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ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS WHEN NO GOOD APPROXIMATION TO THE SOLUTION IS AVAILABLE

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Iterative methods for solving nonlinear equations when no good approximation to the solution is available

by

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ABSTRACT

A class of iterative methods for solving nonlinear systems of equations is constructed. The methods are based on the solution (by A-stable integration techniques) of an initial value problem which is related to the nonlinear problem to be solved. The objective of these methods is to solve nonlinear problems when no good initial approximation to the unknown solution is available. At the end of the paper some numerical examples are given.

KEY WORDS & PHRASES: *numerical analysis, nonlinear equations, Newton's method*

1. INTRODUCTION

Let X be a Banach space and $F: X \to X$ a nonlinear operator. In this note we shall be concerned with iterative methods for solving the equation

$$(1.1) \qquad F(x) = 0.$$

Suppose that $x^* \in X$ is a solution of (1.1). A well-known method for approximating x^* is *Newton's method* defined by

$$(1.2) \qquad x_{k+1} = x_k - [F'(x_k)]^{-1}F(x_k) \qquad k = 0,1,\ldots .$$

In (1.2) $x_0 \in X$ is a given approximation of x^*, and $F'(x)$ denotes the Fréchet-derivative of F at x. If the starting point x_0 is remote from x^* then the sequence $\{x_k\}$ defined in (1.2) generally will not converge to x^*.

In this note we propose a class of iterative methods which may be expected to be suitable for cases where the starting point x_0 is remote from x^* also.

Consider for $x_0 \in X$ the initial value problem

$$(1.3) \qquad \begin{aligned} \dot{x}(t) &= -[F'(x(t))]^{-1}F(x_0) \qquad t \in [0,1] \\ x(0) &= x_0. \end{aligned}$$

If F satisfies some smoothness conditions (see [7]) then (1.3) has a unique solution $x(t)$, satisfying $F(x(t)) = (1-t)F(x_0)$ for all $t \in [0,1]$. Therefore $x(1)$ is a solution to the equation (1.1). It is assumed that the solution x^* we are looking for satisfies $x(1) = x^*$.

Computing the solution $x(t)$ to (1.3) at $t = 1$ by means of a given numerical integration procedure, we obtain an approximation, say $x_1 \approx x(1)$, which is uniquely determined by x_0. We thus have $x_1 = G(x_0)$, where the operator G depends only on F and the given numerical integration procedure. Solving the initial value problem (1.3) once more, with x_0 replaced by x_1, by the same numerical integration procedure, we obtain an approximation $x_2 \approx x(1)$ which is related to x_1 by $x_2 = G(x_1)$, etc.

The iterative methods to be considered in this note are all of the general type

$$(1.4) \qquad x_{k+1} = G(x_k) \qquad (k = 0,1,2,\ldots),$$

where G depends on F and on some numerical integration procedure.

If we use the most simple explicit integration procedure, Euler's rule, with steplength h = 1, method (1.4) reduces to Newton's method (1.2). When the starting point x_0 is close to x^*, with rather weak assumptions on F, the Newton iterates $\{x_k\}$ converge quadratically to x^*. This implies that although Euler's rule with stepsize h = 1 is a crude first order integration technique, a very good approximation x_1 to x(1) is obtained when integrating (1.3) for x_0 close to x^*. Furthermore we observe that if for an $x_0 \in X$ Newton's method does not generate a sequence $\{x_k\}$ that converges to x^*, we may also consider this phenomenon as a failure of Euler's rule (with stepsize h = 1) in solving (1.3). This failure is due to unstable behaviour of Euler's rule when applied to the initial value problem (1.3). These observations suggest that it may be of interest to use highly stable integration procedures for solving (1.3) instead of very accurate ones. Summarizing we may say that we would like to use integration techniques which need not be very accurate (first order accuracy being sufficient) but which prevent the numerical approximation x_1 from getting too far away from x(1). This suggests that it may be of interest to use A-stable integration techniques (cf.[6]) for solving (1.3).

The use of A-stable integration techniques in cases where Newton's method does not generate a sequence $\{x_k\}$ that converges to x^* has been a subject of earlier investigations (cf.[1]; in that paper Euler's rule was used as predictor and the Trapezoidal rule as corrector).

As far as numerical integration methods for solving initial value problems are concerned we shall use the concepts described in [6].

2. ITERATIVE METHODS THAT ARE BASED ON A-STABLE INTEGRATION TECHNIQUES

For a given $x_0 \in X$, let

$$f: X \to X,$$

(2.1)

$$f(x) = -[F'(x)]^{-1}F(x_0), \qquad \text{for all } x \in X.$$

Then (1.3) is equivalent to

$$\dot{x}(t) = f(x(t)), \qquad t \in [0,1],$$

(2.2)

$$x(0) = x_0.$$

For $q \geq 1$ and $\alpha \in \mathbb{R}$ consider the following scheme for solving (2.2)

$$y_0 = x_0,$$

(2.3)

$$y_{j+1} = y_j + \frac{1}{q}\{(1-\alpha)f(y_{j+1}) + \alpha f(y_j)\}, \qquad j = 0,1,\ldots,q-1.$$

y_j is an approximation to $x(\frac{j}{q})$, $j = 1,\ldots,q$. For each $\alpha \in [0,\frac{1}{2}]$ it can be shown that the methods used in (2.3) are A-stable. When $\alpha = \frac{1}{2}$, (2.3) reduces to the well-known *Trapezoidal rule*. This method is of second order. When $\alpha = 0$, the integration method (2.3) is called the *backward Euler method*. This method is not as accurate as the Trapezoidal rule (it is of first order) but it has better stability behaviour (cf [6], p.235).

We note that the integration method (2.3) is implicit for $\alpha \neq 1$. Suppose we have found approximations y_i for $i = 1,2,\ldots,j < q$. Then in order to compute y_{j+1}, we have to solve the (in general nonlinear) problem $H_j(z) = 0$, where

$$H_j(z) \equiv z - y_j - \frac{1}{q}\{(1-\alpha)f(z)+\alpha f(y_j)\}.$$

Instead of solving $H_j(z) = 0$ exactly (which will in general be impossible) we content ourselves with approximations z_j to y_j $(j = 1,\ldots,q)$. We might for example use an explicit integration method (e.g. Euler's rule) as predictor and use the implicit method as corrector. However in this way the good stability properties of the implicit methods are generally spoilt (see, for example [6], p.235). We shall therefore obtain $\{z_j\}$ as follows. Let $z_0 = y_0$. For $0 \le j < q$ let z_j be the approximation to y_j. Then in the expression defining $H_j(z)$ we replace y_j by z_j and z_{j+1} is the first Newton iterate for the problem $H_j(z) = 0$ with starting point z_j. (We suppose $H_j'(z_j)$ exists and is invertible). This means that

$$z_0 = x_0$$

(2.4)

$$z_{j+1} = z_j + \frac{1}{q}[I - \frac{1}{q}(1-\alpha)f'(z_j)]^{-1}f(z_j), \qquad j = 0,1,\ldots,q-1;$$

where

$$f'(z) = [F'(z)]^{-1}F''(z)[F'(z)]^{-1}F(x_0) \qquad \text{for all } z \in X.$$

Therefore, the iterative method which is based on the integration technique (2.4) has an interation function G defined by

$$G(x) = z_q(x), \qquad \text{where } x \in X \text{ and}$$

(2.5)
$$z_0(x) = x,$$

$$z_{j+1}(x) =$$

$$= z_j(x) - \frac{1}{q}\{I - \frac{1}{q}(1-\alpha)[F'(z_j(x))]^{-1}F''(z_j(x))[F'(z_j(x))]^{-1}F(x)\}^{-1}$$

$$F'(z_j(x))F(x) \quad \text{for } j = 0,1,\ldots,q-1.$$

It can be shown that if F satisfies some smoothness conditions then all methods of type (2.5) are (at least) quadratically convergent.

We notice that for $\alpha = \frac{1}{2}$ and $q = 1$ (2.5) becomes the iteration function of a method which is known in the literature as the *method of tangent hyperbolas* (cf[8], p.188 for a bibliography on this method). Furthermore, for

$\alpha = 0$ and $q = 1$ (2.5) is the iteration function of a method which has been investigated in [3]. In that paper it was supposed that $X = \mathbb{R}'$. In the computations that were performed on some problems in $\mathbb{R}'$, the method exhibited convergence behaviour better than the convergence behaviour of Newton's method, especially when the starting points were not close to the desired solution.

In higher dimensional vectorspaces computation of $F''(z)$ requires in general an exorbitant amount of work. Therefore methods with iteration function of type (2.5) are rather cumbersome from the computational point of view. In the next section we shall modify (2.5) in such a way that $F''(z_j(x))$ need not be computed.

3. ITERATIVE METHOD'S WHICH REQUIRE NO EVALUATION OF THE SECOND DERIVATIVE

In this section we shall construct a class of derative methods which are related to the iterative methods of type (2.5). However these methods do not require the evaluation of the second derivative of F.

Let $x,z \in X$. For any $\varepsilon > 0$ a $\rho > 0$ exists such that for all θ, $0 < |\theta| < \rho$,

$$\frac{1}{\theta}\| F'(z+\theta[F'(z)]^{-1}F(x)-F'(z)-\theta F''(z)[F'(z)]^{-1}F(x)\| < \varepsilon .$$

When θ is small, $\frac{1}{\theta}\{F'(z+\theta[F'(z)]^{-1}F(x))-F'(z)\}$ is therefore approximately equal to $F''(z)[F'(z)]^{-1}F(x)$. Thus we can approximate the iteration function G, defined in (2.5) by an iteration function $\tilde{G}$,

$$\tilde{G}(x) = \tilde{z}_q(x), \qquad \text{where } x \in X \text{ and}$$

$$\tilde{z}_0(x) = x, \tag{3.1}$$

$$\tilde{z}_{j+1}(x) = \tilde{z}_j(x)-\frac{1}{q}[F'(\tilde{z}_j(x))-\frac{1-\alpha}{q\theta}\{F'(\tilde{z}_j(x)+\theta[F'(\tilde{z}_j(x))]^{-1}F(x)) - F'(\tilde{z}_j(x))\}]^{-1}F(x), \qquad j = 0,1,\ldots,q-1.$$

It can be shown that if F satisfies some smoothness conditions then all methods of type (3.1) are (at least) quadratically convergent.

We notice that for $q = 1$, methods of type (3.1) have been investigated by several authors. See for example ([9],p.164), where it is assumed that $X = \mathbb{R}'$. For the case that X is an arbitrary Banach space an example is given, for instance, in [5]. In that paper the iterative method (3.1) with $\theta = \frac{1}{2}$ and $\alpha = \frac{3}{2}$ is considered. However, just as with the method of tangent hyperbolas, the main purpose of investigating these iterative methods was their convergence behaviour *near* x^*. In this note we are mainly interested in the convergence behaviour of iterative methods when the starting point x_0 is remote from x^*.

4. NUMERICAL EXAMPLES

The iterative methods (3.1) have been applied to two problems in which Newton's method fails.

We noticed previously that nonconvergence of Newton's method, starting in $x_0 \in X$, may be conceived as an instability of Euler's rule with stepsize $h = 1$, when integrating the initial value problem (1.3). At first sight, instead of our way of handling the problem by using method (2.3), (2.4) or (3.1) with $0 \le \alpha \le \frac{1}{2}$, one might hold to Euler's rule, while improving its stability behaviour by using a smaller stepsize h. Iterative methods which are based on this integration technique have an iteration function $\hat{G}$, where

$$\hat{G}(x) = \hat{z}_q(x), \qquad q \ge 1 \text{ is given, } x \in X, \text{ and}$$

$$\hat{z}_0(x) = x, \tag{4.1}$$

$$\hat{z}_{j+1}(x) = \hat{z}_j(x) - \frac{1}{q}[F'(\hat{z}_j(x))]^{-1}F(x), \qquad j = 0,1,\ldots,q-1.$$

Note that (4.1) can be obtained from (3.1) by choosing $\alpha = 1$. In order to find out which approach is best, we tested both methods of type (3.1) (with $\alpha = 0$ and $\alpha = \frac{1}{2}$) and type (4.1).

<u>PROBLEM 1</u>. This problem arises from a finite-difference approach to the two-point boundary value problem

$$\frac{d}{ds}\{s^2 \frac{d}{ds} U(s)\} - s^2 f(U(s)) = 0 \qquad 0 < s < 1$$

(4.2)

$$U'(0) = 0, \qquad U(1) = 1,$$

where $f(u) = \varepsilon^{-1} \frac{u}{u+\kappa}$ (see [4], pp.162-168).

For $n \geq 1$, ε and κ positive consider the (n+1)-dimensional problem $F(x) = 0$, where for $x = (\xi_0,\xi_1,\ldots,\xi_n)^t$ and $F(x) = (\phi_0(x),\phi_1(x),\ldots,\phi_n(x))^t$

$$\phi_0(x) = [s_{\frac{1}{2}}]^2(\xi_0-\xi_1)$$

$$\phi_j(x) = -[s_{j-\frac{1}{2}}]^2\xi_{j-1}+([s_{j-\frac{1}{2}}]^2+[s_{j+\frac{1}{2}}]^2)\xi_j-[s_{j+\frac{1}{2}}]^2\xi_{j+1}+\Delta^2[s_j]^2 f(\xi_j)$$

$$(j = 1,\ldots,n-1)$$

$$\phi_n(x) = -[s_{n-\frac{1}{2}}]^2\xi_{n-1}+([s_{n-\frac{1}{2}}]^2+[s_{n+\frac{1}{2}}]^2)\xi_n-[s_{n+\frac{1}{2}}]^2+\Delta^2[s_n]^2 f(\xi_n).$$

$$\Delta = (n+1)^{-1} \quad \text{and} \quad s_i = i.\Delta (i = \tfrac{1}{2},1,\ldots,n+\tfrac{1}{2}).$$

On physical grounds the solution of (4.2) looked for, should be positive. It can be shown that $F(x) = 0$ has a unique positive solution x^*. The starting point x_0 was chosen to be $x_0 = (\xi_0^0,\xi_1^0,\ldots,\xi_n^0)^t$, where $\xi_j^0 = (1-\varepsilon\kappa)[s_j]^2 + \varepsilon\kappa$ $(j = 0,1,\ldots,n)$. It can be shown that for the solution $x(t)$ of (1.3), $x(1) = x^*$.

The computations were performed for $n = 100$ and $\kappa = 0.1$. For ε we took $\varepsilon = 0.1$, 0.05, 0.01 and 0.001.

PROBLEM 2. As a second example we consider the problem $F(x) = 0$, where for $x = (\xi_1,\xi_2)^t$,

$$F(x) = \begin{pmatrix} \frac{1}{2}\{\sin(\xi_1\xi_2) - \frac{1}{2\pi}\xi_2 - \xi_1\} \\ \\ (1-\frac{1}{4\pi})(e^{2\xi_1}-e) + \frac{e}{\pi}\xi_2 - 2e\xi_1 \end{pmatrix},$$

which was found in [2]. The starting point is $x_0 = (0.4,3)^t$. The solution curve $x(t)$, $t \in [0,1]$ of (1.3) terminates at the solution $x^* = x(1) = (0.299449,2.83693)^t$ of $F(x) = 0$. This equation also has a solution $(0.5,\pi)^t$ and, moreover, Newton's method starting at x_0 converges to the further solution $(-0.26,0.62)^t$.

All computations were performed on a CD Cyber 73-28 computer. Any iterative process was considered to yield a sequence $\{x_k\}$ converging to a solution of the equation $F(x) = 0$ whenever for some $k \geq 1$

$$\|x_k - x_{k-1}\| \leq 10^{-6}(1+\|x_k\|) \text{ or } \|F(x_k)\| \leq 10^{-6}.$$

The norm used was the Euclidian norm. Only if a method succeeded in generating a sequence $\{x_k\}$ that converged to the desired solution x^*, the number of iteration steps, required for the stopping criterion to be satisfied is given.

Table 4.1

Method 3.1, $\theta = 10^{-4}$.

Problem 1, $\kappa = 0.1$, $n = 100$.

method		problem			
q	α	ε=0.1	ε=0.05	ε=0.01	ε=0.001
1	0	2	4	5	7
1	0.5	2	3	FAILURE	FAILURE
2	0	2	3	4	5
2	0.5	2	2	3	4
4	0	2	3	4	4
4	0.5	1	2	2	2

Table 4.2

Method 3.1, $\theta = 10^{-4}$.

Problem 2.

method		
q	α	
1	0	7
1	0.5	5
2	0	5
2	0.5	3
4	0	4
4	0.5	3

Table 4.3

Method 4.1.

Problem 1, $\kappa = 0.1$, $n = 100$.

method	problem			
q	ε=0.1	ε=0.05	ε=0.01	ε=0.001
1	3	FAILURE	FAILURE	FAILURE
2	2	4	FAILURE	FAILURE
4	2	3	4	FAILURE
8	2	3	3	FAILURE

Table 4.4

Method 4.1.

Problem 2.

method	
q	
1	FAILURE
2	FAILURE
4	4
8	3

REMARK. We have given the results of method (3.1) only for $\theta = 10^{-4}$. The computations for method (3.1) were also performed for $\theta = 10^{-3}$ and $\theta = 10^{-5}$. Apart from some minor differences in the number of iteration steps the results were the same.

CONCLUSION

The methods of type (3.1) that were tested, appear to have a convergence behaviour superior to the convergence behaviour of Newton's method (which is equivalent to (4.1) with q=1), especially when the starting point x_0 is remote from x^*. Even in comparison with iterative methods that are based on Euler's rule with small stepsizes ((4.1) with q>1) method (3.1) is found out to be better. The work per step required for method (4.1) (with $\alpha \neq 1$) is roughly twice the work required for method (3.1) - with the same $q \geq 1$. Even if we compare methods (4.1) with methods of type (3.1) that require the same amount of work per iteration step the former (especially with α=0) turn out to be better.

Among the methods of type (3.1), the ones that are based on the backward Euler method (α=0) appear to be more reliable than the ones that are based on the Trapezoidal rule ($\alpha=\frac{1}{2}$). However if small stepsizes are taken (i.e. q large) then the latter methods generate sequences that converge faster to the solution than the former ones.

We note that the practical significance of the methods (3.1) might be increased by using approximations $\partial F(z)$ to $F'(z)$ in the formulas involved (in much the same way as this is done for Newton's method, see for example [8]).

In the future we intend to investigate such approximations of the methods (3.1). We also plan to investigate the methods (3.1) in more detail, both theoretically and by more extensive numerical experiments.

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