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AFDELING NUMERIEKE WISKUNDE
(DEPARTMENT OF NUMERICAL MATHEMATICS)

NW 48/77 SEPTEMBER

P.J. VAN DER HOUWEN & H.J.J. TE RIELE

BACKWARD DIFFERENTIATION FORMULAS FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

I CONVERGENCE AND STABILITY

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Backward differentiation formulas for Volterra integral equations of the second kind

I Convergence and Stability

by

P.J. van der Houwen & H.J.J. te Riele

ABSTRACT

In this report backward differentiation formulas are studied for nonlinear systems of Volterra integral equations of the second kind. Consistency, convergence and, in particular, stability are investigated. For a standard class of model equations, the formulas have the same stability properties as the well-known Curtiss-Hirschfelder formulas for ordinary initial value problems. For a much wider class of model equations the stability regions of the formulas are derived.

KEY WORDS & PHRASES: *Volterra integral equations, backward differentiation formula, stability*

CONTENTS

1. INTRODUCTION

In this paper we study backward differentiation formulas for Volterra integral equations of the second kind. These formulas are analogues of the well-known Curtiss-Hirschfelder formulas for first order ordinary initial value problems.

For the standard class of model integral equations the stability analysis of the formulas reduces to that of the Curtiss-Hirschfelder formulas which are known to have excellent stability properties. We present a stability analysis for a much wider class of model equations; for this class, the stability properties of the formulas turn out to be very satisfactory. The most important difference between the two classes of model equations is demonstrated by differentiating them with respect to the independent variable. The standard class of integral equations then reduces to ordinary differential equations, whereas in our case again a class of integral equations is obtained. Numerical experiments to test the stability theory will be published in the near future.

It should be remarked that DE HOOG and WEISS [3] have given an implicit block-Runge-Kutta method which has stability properties comparable with our formulas.BAKER and KEECH [1] have analysed the stability of this Runge-Kutta type method but only for the standard class of model equations.

We believe that our (multistep) method is much cheaper than that of De Hoog and Weiss, but this assertion will be approved in the numerical experiments.

In this paper existence and uniqueness of a solution of the Volterra integral equation to be considered will be assumed. Moreover, the kernel of the equation will be assumed to exist in a strip *outside* the usual domain of definition.

2. DERIVATION OF THE COMPUTATIONAL SCHEME

Suppose we are given the system of nonlinear integral equations

$$\vec{f}(x) = \vec{g}(x) + \int_{x_0}^{x} \vec{K}(x,\xi,\vec{f}(\xi))d\xi, \tag{2.1}$$

where $\vec{g}(x)$ and $\vec{K}(x,\xi,f)$ are prescribed vector functions which are sufficiently differentiable with respect to x. By differentiation this equation can be transformed into the integro-differential equation

$$\vec{f}'(x) - \vec{g}'(x) - \int_{x_0}^{x} \vec{K}'(x,\xi,\vec{f}(\xi))d\xi = \vec{K}(x,x,\vec{f}(x)), \tag{2.2}$$

where $\vec{f}'$, $\vec{g}'$ and $\vec{K}'$ denote the derivatives with respect to x of the functions $\vec{f}$, $\vec{g}$ and $\vec{K}$. Let $x_n = x_0 + nh$, $n = 0,1,\ldots.$, denote reference points on the x-axis, h being the integration step. Let us consider the relation

$$\vec{f}'(x_{n+1}) - \vec{g}'(x_{n+1}) - \int_{x_0}^{x_{n+1}} \vec{K}'(x_{n+1},\xi,\vec{f}(\xi))d\xi = \vec{K}(x_{n+1},x_{n+1},\vec{f}(x_{n+1})). \tag{2.2'}$$

By replacing the derivatives in the left hand side by a *backward differentiation formula* on the reference points x_n, we obtain a scheme of the form

$$\begin{aligned}
&[\vec{f}_{n+1} - \vec{g}(x_{n+1}) - \int_{x_0}^{x_{n+1}} \vec{K}(x_{n+1},\xi,\vec{f}(\xi))d\xi] \\
&\quad - \sum_{\ell=1}^{k} a_\ell [\vec{f}_{n+1-\ell} - \vec{g}(x_{n+1-\ell}) - \int_{x_0}^{x_{n+1}} \vec{K}(x_{n+1-\ell},\xi,\vec{f}(\xi))d\xi] = \\
&\quad = b_0\, h\, \vec{K}(x_{n+1},x_{n+1},\vec{f}_{n+1}),
\end{aligned} \tag{2.3}$$

where $\vec{f}_{n+1-\ell}$ denotes a numerical approximation to $\vec{f}(x_{n+1-\ell})$ and where the coefficients a_ℓ and b_0 define the differentiation formula used. Scheme (2.3) is, in fact, identical to the well-known Curtiss-Hirschfelder formula applied to the "differential equation" (2.2). The values of a_ℓ and b_0 are listed in table 2.1 (cf. e.g. [5, p.242]).

Table 2.1. Coefficients of backward differentiation formulas

k	b_0	a_1	a_2	a_3	a_4	a_5	a_6
1	1	1					
2	2/3	4/3	-1/3				
3	6/11	18/11	-9/11	2/11			
4	12/25	48/25	-36/25	16/25	-3/25		
5	60/137	300/137	-300/137	200/137	-75/137	12/137	
6	60/147	360/147	-450/147	400/147	-225/147	72/147	-10/147

By defining the function

$$\vec{F}_{n+1}(x) = \vec{g}(x) + \int_{x_0}^{x_{n+1}} \vec{K}(x,\xi,\vec{f}(\xi))d\xi, \tag{2.4}$$

we may write (2.3) in a more compact form:

$$\vec{f}_{n+1} = \sum_{\ell=1}^{k} a_\ell \vec{f}_{n+1-\ell} + b_0 h \vec{K}(x_{n+1},x_{n+1},\vec{f}_{n+1}) + \tag{2.3'}$$

$$+ \vec{F}_{n+1}(x_{n+1}) - \sum_{\ell=1}^{k} a_\ell \vec{F}_{n+1}(x_{n+1-\ell}).$$

Notice that this formula reduces to the familiar Curtiss-Hirschfelder formula for ordinary differential equations when $\vec{K}$ and $\vec{g}$ do not depend on x.

In order to give a step-by-step formula for $\vec{f}_{n+1}$, we have to specify the formula for approximating $\vec{F}_{n+1}(x)$. Let us define the quadrature rule

$$\vec{F}_{n+1}(x) \cong \tilde{\vec{F}}_{n+1}(x) = \vec{g}(x) + \sum_{j=0}^{n+1} w_{n+1j} \vec{K}(x,x_j,\vec{f}_j), \quad n = 0,1,\ldots, \tag{2.5}$$

where w_{n+1j} are weights satisfying the condition

$$w_{n+1j} = w_{nj}, \quad j = 0,1,\ldots,n-k. \tag{2.6}$$

Then, we have the recurrence relation

$$(2.7)\qquad \tilde{\vec{F}}_{n+1}(x) = \tilde{\vec{F}}_n(x) + \sum_{j=n+1-k}^{n} (w_{n+1j}-w_{nj})\vec{K}(x,x_j,\vec{f}_j) + w_{n+1n+1}\,\vec{K}(x,x_{n+1},\vec{f}_{n+1}).$$

From (2.3'), (2.5) and (2.7) the following computational scheme can be derived

$$(2.8)\quad \begin{cases} \vec{f}_{n+1} = \tilde{\vec{F}}_{n+1}(x_{n+1}) + b_0\, h\vec{K}(x_{n+1},x_{n+1},\vec{f}_{n+1}) + \\ \qquad + \sum_{\ell=1}^{k} a_\ell\, [\vec{f}_{n+1-\ell} - \tilde{\vec{F}}_{n+1}(x_{n+1-\ell})], \\ \tilde{\vec{F}}_{n+1}(x_{n+1-\ell}) = \tilde{\vec{F}}_n(x_{n+1-\ell}) + \sum_{j=n+1-k}^{n} \nabla w_{n+1j}\, \vec{K}(x_{n+1-\ell},x_j,\vec{f}_j) + \\ \qquad + w_{n+1n+1}\, \vec{K}(x_{n+1-\ell},x_{n+1},\vec{f}_{n+1}), \quad \ell = 1,2,\ldots,k, \\ \tilde{\vec{F}}_{n+1}(x_{n+1}) = \vec{g}(x_{n+1}) + \sum_{j=0}^{n} w_{n+1j}\, \vec{K}(x_{n+1},x_j,\vec{f}_j) + \\ \qquad + w_{n+1n+1}\, \vec{K}(x_{n+1},x_{n+1},\vec{f}_{n+1}), \end{cases}$$

where

$$\nabla w_{n+1j} = w_{n+1j} - w_{nj}.$$

In the n-th step, this scheme requires the evaluation of $\vec{g}(x_{n+1})$, the evaluation of $\vec{K}(x_{n+1},x_0,\vec{f}_0),\ldots,\ \vec{K}(x_{n+1},x_n,\vec{f}_n)$, the solution of a system of equations for $\vec{f}_{n+1}$, and, finally, the evaluation of $\vec{K}(x_{n-k+2},x_{n+1},\vec{f}_{n+1}),\ldots\ ,\vec{K}(x_{n+1},x_{n+1},\vec{f}_{n+1})$. Notice that some values of $\vec{K}(x,\xi,\vec{f})$ are needed for $x < \xi$. If the kernel is not defined there, its values should be approximated by extrapolation.

In order to start this scheme, we must precompute (approximations of) the quantities

$$\vec{f}_j, \quad \text{for} \quad j = 0,1,\ldots,k-1,$$

$$\vec{K}(x_i,x_j,\vec{f}_j), \quad \text{for} \quad i,j = 0,1,\ldots,k-1, \quad \text{and}$$

$$\tilde{F}_{k-1}(x_j), \quad \text{for} \quad j = 0,1,\ldots,k-1.$$

It may be interesting to compare scheme (2.8) with the multistep methods usually considered for equations of type (2.1). In our notation these methods read

$$(2.9) \qquad \vec{f}_{n+1} = \vec{g}(x_{n+1}) + \sum_{j=0}^{n+1} w_{n+1j}\, \vec{K}(x_{n+1},x_j,\vec{f}_j) = \vec{\tilde{F}}_{n+1}(x_{n+1}).$$

Thus, scheme (2.8) contains much more information from the "past", i.e., the vectors $\vec{f}_j$ and $\vec{\tilde{F}}_{n+1}(x_j)$, $j \le n$, than scheme (2.9).

For the quadrature formula (2.5) we choose the Gregory formula, which is given by (see e.g. [6])

$$(2.10) \qquad \int_{x_0}^{x_m} \varphi(\xi)d\xi \approx h(\tfrac{1}{2}\varphi_0+\varphi_1+\ldots+\varphi_{m-1}+\tfrac{1}{2}\varphi_m) +$$

$$- h[c_1(\nabla\varphi_m - \Delta\varphi_0) + c_2(\nabla^2\varphi_m + \Delta^2\varphi_0) + \ldots$$

$$+ c_r(\nabla^r\varphi_m + (-1)^r\Delta^r\varphi_0)],$$

where $m \ge r$. The first four values of c_r are given by

$$c_1 = \frac{1}{12}, \quad c_2 = \frac{1}{24}, \quad c_3 = \frac{19}{720}, \quad c_4 = \frac{3}{160}.$$

The error in the approximation (2.10) is of the form $O(h^{r+2})$ (see [6]). For future reference, we give the matrices $W = (w_{ij})$ for r = 0, 1, 2, 3 and 4.

<u>Second order formula</u> (r=0)

$$
(2.11) \qquad W = \frac{h}{2}\begin{bmatrix}
1 & 1 & & & \bigcirc & \\
1 & 2 & 1 & & & \\
. & 2 & 2 & 1 & & \\
. & . & . & . & & \\
. & . & . & & . & \\
. & . & . & & & . \\
1 & 2 & 2\ldots & 2 & & 1
\end{bmatrix};
$$

<u>Third order formula</u> (r=1)

$$
(2.12) \qquad W = \frac{h}{12}\begin{bmatrix}
6 & 6 & & & & & & \\
5 & 14 & 5 & & & \bigcirc & & \\
5 & 13 & 13 & 5 & & & & \\
. & 13 & 12 & 13 & 5 & & & \\
. & . & 12 & 12 & 13 & 5 & & \\
. & . & . & . & & . & & \\
. & . & . & . & & & . & \\
. & . & . & . & & & & . \\
5 & 13 & 12 & 12\ldots 12 & & 13 & & 5
\end{bmatrix};
$$

<u>Fourth order formula</u> (r=2)

$$
(2.13) \qquad W = \frac{h}{24}\begin{bmatrix}
8 & 32 & 8 & & & & & & \\
9 & 27 & 27 & 9 & & & \bigcirc & & \\
9 & 28 & 22 & 28 & 9 & & & & \\
. & 28 & 23 & 23 & 28 & 9 & & & \\
. & . & 23 & 24 & 23 & 28 & 9 & & \\
. & . & . & 24 & 24 & 23 & 28 & 9 & \\
. & . & . & . & . & & & . & \\
. & . & . & . & . & & & & . \\
. & . & . & . & . & & & & & . \\
9 & 28 & 23 & 24 & 24\ldots & 24 & 23 & 28 & 9
\end{bmatrix};
$$

Fifth order formula (r = 3)

(2.14) $$\frac{720}{h} W =$$

$$\begin{bmatrix}
270 & 810 & 810 & 270 & & & & & & & \\
251 & 916 & 546 & 916 & 251 & & & \bigcirc & & & \\
251 & 897 & 652 & 652 & 897 & 251 & & & & & \\
\cdot & 897 & 633 & 758 & 633 & 897 & 251 & & & & \\
\cdot & \cdot & 633 & 739 & 739 & 633 & 897 & 251 & & & \\
\cdot & \cdot & \cdot & 739 & 720 & 739 & 633 & 897 & 251 & & \\
\cdot & \cdot & \cdot & \cdot & 720 & 720 & 739 & 633 & 897 & 251 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \ddots & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\
251 & 897 & 633 & 739 & 720 & 720 & \cdots & 720 & 739 & 633 & 897 \;\; 251
\end{bmatrix};$$

Sixth order formula (r = 4)

(2.15) $$\frac{1440}{h} W =$$

$$\begin{bmatrix}
448 & 2048 & 768 & 2048 & 448 & & & & & & & & & \\
475 & 1875 & 1250 & 1250 & 1875 & 475 & & & & \bigcirc & & & & \\
475 & 1902 & 1077 & 1732 & 1077 & 1902 & 475 & & & & & & & \\
\cdot & 1902 & 1104 & 1559 & 1559 & 1104 & 1902 & 475 & & & & & & \\
\cdot & \cdot & 1104 & 1586 & 1386 & 1586 & 1104 & 1902 & 475 & & & & & \\
\cdot & \cdot & \cdot & 1586 & 1413 & 1413 & 1586 & 1104 & 1902 & 475 & & & & \\
\cdot & \cdot & \cdot & \cdot & 1413 & 1440 & 1413 & 1586 & 1104 & 1902 & 475 & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1440 & 1440 & 1413 & 1586 & 1104 & 1902 & 475 & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \ddots & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & \\
475 & 1902 & 1104 & 1586 & 1413 & 1440 & 1440 & \cdots & 1440 & 1413 & 1586 & 1104 & 1902 & 475
\end{bmatrix}.$$

3. CONSISTENCY AND CONVERGENCE

It is well known that a k-step backward differentiation formula is of order k as $h \to 0$. Hence, formula (2.3') has a local error of $O(h^{k+1})$, so that the following relation holds for the exact solution $\vec{f}(x)$:

$$(3.1) \qquad \vec{f}(x_{n+1}) = \vec{F}_{n+1}(x_{n+1}) + b_0 h\vec{K}(x_{n+1},x_{n+1},\vec{f}(x_{n+1})) + + \sum_{\ell=1}^{k} a_\ell[\vec{f}(x_{n+1-\ell}) - \vec{F}_{n+1}(x_{n+1-\ell})] + O(h^{k+1}).$$

Furthermore, let the weights w_{nj} define a quadrature rule with error of order $q + 1$, i.e.

$$(3.2) \qquad \vec{F}_{n+1}(x) = \vec{g}(x) + \sum_{j=0}^{n+1} w_{n+1j}\, \vec{K}(x,x_j,\vec{f}(x_j)) + C_{n+1}(x,h)h^{q+1},$$

where $C_{n+1}(x,h)$ is a bounded function as $h \to 0$.

The local error of the actual scheme (2.8) is now easily derived. Assuming that the "localizing" condition

$$\vec{f}_j = \vec{f}(x_j) \quad \text{for} \quad j = 0,1,\ldots,n$$

is satisfied we may derive from (2.8)

$$\begin{aligned}\vec{f}_{n+1} = \vec{g}(x_{n+1}) + \sum_{j=0}^{n} w_{n+1j}\vec{K}(x_{n+1},x_j,\vec{f}(x_j)) + w_{n+1n+1}\vec{K}(x_{n+1},x_{n+1},\vec{f}_{n+1}) \\ + b_0 h\vec{K}(x_{n+1},x_{n+1},\vec{f}_{n+1}) + \sum_{\ell=1}^{k} a_\ell\, \vec{f}(x_{n+1-\ell}) + \\ - \sum_{\ell=1}^{k} a_\ell[\vec{g}(x_{n+1-\ell}) + \sum_{j=0}^{n} w_{n+1j}\vec{K}(x_{n+1-\ell},x_j,\vec{f}(x_j))] + \\ - \sum_{\ell=1}^{k} a_\ell\, w_{n+1n+1}\, \vec{K}(x_{n+1-\ell},x_{n+1},\vec{f}_{n+1}).\end{aligned}$$

Using relation (3.2) yields

$$\vec{f}_{n+1} = \vec{F}_{n+1}(x_{n+1}) + w_{n+1n+1}[\vec{K}(x_{n+1},x_{n+1},\vec{f}_{n+1}) - \vec{K}(x_{n+1},x_{n+1},\vec{f}(x_{n+1})]$$

$$+ b_0 h\,\vec{K}(x_{n+1},x_{n+1},\vec{f}(x_{n+1})) + b_0 h[\vec{K}(x_{n+1},x_{n+1},\vec{f}_{n+1}) - \vec{K}(x_{n+1},x_{n+1},\vec{f}(x_{n+1})]$$

$$+ \sum_{\ell=1}^{k} a_\ell[\vec{f}(x_{n+1-\ell}) - \vec{F}_{n+1}(x_{n+1-\ell})] +$$

$$- \sum_{\ell=1}^{k} a_\ell\, w_{n+1n+1}[\vec{K}(x_{n+1-\ell},x_{n+1},\vec{f}_{n+1}) - \vec{K}(x_{n+1-\ell},x_{n+1},\vec{f}(x_{n+1}))] +$$

$$- [C_{n+1}(x_{n+1},h) - \sum_{\ell=1}^{k} a_\ell\, C_{n+1}(x_{n+1-\ell},h)]h^{q+1}.$$

Finally, by virtue of (3.1) and assuming that the function $C_{n+1}(x,h)$ satisfies a Lipschitz condition with respect to x, we obtain for the local error the relation

$$(3.3)\quad \vec{f}_{n+1} - \vec{f}(x_{n+1}) = (w_{n+1n+1}+b_0 h)[\vec{K}(x_{n+1},x_{n+1},\vec{f}_{n+1}) - \vec{K}(x_{n+1},x_{n+1},\vec{f}(x_{n+1}))]$$

$$- \sum_{\ell=1}^{k} a_\ell\, w_{n+1n+1}[\vec{K}(x_{n+1-\ell},x_{n+1},\vec{f}_{n+1}) - \vec{K}(x_{n+1-\ell},x_{n+1},\vec{f}(x_{n+1}))]$$

$$+ O(h^{k+1}) + O(h^{q+2}).$$

In order to derive an estimate for the local error from this formula, we assume that $\vec{K}$ satisfies a Lipschitz condition of the form

$$\|\vec{K}(x_j,x_{n+1},\vec{u}) - \vec{K}(x_j,x_{n+1},\vec{v})\| \le L_{jn+1}\, \|\vec{u} - \vec{v}\|$$

with respect to some norm $\|.\|$. From (3.3) we then obtain the inequality

$$(3.4)\quad \|\vec{f}_{n+1}-\vec{f}(x_{n+1})\| \le L\|\vec{f}_{n+1}-\vec{f}(x_{n+1})\| + O(h^{k+1}) + O(h^{q+2}),$$

where

$$L = |w_{n+1n+1} + b_0 h|\, L_{n+1n+1} + |w_{n+1n+1}| \sum_{\ell=1}^{k} |a_\ell|\, L_{n+1-\ell\, n+1}.$$

Since $L = O(h)$ as $h \to 0$, the inequality $L < 1$ holds for sufficiently small values of h. Hence,

$$(3.4') \qquad \|\vec{f}_{n+1} - \vec{f}(x_{n+1})\| \leq O(h^{k+1}) + O(h^{q+2}) \quad \text{as } h \to 0.$$

This implies that scheme (2.8) has a global error of order p where

$$(3.5) \qquad p = \min(k,\ q + 1).$$

A consequence of this result is that one should combine a k-step backward differentiation formula with a quadrature rule for $\vec{F}_{n+1}(x)$ of order $q = k-1$, i.e. a quadrature formula with error $O(h^k)$.

4. STABILITY

4.1. The variational equation

The stability analysis of integration formulas for second kind Volterra equations is usually based on the analysis of the numerical scheme when applied to equations with the kernel function (cf. [1])

$$(4.1) \qquad \vec{K}(x,\xi,\vec{f}) = \alpha\vec{f}, \quad \alpha \text{ some constant.}$$

In our case of formula (2.8), such a stability analysis is very simple: since the kernel function (4.1) does not depend on x it follows that

$$\tilde{\vec{F}}_{n+1}(x_j) = \tilde{\vec{F}}_{n+1}(x_{n+1}) + \vec{g}(x_j) - \vec{g}(x_{n+1})$$

for all values of j; hence the stability analysis of scheme (2.8) reduces to that of a Curtiss-Hirschfelder formula of which the stability behaviour is extensively investigated (see e.g. [2]) and which is known to have excellent stability properties.

A more general class of kernel functions was investigated in [4] for a class of Runge-Kutta type formulas and the class of multistep formulas of the form (2.9). These kernel functions satisfy the conditions

$$\frac{\partial\vec{K}}{\partial\vec{f}}(x,\xi,\vec{f}) \cong \frac{\partial\vec{K}}{\partial\vec{f}}(x,x_n,\vec{f}_n)$$

for $(\xi,\vec{f})$ in the neighbourhood of $(x_n,\vec{f}_n)$ and

$$\frac{\partial^2\vec{K}}{\partial x\partial\vec{f}}(x,x_j,\vec{f}_j) \cong \frac{\partial^2\vec{K}}{\partial x\partial\vec{f}}(x_n,x_j,\vec{f}_j), \quad j = 0,1,\ldots,n$$

for x in the neighbourhood of x_n. In other words, the function $\vec{K}$ is required to behave more or less as

(4.2) $\qquad \vec{K}(x,\xi,\vec{f}) = (\alpha+\beta x)\vec{f}, \quad \alpha$ and β constant.

In this paper we investigate a still larger class of kernel functions, i.e. we will consider functions of the type

(4.3) $\qquad \vec{K}(x,\xi,\vec{f}) = \vec{A}(\xi,\vec{f}) + x\,H\,\vec{f},$

where $\vec{A}$ is an arbitrary vector function and H an arbitrary matrix. Substitution of (4.3) into (2.5) yields

$$\vec{\tilde{F}}_{n+1}(x) = \vec{g}(x) + \sum_{j=0}^{n+1} w_{n+1j}\,[\vec{A}(x_j,\vec{f}_j) + x\,H\,\vec{f}_j]$$

so that scheme (2.8) becomes

(4.4)
$$\begin{aligned}\vec{f}_{n+1} = \vec{g}(x_{n+1}) - \sum_{\ell=1}^{k} a_\ell\,\vec{g}(x_{n+1-\ell}) + \sum_{j=0}^{n+1} w_{n+1j}\,x_{n+1}\,H\vec{f}_j\,+ \\ + b_0\,h[\vec{A}(x_{n+1},\vec{f}_{n+1}) + x_{n+1}\,H\,\vec{f}_{n+1}] + \\ + \sum_{\ell=1}^{k} a_\ell\,[\vec{f}_{n+1-\ell} - \sum_{j=0}^{n+1} w_{n+1j}\,x_{n+1-\ell}\,H\,\vec{f}_j].\end{aligned}$$

Now suppose that the vectors $\vec{f}_j$, $j = 0,1,\ldots,n$ are perturbed by perturbations $\Delta\vec{f}_j$ *) , then (4.4) yields for the resulting perturbation $\Delta\vec{f}_{n+1}$ of $\vec{f}_{n+1}$ the formula (provided that $\Delta\vec{f}_j$ is sufficiently small)

*) In this section the meaning of Δ differs from that in formula (2.10) .

$$\Delta\vec{f}_{n+1} = H \sum_{j=0}^{n+1} w_{n+1j} [x_{n+1} - \sum_{\ell=1}^{k} a_\ell x_{n+1-\ell}]\Delta\vec{f}_j +$$

$$+ b_0 h[\frac{\partial\vec{A}}{\partial\vec{f}} (x_{n+1},\vec{f}_{n+1}) + x_{n+1} H]\Delta\vec{f}_{n+1} +$$

$$+ \sum_{\ell=1}^{k} a_\ell \Delta\vec{f}_{n+1-\ell} =$$

$$= b_0 hH \sum_{j=0}^{n+1} w_{n+1j} \Delta\vec{f}_j +$$

$$+ b_0 h \frac{\partial\vec{K}}{\partial\vec{f}} (x_{n+1},x_{n+1},\vec{f}_{n+1}) \Delta\vec{f}_{n+1} +$$

$$+ \sum_{\ell=1}^{k} a_\ell \Delta\vec{f}_{n+1-\ell},$$

where we used the relation $\sum_{\ell=1}^{k} \ell a_\ell = b_0$.
Defining the quantities

$$J_{n+1} = \frac{\partial\vec{K}}{\partial\vec{f}} (x_{n+1},x_{n+1},f_{n+1}),$$

(4.6)

$$\Delta\vec{S}_{n+1} = \sum_{j=0}^{n+1} w_{n+1j} \Delta\vec{f}_j,$$

and observing that

$$\Delta\vec{S}_{n+1} = \Delta\vec{S}_n + \sum_{j=n-k+1}^{n} \nabla w_{n+1j}\Delta\vec{f}_j + w_{n+1n+1}\Delta\vec{f}_{n+1}, \tag{4.7}$$

we can represent (4.5) and (4.7) in the form

$$A_n \Delta\vec{V}_{n+1} = B_n \Delta\vec{V}_n, \tag{4.5'}$$

where

$$\Delta\vec{V}_n = (\Delta\vec{S}_n,\Delta\vec{f}_n,\Delta\vec{f}_{n-1}, \dots ,\Delta\vec{f}_{n-k+1})^T$$

and where A_n and B_n are the matrices

$$A_n = \begin{bmatrix} I & -w_{n+1n+1}I & 0 & & \bigcirc \\ -b_0 h\, H & I-b_0 hJ_{n+1} & 0 & & \\ 0 & 0 & I & & \\ & \bigcirc & & \ddots & \\ & & & & I \end{bmatrix},$$

$$B_n = \begin{bmatrix} I & \nabla w_{n+1n}I & \nabla w_{n+1n-1}I & \cdots & \nabla w_{n+1n+1-k}I \\ 0 & a_1 I & a_2 I & \cdots & a_k I \\ 0 & I & & \bigcirc & \\ & \bigcirc & \ddots & & \\ & & & I & 0 \end{bmatrix}.$$

The vector of perturbations $\Delta\vec{V}_{n+1}$ remains bounded in some norm $\|.\|$ when

(4.8) $\quad \|A_n^{-1}\, B_n\| \leq 1.$

A necessary condition to satisfy this inequality is the requirement that all eigenvalues ζ of $A_n^{-1}B_n$ are within or on the unit circle, i.e. the condition that the roots of the characteristic equation

(4.9) $\quad \det(B_n - \zeta A_n) = 0$

are within or on the unit circle.

4.2. Results for scalar integral equations

In this section we derive some results for scalar integral equations. The matrices A_n and B_n then are $(k+1)*(k+1)$ matrices and can be analyzed without restrictions on J_{n+1} and H.

The analysis of vector integral equations is considerably more difficult and is still subject of investigation.

Due to the special form of the matrices A_n and B_n, it is possible to work out (4.9) as follows. We write $a_i = a_i'/a$ for $i = 1,2,\ldots,k$, where a_i' are integers and a is a positive integer which is as small as possible; we also write $b_0 = b_0'/a$. Moreover, we define

$$u_i' = \frac{b_0'}{h} \nabla w_{n+1,n+1-i}, \quad \text{for} \quad i = 0,1,\ldots,k.$$

Then evaluation of the determinant (4.9) yields the characteristic equation

$$(4.10)\qquad \begin{aligned} &c_0\zeta^{k+1} + c_1\zeta^k + \ldots + c_{k+1} = 0, \quad \text{where} \\ &c_0 = u_0 y + b_0' z - a, \quad y = h^2 H, \quad z = hJ_{n+1}, \\ &c_1 = u_1 y - b_0' z + a + a_1', \\ &c_i = u_i y + a_i' - a_{i-1}', \quad \text{for} \quad i = 2,3,\ldots,k, \\ &c_{k+1} = -a_k'. \end{aligned}$$

From (3.5) we know that if we choose the quadrature rule for the calculation of $\tilde{F}_{n+1}(x)$ to have a global error of order k, then scheme (2.8) also has a global error of order k. So if k = 2 we choose the weights as in (2.11) and, in general, for a backward differentiation formula of order k we choose a Gregory formula with (k-2)-th order difference correction as quadrature formula.

As an example, we derive here the stability polynomials for k = 2 and k = 3.

<u>k = 2</u> We have $a = 3$, $a_1' = 4$, $a_2' = -1$, $b_0' = 2$; moreover, we have $\nabla w_{n+1,n+1} = \nabla w_{n+1,n} = h/2$ and $\nabla w_{n+1,n-1} = 0$, so that $u_0 = u_1 = 1$ and $u_2 = 0$. Hence, in (4.10) we have

$$\begin{aligned} c_0 &= y + 2z - 3, \\ c_1 &= y - 2z + 7, \\ c_2 &= -5, \\ c_3 &= 1. \end{aligned}$$

k = 3 We have a = 11, $a_1' = 18$, $a_2' = -9$, $a_3' = 2$, $b_0' = 6$; moreover, we have $\nabla w_{n+1,n+1} = 5h/12$, $\nabla w_{n+1,n} = 8h/12$, $\nabla w_{n+1,n-1} = -h/12$, $\nabla w_{n+1,n-2} = 0$, so that $u_0 = 5/2$, $u_1 = 4$, $u_2 = -1/2$ and $u_3 = 0$. Hence in (4.10) we have

$$c_0 = \tfrac{1}{2}(5y + 12\ z - 22),$$

$$c_1 = 4y - 6\ z + 29,$$

$$c_2 = -\tfrac{1}{2}(y + 54),$$

$$c_3 = 11,$$

$$c_4 = -2.$$

For k = 2(1)6 we have computed the stability regions in the (z,y)-plane of scheme (2.8) using the test kernel given in (4.3). Here, the stability region is defined as the set of points (z,y) for which (4.10) has its roots within or on the unit circle. All calculations are based on the application of the Schur-criterion ([5, pp.77-9]). The regions are displayed in figures 4.1 to 4.5. The regions found in the first quadrant of the (z,y)-plane are not displayed for the following reason. It is well-known that a differential equation $\vec{f}' = \vec{R}(x,\vec{f})$ is inherently stable when its Jacobian has negative eigenvalues. Now differentiating the scalar version of (2.1) with respect to x yields the "differential equation"

$$f'(x) = g'(x) + K(x,x,f(x)) + \int_{x_0}^{x} \frac{\partial K}{\partial x}\,(x,\xi,f(\xi))d\xi.$$

Restricting the kernel function K to the class (4.3) and introducing the new variable

$$z = \int_{x_0}^{x} f(\xi)d\xi$$

yields the system

$$\begin{aligned} f'(x) &= g'(x) + K(x,x,f(x)) + Hz(x) \\ z'(x) &= f(x) \end{aligned} \qquad (4.11)$$

The eigenvalues of the Jacobian are negative when

$$(4.12) \qquad J_{n+1} = \frac{\partial K}{\partial f} < 0 \ , \ H = \frac{\partial^2 K}{\partial x \partial f} < 0 \ ,$$

from which we may conclude that the integral equation is certainly not stable in the first quadrant of the (z,y)-plane.

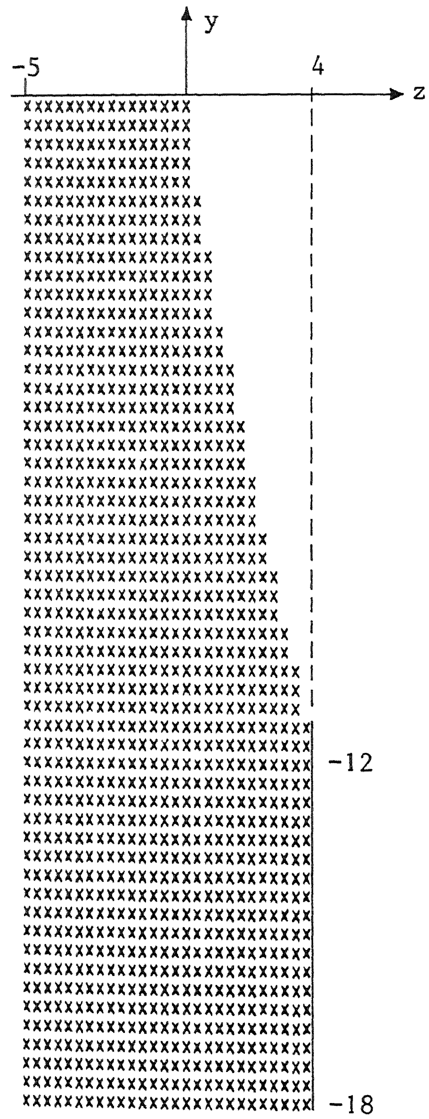
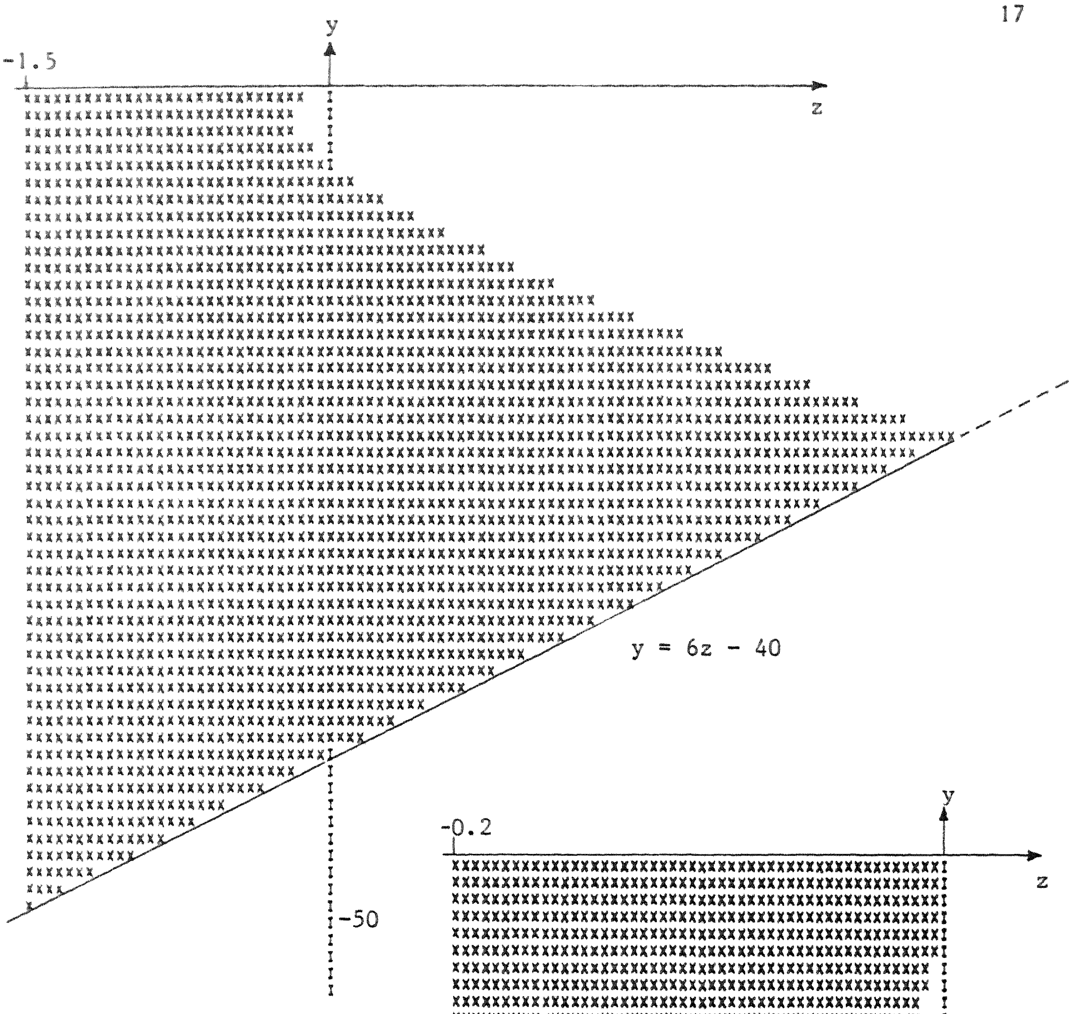
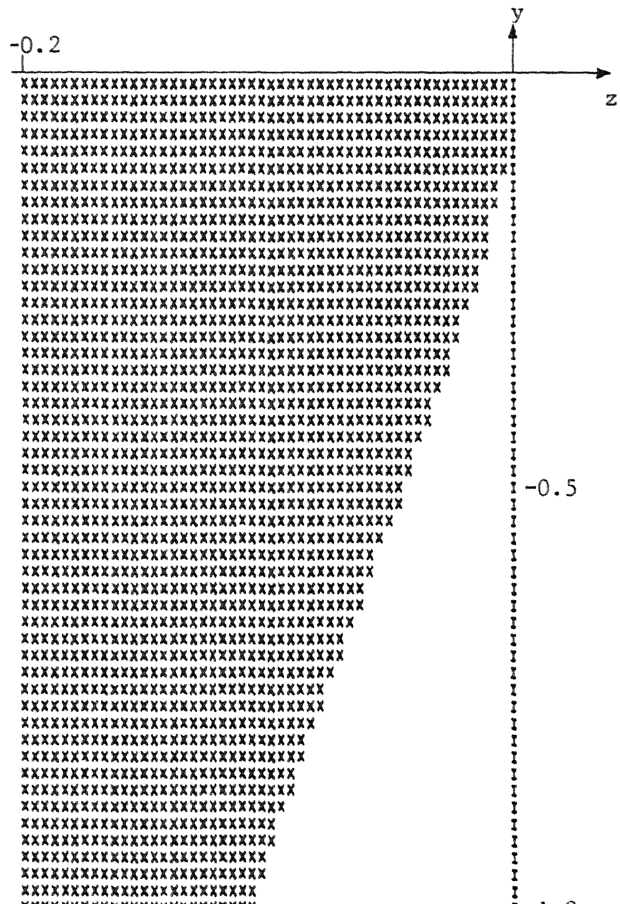


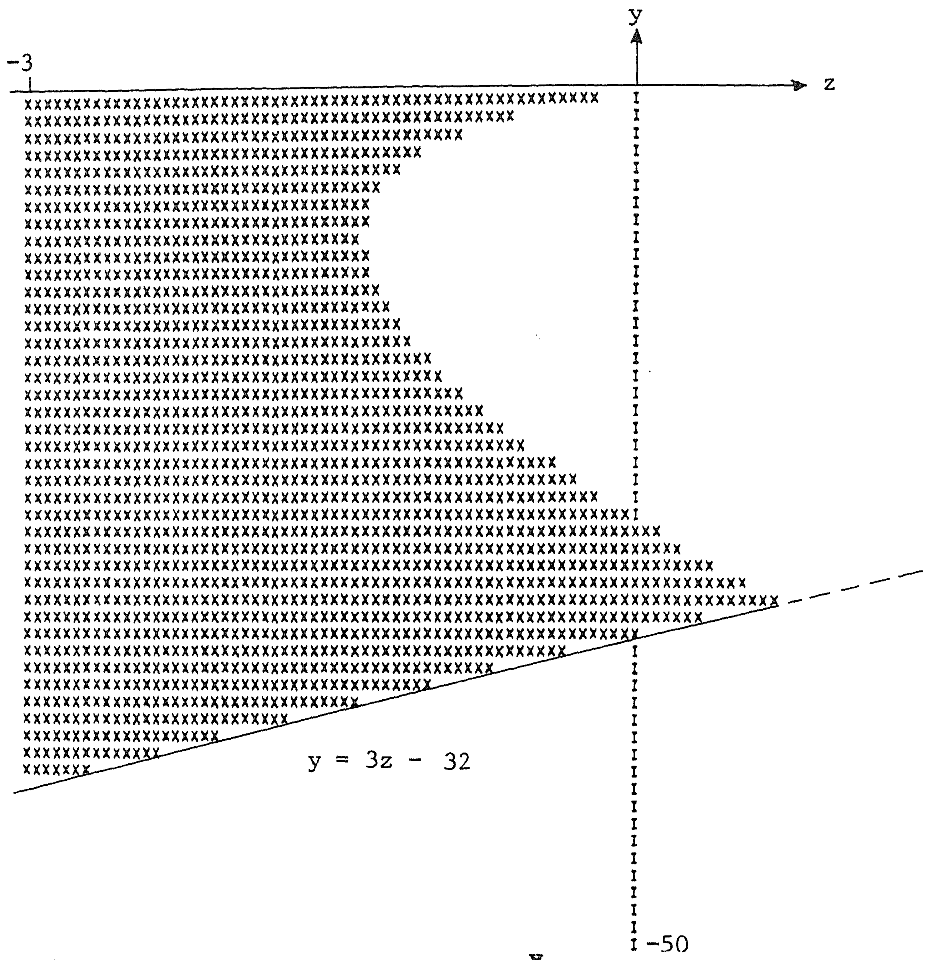
Figure 4.1 k=2





Detail near the origin

Figure 4.2 k=3



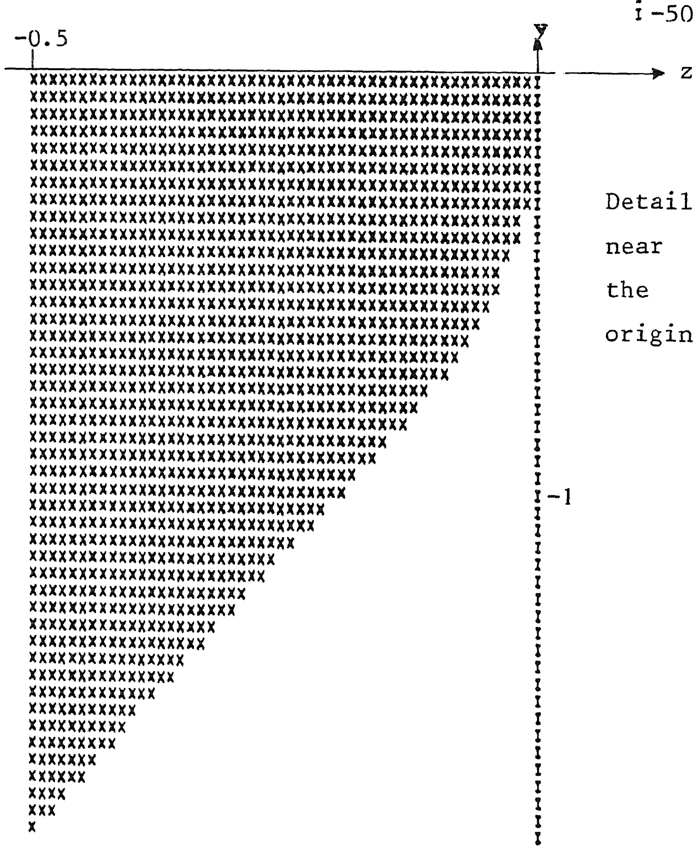
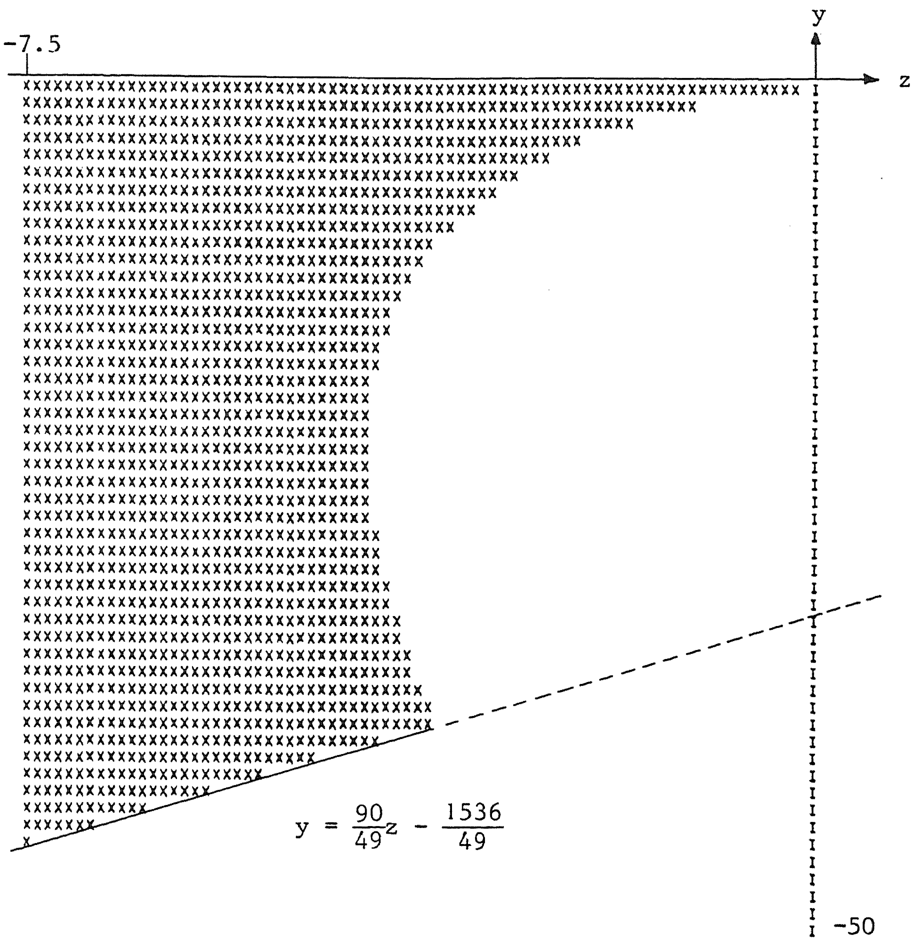


Figure 4.3 k=4


y
-7.5
z
$y = \frac{90}{49}z - \frac{1536}{49}$
-50

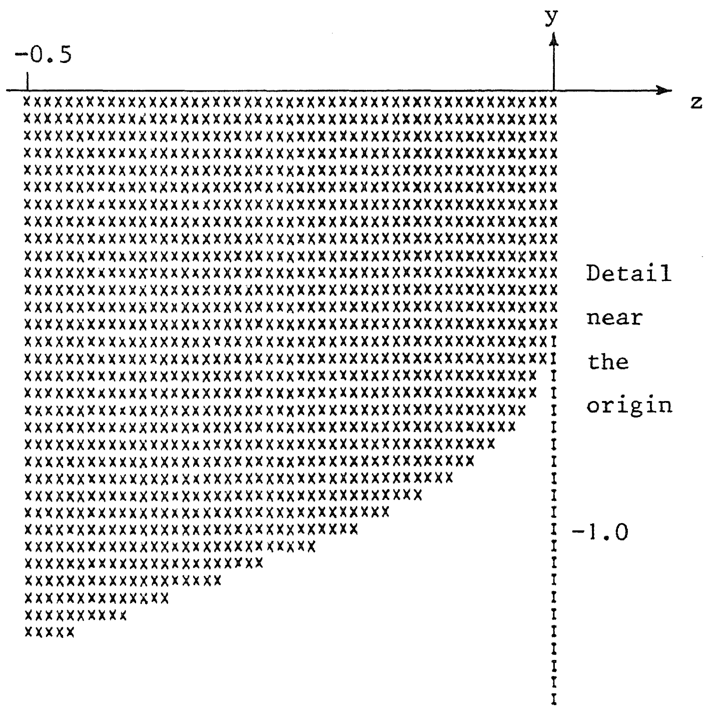
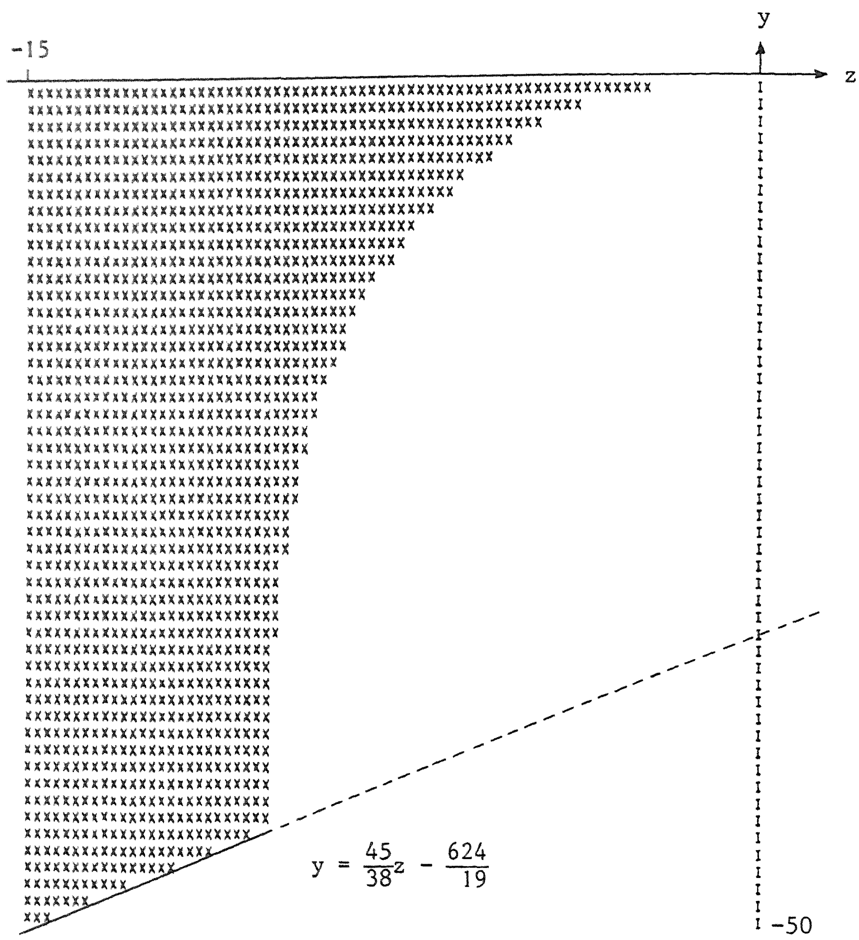

y
-0.5
z
Detail
near
the
origin
-1.0

Figure 4.4 k=5


-15
y
z
$y = \frac{45}{38}z - \frac{624}{19}$
-50

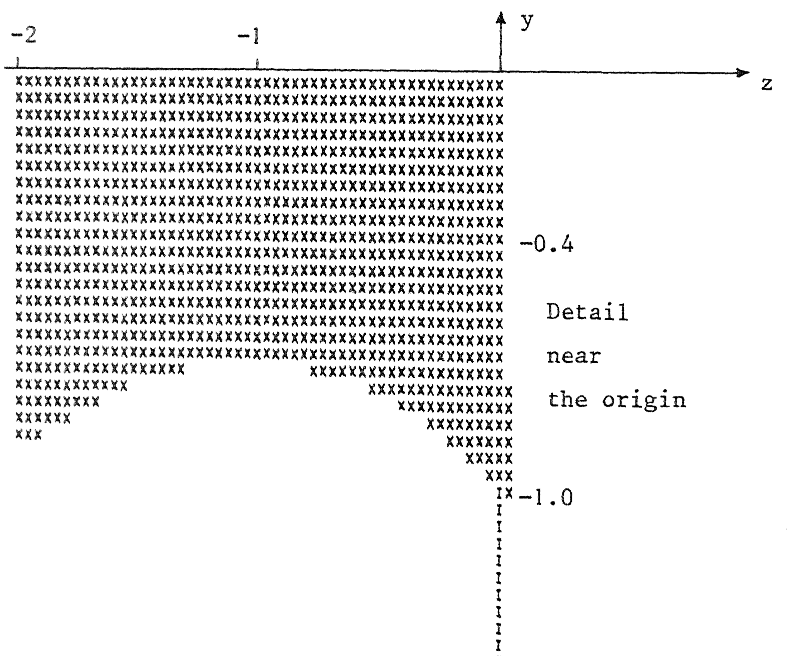

-2
-1
y
z
-0.4
Detail
near
the origin
-1.0

Figure 4.5 k=6

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