352418.06806263
37	1.00373874	206325152546206301.86968026
37	1.00343310	5478708916576279669.09860138
38	1.00612140	2589158977352418.05885220
38	1.01062213	206325152546206301.86203972
38	1.00963417	5478708916576279669.09024589
39	1.00801942	2589158977352418.04998790
39	1.01207617	206325152546206301.85241258
39	1.01045689	5478708916576279669.08040409
40	1.00138033	2589158977352418.04412159
40	1.00341149	206325152546206301.84834351
40	1.00152700	5478708916576279669.07653356
41	1.00099738	2589158977352418.05290762
41	1.00386682	206325152546206301.83891350
47	1.00039216	20749499.96408269