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NON-LINEAR SPLITTING METHODS FOR SEMI-DISCRETIZED
PARABOLIC DIFFERENTIAL EQUATIONS

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Non-linear splitting methods for semi-discretized parabolic differential equations

by

P.J. van der Houwen and J.G. Verwer

ABSTRACT

The purpose of the paper is to define splitting methods for systems of ordinary differential equations originating from semi-discretization of scalar parabolic differential equations. Attention is focussed on explicit systems not satisfying a simple linear splitting relation. By introducing non-linear splitting relations splitting methods are defined for arbitrary non-linear parabolic problems, provided the semi-discretization of these problems leads to an explicit system of ordinary equations. The greater part of these methods are discussed in the literature for linear problems.

KEY WORDS & PHRASES: *Numerical analysis, Ordinary differential equations, Parabolic differential equations, Method of lines, Splitting methods*

CONTENTS

1. INTRODUCTION

In the numerical treatment of partial differential equations splitting is referred to as a method of breaking down a complicated (multi-dimensional) process into a series of simple (one-dimensional) processess. Well-known examples are the alternating direction, the locally one-dimensional, and the hopscotch methods ([3,9]). In the literature, these methods, when applied to time-dependent problems, are usually treated as direct grid methods. The idea of splitting can also be applied in conjunction with the method of lines, an approach followed in [8]. In that paper we considered systems

$$\frac{d\vec{y}}{dx} = \vec{f}(\vec{y}), \tag{1.1}$$

of which $\vec{f}(\vec{y})$ can be linearly splitted into k terms i.e.

$$\vec{f}(\vec{y}) = \sum_{i=1}^{k} \vec{f}_i(\vec{y}). \tag{1.2}$$

For systems of this type we defined a wide class of integration formulas, which was shown to contain known splitting schemes by identifying the functions $\vec{f}_i$ appropriately. In particular, we paid attention to functions $\vec{f}$ originating from semi-discretization of parabolic equations.

In this paper we also assume that (1.1) originates from semi-discretization of parabolic equations, but here $\vec{f}$ is supposed to satisfy a non-linear splitting relation of the type

$$\vec{f}(\vec{y}) = \sum_{j=1}^{m} \vec{F}_j(\vec{y},\vec{y}), \tag{1.3}$$

where the functions $\vec{F}_j$ are still to be prescribed and are called splitting functions. This, at first sight somewhat strange splitting relation is introduced to extend results from [8] to semi-discretized equations not satisfying (1.2) with "simple" functions $\vec{f}_i$, e.g. functions with a tri-diagonal Jacobian matrix. The functions $\vec{F}_j$ are also assumed to be "simple", e.g. having tridiagonal Jacobian matrices with respect to both arguments. To give an example, by using these non-linear splitting functions $\vec{F}_j$ in conjunction with the method of lines approach, alternating direction methods are defined for functions $\vec{f}$ with an arbitrary non-linear 5-point coupling.

2. A CLASS OF NON-LINEAR SPLITTING SCHEMES

Let $\vec{y}_n$ denote the numerical approximation to the analytical solution $\vec{y}$ at $x = x_n$. Let $h_n = x_{n+1} - x_n$, i.e. the n-th integration step-size. We now define one-step integration formulas of the type

$$
\begin{aligned}
&\vec{y}^{(0)}_{n+1} = \vec{y}_n, \\
(2.1)\qquad &\vec{y}^{(j)}_{n+1} = \vec{y}^{(j-1)}_{n+1} + h_n \sum_{k,\ell=0}^{j} \lambda_{jk\ell}\, \vec{F}_j(\vec{y}^{(k)}_{n+1}, \vec{y}^{(\ell)}_{n+1}), \quad j = 1(1)m, \\
&\vec{y}_{n+1} = \vec{y}^{(m)}_{n+1}, \quad m \geq 2,
\end{aligned}
$$

where the functions $\vec{F}_j(\vec{u},\vec{v})$ satisfy relation (1.3). In particular, it is assumed that we are able to choose these functions in such a way that the Jacobians with respect to both arguments are "simple", e.g. tridiagonal. By avoiding the occurrence of $\vec{F}_j(\vec{y}^{(j)}_{n+1}, \vec{y}^{(j)}_{n+1})$ at the j-th stage, i.e. by setting $\lambda_{jjj} = 0$, we then obtain a computationally "simple" process. Observe that in (2.1) the number of stages is equal to the number of splitting functions $\vec{F}_j$.

2.1. The amplification matrix

Let us introduce the new parameters

$$
(2.2)\qquad L_{jk} = \sum_{\ell=0}^{j} \lambda_{jk\ell}, \quad M_{j\ell} = \sum_{k=0}^{j} \lambda_{jk\ell},
$$

and let $\Delta\vec{y}^{(k)}_{n+1}$ denote a perturbation of $\vec{y}^{(k)}_{n+1}$. By writing

$$
\Delta\vec{F}_j(\vec{y}^{(k)}_{n+1}, \vec{y}^{(\ell)}_{n+1}) \simeq J_j \Delta\vec{y}^{(k)}_{n+1} + K_j \Delta\vec{y}^{(\ell)}_{n+1},
$$

where J_j and K_j represent the partial derivatives of $\vec{F}_j(\vec{u},\vec{v})$ with respect to $\vec{u}$ and $\vec{v}$ at the point $(\vec{y}_n, \vec{y}_n)$, we then obtain the first order variational equations

$$\begin{aligned}\Delta\vec{y}^{(j)}_{n+1} &= \Delta\vec{y}^{(j-1)}_{n+1} + h_n \sum_{k,\ell=0}^{j} \lambda_{jk\ell}\, [J_j \Delta\vec{y}^{(k)}_{n+1} + K_j \Delta\vec{y}^{(\ell)}_{n+1}] \\ &= \Delta\vec{y}^{(j-1)}_{n+1} + \sum_{k=0}^{j} L_{jk}\, h_n J_j\, \Delta\vec{y}^{(k)}_{n+1} + \sum_{\ell=0}^{j} M_{j\ell}\, h_n K_j\, \Delta\vec{y}^{(\ell)}_{n+1} \\ &= \Delta\vec{y}^{(j-1)}_{n+1} + \sum_{\ell=0}^{j} [L_{j\ell}\, h_n J_j + M_{j\ell}\, h_n K_j] \Delta\vec{y}^{(\ell)}_{n+1}.\end{aligned}$$

Performing the recurrence yields

(2.3) $$\Delta\vec{y}_{n+1} = R_m \Delta\vec{y}_n,$$

R_m being the amplification matrix defined by the formal relations

(2.4) $$\begin{aligned}&R_0 = I, \quad I \text{ the unit matrix},\\ &R_j = R_{j-1} + h_n \sum_{\ell=0}^{j} [L_{j\ell}\, J_j + M_{j\ell}\, K_j] R_\ell, \quad j = 1(1)m.\end{aligned}$$

In the greater part of the known applications R_m is factorized. We also impose this property. Consequently, we shall assume that

(2.5) $$L_{j\ell} = M_{j\ell} = 0, \quad \ell = 0,\ldots,j-2; \quad j = 2,\ldots,m,$$

yielding the formal expression

(2.6) $$R_m = \prod_{j=m}^{1} [I - h_n(L_{jj}J_j + M_{jj}K_j)]^{-1} [I + h_n(L_{jj-1}J_j + M_{jj-1}K_j)].$$

When discussing particular schemes in following sections we confine ourselves to schemes which are well known when applied to linear problems. As a consequence, we omit a stability analysis of the corresponding amplification matrices. For such an analysis we refer to the given literature.

2.2 The order conditions

The order conditions will be derived for orders $p = 1$ and $p = 2$. Expanding $\vec{y}^{(j)}_{n+1}$ in powers of h_n at $x = x_n$ yields

$$\vec{y}_{n+1}^{(j)} = \vec{y}_{n+1}^{(j-1)} + h_n \sum_{k=0}^{j} L_{jk} \vec{F}_j(\vec{y}_n,\vec{y}_n) +$$
$$h_n \sum_{k=0}^{j} [L_{jk}J_j + M_{jk}K_j](\vec{y}_{n+1}^{(k)} - \vec{y}_n) + 0(h_n^3).$$

Using conditions (2.5) this expression is simplified to the formal relation

$$\text{(2.7)} \qquad \vec{y}_{n+1}^{(j)} - \vec{y}_n = [I-h_n(L_{jj}J_j + M_{jj}K_j)]^{-1} *$$
$$[h_n(L_{jj} + L_{jj-1}) \vec{F}_j(\vec{y}_n,\vec{y}_n) +$$
$$(I+h_n(L_{jj-1}J_j + M_{jj-1}K_j))(\vec{y}_{n+1}^{(j-1)} - \vec{y}_n)] + 0(h_n^3).$$

For j = 1 (2.7) reads

$$\text{(2.7')} \qquad \vec{y}_{n+1}^{(1)} - \vec{y}_n = h_n(L_{11}+L_{10})[I-h_n(L_{11}J_1+M_{11}K_1)]^{-1}\vec{F}_1(\vec{y}_n,\vec{y}_n) + 0(h_n^3).$$

Let us now assume that for every j, $\vec{y}_{n+1}^{(j)} - \vec{y}_n$ can be expanded as

$$\text{(2.8)} \qquad \vec{y}_{n+1}^{(j)} - \vec{y}_n = h_n\vec{a}^{(j)} + h_n^2 \vec{b}^{(j)} + 0(h_n^3).$$

Substitution of this expression into (2.7) yields

$$h_n\vec{a}^{(j)} + h_n^2 \vec{b}^{(j)} = h_n[I+h_n(L_{jj}J_j + M_{jj}K_j)] *$$
$$[(L_{jj} + L_{jj-1}) \vec{F}_j(\vec{y}_n,\vec{y}_n) +$$
$$(I+h_n(L_{jj-1}J_j + M_{jj-1}K_j))(\vec{a}^{(j-1)}+h_n\vec{b}^{(j-1)})] +$$
$$0(h_n^3).$$

Hence, if assumption (2.8) applies $\vec{a}^{(j)}$ and $\vec{b}^{(j)}$ has to satisfy the recurrence relations

$$\vec{a}^{(j)} = \vec{a}^{(j-1)} + (L_{jj} + L_{jj-1}) \vec{F}_j(\vec{y}_n,\vec{y}_n),$$

$$\vec{b}^{(j)} = \vec{b}^{(j-1)} + [(L_{jj-1} + L_{jj})J_j + (M_{jj-1} + M_{jj})K_j]\vec{a}^{(j-1)} + (L_{jj-1} + L_{jj})(L_{jj}J_j + M_{jj}K_j)\vec{F}_j(\vec{y}_n,\vec{y}_n), \tag{2.10}$$

where

$$\vec{a}^{(1)} = (L_{10} + L_{11})\vec{F}_1(\vec{y}_n,\vec{y}_n),$$

(2.10')

$$\vec{b}^{(1)} = (L_{11}J_1 + M_{11}K_1)\vec{a}^{(1)}.$$

The local analytical solution through the point $(x_n,\vec{y}_n)$ expands as

$$\vec{y}(x_{n+1}) = \vec{y}_n + h_n\vec{f}(\vec{y}_n) + \tfrac{1}{2}h_n^2 \frac{\partial \vec{f}}{\partial \vec{y}}(\vec{y}_n)\vec{f}(\vec{y}_n) + O(h_n^3). \tag{2.11}$$

Using

$$\vec{f}(\vec{y}_n) = \sum_{j=1}^{m} \vec{F}_j(\vec{y}_n,\vec{y}_n),$$

(2.12)

$$\frac{\partial \vec{f}}{\partial \vec{y}}(\vec{y}_n)\vec{f}(\vec{y}_n) = \sum_{i=1}^{m}(J_i + K_i)\sum_{j=1}^{m}\vec{F}_j(\vec{y}_n,\vec{y}_n),$$

a comparison of (2.8) and (2.11) yields the order conditions

$$p = 1:\quad \vec{a}^{(m)} = \sum_{j=1}^{m}\vec{F}_j(\vec{y}_n,\vec{y}_n), \tag{2.13}$$

$$p = 2:\quad \vec{b}^{(m)} = \tfrac{1}{2}\sum_{i=1}^{m}(J_i + K_i)\sum_{j=1}^{m}\vec{F}_j(\vec{y}_n,\vec{y}_n). \tag{2.14}$$

The first order condition (2.13) can always be satisfied by appropriate values of the coefficients $L_{j\ell}$ and $M_{j\ell}$, irrespective the form of the splitting functions $\vec{F}_j$. Condition (2.14) is a more complicated one. Second order consistency can only be obtained for special choices of the functions $\vec{F}_j$.

<u>REMARK 2.1</u> We do not give a special convergence proof of method (2.11), as it is a one-step integration method of the type $\vec{y}_{n+1} = \vec{y}_n + h_n\,\vec{\Phi}\,(h_n,\vec{y}_n,\vec{y}_{n+1})$. Convergence results for one-step methods

defined by general increment functions $\vec{\Phi}$ are known in the literature (see e.g. [4] or [7]).

3. EXAMPLES OF SPLITTING FUNCTIONS

Before giving particular schemes from class (2.1) we first give examples of splitting functions for systems

$$\frac{d\vec{y}}{dx} = \vec{f}(\vec{y}), \tag{3.1}$$

originating from semi-discretization of two and three-dimensional parabolic equations (scalar ones). With the exceptions of a few these functions define splittings known from the literature. In this context we observe however that we admit arbitrary non-linearities, provided some coupling between components of $\vec{y}$ has been prescribed. In the literature splittings are usually defined for linear problems.

3.1 Splittings for semi-discretized two-dimensional parabolic equations

In this section we give examples of splitting functions $\vec{F}_1$ and $\vec{F}_2$ such that

$$\vec{f}(\vec{y}) = \vec{F}_1(\vec{y},\vec{y}) + \vec{F}_2(\vec{y},\vec{y}). \tag{3.2}$$

If the index j is omitted the functions $\vec{F}_1$ and $\vec{F}_2$ are chosen identically. The two-stage schemes using these functions will be given in section 4.1. It is agreed that the $\vec{u}$-argument of $\vec{F}_1(\vec{u},\vec{v})$ occurs implicitly in the computation of $\vec{y}^{(1)}_{n+1}$, while the $\vec{v}$-argument of $\vec{F}_2(\vec{u},\vec{v})$ occurs implicitly in the computation of $\vec{y}^{(2)}_{n+1}$.

The components of $\vec{y}$ and $\vec{f}$ are supposed to be arranged in a two-dimensional array. Each array element is then associated to a gridpoint of the two-dimensional grid imposed on the region under consideration. Such a grid is not necessarily rectangular, but may be of any shape. It also may contain "holes". In fig. 3.1 an example of such an array is given.

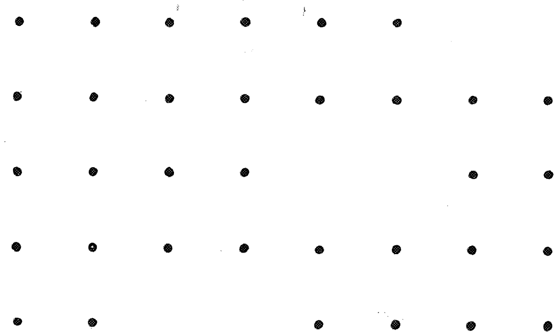

Fig.3.1 Two-dimensional arrangement of the components of $\vec{y}$ and $\vec{f}$

In order to define the splitting functions $\vec{F}_j(\vec{u},\vec{v})$ it is convenient to divide the set of gridpoint into four subsets as shown in fig.3.2. Related to these subsets we then define operators P_o, $P_\bullet$, P_+ and P_x on vectors $\vec{u}$, which leave the components of $\vec{u}$ corresponding to o, $\bullet$, + and x gridpoints unchanged and substitute a zero for all other components. Furthermore, we give functions for 5-point and 9-point coupled equations.

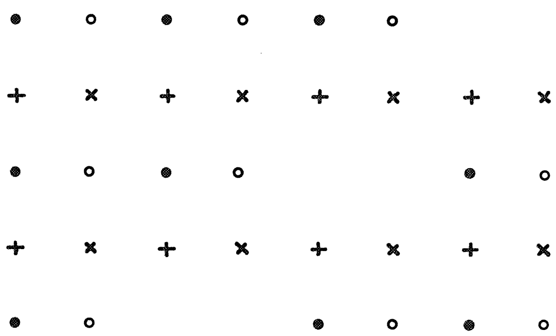

Fig.3.2 Four subsets of gridpoints

3.1.1 Odd-even hopscotch splittings

The most simple splitting function for 5-point coupled functions $\vec{f}$ is given by ($\vec{F} = \vec{F}_1 = \vec{F}_2$)

$$(3.3) \qquad \vec{F}(\vec{u},\vec{v}) = \tfrac{1}{2}(P_o+P_+)\,\vec{f}(\vec{u}) + \tfrac{1}{2}(P_\bullet+P_x)\vec{f}(\vec{v}).$$

By computing $\vec{y}_{n+1}^{(1)}$ (see scheme (4.1')) first at the $\bullet$ and x points, and then at o and + points, only scalar equations are to be solved. The same holds for $\vec{y}_{n+1}^{(2)}$ when the computing order is reversed. This type of splitting is known as the odd-even hopscotch splitting ([2]).

An alternative odd-even hopscotch splitting for 5-point coupled functions is obtained by splitting the argument of $\vec{f}$:

$$\vec{F}(\vec{u},\vec{v}) = \tfrac{1}{2}\vec{f}((P_o+P_+)\ \vec{u} + (P_\bullet+P_x)\vec{v}). \tag{3.4}$$

Here, also scalar equations have to be solved, provided the order in which the solutions at the gridpoints are computed is the reversed of (3.3).

In case of 9-point coupled functions $\vec{f}$, the functions

$$\begin{aligned}\vec{F}_1(\vec{u},\vec{v}) = {} & \tfrac{1}{2}(P_\bullet+P_x)\vec{f}(\vec{v}) + \\ & \tfrac{1}{2}P_o\vec{f}((P_o+P_\bullet+P_x)\vec{u} + P_+\vec{v}) + \\ & \tfrac{1}{2}P_+\vec{f}((P_++P_\bullet+P_x)\vec{u} + P_o\vec{v}),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\vec{F}_2(\vec{u},\vec{v}) = {} & \tfrac{1}{2}(P_o+P_+)\vec{f}(\vec{u}) + \\ & \tfrac{1}{2}P_\bullet\vec{f}((P_\bullet+P_o+P_+)\vec{v} + P_x\vec{u}) + \\ & \tfrac{1}{2}P_x\vec{f}((P_x+P_o+P_+)\vec{v} + P_\bullet\vec{u}),\end{aligned} \tag{3.7}$$

also define a splitting of the odd-even hopscotch type. As far as we know, both (3.4) and (3.6),(3.7) are not mentioned in the literature.

3.1.2 Line hopscotch splittings

A splitting which applies to *five-point* as well as to *nine-point* coupled functions $\vec{f}$ is presented by

$$\vec{F}(\vec{u},\vec{v}) = \tfrac{1}{2}(P_o+P_\bullet)\vec{f}(\vec{v}) + \tfrac{1}{2}(P_++P_x)\vec{f}(\vec{u}). \tag{3.8}$$

By solving first the o and $\bullet$ components and then the + and x components in the first stage (see scheme (4.1')) and, vice versa, in the second stage, only *tridiagonal* implicit schemes are to be solved. This type of splitting is known as the *line hopscotch splitting* ([3]). Formula (3.8) defines the splitting along horizontal grid lines. In a similar way the splitting may be defined along vertical grid lines.

An analogue of the line hopscotch splitting (3.8) is obtained by splitting the argument $\vec{f}$:

$$\vec{F}(\vec{u},\vec{v}) = \tfrac{1}{2}\vec{f}((P_o+P_\bullet)\vec{v} + (P_x+P_+)\vec{u}). \tag{3.9}$$

This splitting also requires the solution of tridiagonal sets of algebraic equations, irrespective whether we have five or nine-point couplings. As far as we know, it has not been discussed in the literature.

3.1.3 Alternating direction splittings

Still more sophisticated formulas can be constructed by splitting both $\vec{f}$ and its argument $\vec{y}$. An example is presented by

$$\begin{aligned}\vec{F}(\vec{u},\vec{v}) = {} & \tfrac{1}{2}P_o\vec{f}((\tfrac{1}{2}P_o+P_\bullet)\vec{u} + (P_x+\tfrac{1}{2}P_o)\vec{v}) + \\ & \tfrac{1}{2}P_x\vec{f}((\tfrac{1}{2}P_x+P_+)\vec{u} + (P_o+\tfrac{1}{2}P_x)\vec{v}) + \\ & \tfrac{1}{2}P_\bullet\vec{f}((\tfrac{1}{2}P_\bullet+P_o)\vec{u} + (P_++\tfrac{1}{2}P_\bullet)\vec{v}) + \\ & \tfrac{1}{2}P_+\vec{f}((\tfrac{1}{2}P_++P_x)\vec{u} + (P_\bullet+\tfrac{1}{2}P_+)\vec{v}),\end{aligned} \tag{3.10}$$

which represents an alternating direction splitting ([6]). Here, tridiagonal systems of algebraic equations are to be solved alternatingly along the rows of o$\bullet$o and x+x points, and along the columns of $\bullet$+$\bullet$ and oxo points (see scheme (4.1')), provided $\vec{f}$ is a 5-point coupled function.

For 9-point coupled functions we need non-identical splitting functions (see scheme (4.2)):

$$\vec{F}_1(\vec{u},\vec{v}) = \tfrac{1}{2}P_o\vec{f}(\tfrac{1}{2}P_o+P_\bullet)\vec{u} + (P_x+P_+ +\tfrac{1}{2}P_o)\vec{v}) +$$
$$\tfrac{1}{2}P_x\vec{f}(\tfrac{1}{2}P_x+P_+)\vec{u} + (P_o+P_\bullet+\tfrac{1}{2}P_x)\vec{v}) +$$
$$\tfrac{1}{2}P_\bullet\vec{f}(\tfrac{1}{2}P_\bullet+P_o)\vec{u} + (P_x+P_+ +\tfrac{1}{2}P_\bullet)\vec{v}) +$$
$$\tfrac{1}{2}P_+\vec{f}(\tfrac{1}{2}P_+ +P_x)\vec{u} + (P_o+P_\bullet+\tfrac{1}{2}P_+)\vec{v}), \tag{3.11}$$

$$\vec{F}_2(\vec{u},\vec{v}) = \tfrac{1}{2}P_o\vec{f}(\tfrac{1}{2}P_o+P_\bullet+P_+)\vec{u} + (P_x+\tfrac{1}{2}P_o)\vec{v}) +$$
$$\tfrac{1}{2}P_x\vec{f}(\tfrac{1}{2}P_x+P_+ +P_\bullet)\vec{u} + (P_o+\tfrac{1}{2}P_x)\vec{v}) +$$
$$\tfrac{1}{2}P_\bullet\vec{f}(\tfrac{1}{2}P_\bullet+P_o+P_x)\vec{u} + (P_+ +\tfrac{1}{2}P_\bullet)\vec{v}) +$$
$$\tfrac{1}{2}P_+\vec{f}(\tfrac{1}{2}P_+ +P_x+P_o)\vec{u} + (P_\bullet+\tfrac{1}{2}P_+)\vec{v}). \tag{3.12}$$

3.2 A splitting for semi-discretized three-dimensional parabolic equations

In the present section we confine ourselves to one splitting, viz. an alternating direction one. Let us assume that $\vec{f}(\vec{y})$ satisfies

$$\vec{f}(\vec{y}) = \vec{H}(\vec{y},\vec{y},\vec{y}), \tag{3.13}$$

$\vec{H}$ to be found. Further, let us assume that the components of $\vec{y}$ and $\vec{f}$ are arranged in a three-dimensional array, each element of it being associated to a gridpoint of a three-dimensional grid, while the components of $\vec{f}$ satisfy a 7-point coupling. In a similar way as done in section 3.1, we now divide the set of gridpoints into 8 subsets (see fig.3.3) and define the related operators $P_\bullet$, P_o, P_x, P_+, $P_{\bullet\bullet}$, P_{oo}, P_{xx}, P_{++}.

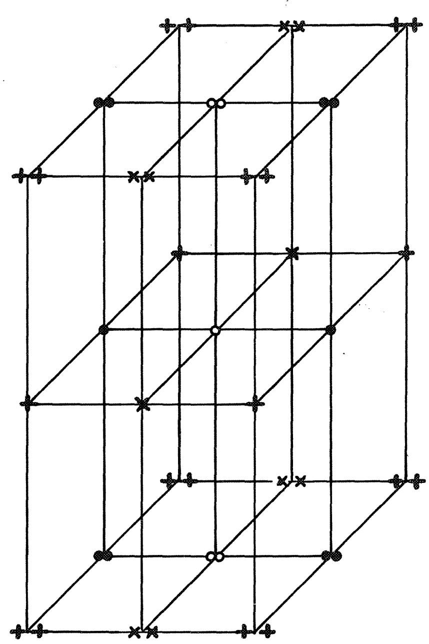

Fig.3.3 Eight sets of gridpoints

The alternating direction splitting of $\vec{f}$ is then defined by

$$
\begin{aligned}
(3.14)\qquad \vec{H}(\vec{u},\vec{v},\vec{w}) = {} & P_{\circ}\vec{f}((\tfrac{1}{3}P_{\circ}+P_{\bullet})\vec{u} + (\tfrac{1}{3}P_{\circ}+P_{\times})\vec{v} + (\tfrac{1}{3}P_{\circ}+P_{\circ\circ})\vec{w}) + \\
& P_{\bullet}\vec{f}((\tfrac{1}{3}P_{\bullet}+P_{\circ})\vec{u} + (\tfrac{1}{3}P_{\bullet}+P_{+})\vec{v} + (\tfrac{1}{3}P_{\bullet}+P_{\bullet\bullet})\vec{w}) + \\
& P_{\times}\vec{f}((\tfrac{1}{3}P_{\times}+P_{+})\vec{u} + (\tfrac{1}{3}P_{\times}+P_{\circ})\vec{v} + (\tfrac{1}{3}P_{\times}+P_{\times\times})\vec{w}) + \\
& P_{+}\vec{f}((\tfrac{1}{3}P_{+}+P_{\times})\vec{u} + (\tfrac{1}{3}P_{+}+P_{\bullet})\vec{v} + (\tfrac{1}{3}P_{+}+P_{++})\vec{w}) + \\
& P_{\circ\circ}\vec{f}((\tfrac{1}{3}P_{\circ\circ}+P_{\bullet\bullet})\vec{u} + (\tfrac{1}{3}P_{\circ\circ}+P_{\times\times})\vec{v} + (\tfrac{1}{3}P_{\circ\circ}+P_{\circ})\vec{w}) + \\
& P_{\bullet\bullet}\vec{f}((\tfrac{1}{3}P_{\bullet\bullet}+P_{\circ\circ})\vec{u} + (\tfrac{1}{3}P_{\bullet\bullet}+P_{++})\vec{v} + (\tfrac{1}{3}P_{\bullet\bullet}+P_{\bullet})\vec{w}) + \\
& P_{\times\times}\vec{f}((\tfrac{1}{3}P_{\times\times}+P_{++})\vec{u} + (\tfrac{1}{3}P_{\times\times}+P_{\circ\circ})\vec{v} + (\tfrac{1}{3}P_{\times\times}+P_{\times})\vec{w}) + \\
& P_{++}\vec{f}((\tfrac{1}{3}P_{++}+P_{\times\times})\vec{u} + (\tfrac{1}{3}P_{++}+P_{\bullet\bullet})\vec{v} + (\tfrac{1}{3}P_{++}+P_{+})\vec{w}).
\end{aligned}
$$

When used in conjunction with scheme (4.8), this splitting requires the solution of tridiagonal systems of non-linear algebraic equations (this method goes back to DOUGLAS [1]).

4. EXAMPLES OF SPLITTING SCHEMES

In the present section we give examples of schemes, using splitting functions from section 3, which may be recognized as known splitting schemes provided a corresponding problem class is chosen. We observe that these schemes can also be given for the non-autonomous equation

$$\frac{d\vec{y}}{dx} = \vec{f}(\vec{x},\vec{y}).$$

As this is not essential in the context of this paper it is omitted.

4.1 Schemes for two-dimensional splitting functions

Let us consider the simple two-stage formula

$$(4.1)\qquad \begin{aligned} \vec{y}^{(1)}_{n+1} &= \vec{y}_n + h_n\vec{F}_1(\vec{y}^{(1)}_{n+1},\vec{y}_n), \\ \vec{y}_{n+1} &= \vec{y}^{(1)}_{n+1} + h_n\vec{F}_2(\vec{y}^{(1)}_{n+1},\vec{y}_{n+1}). \end{aligned}$$

This scheme belongs to class (2.1),(2.5) and is of first order, provided $\vec{F}_1$ and $\vec{F}_2$ satisfy (1.3). In case of identical splitting functions $\vec{F} = \vec{F}_1 = \vec{F}_2$ and

$$(4.1')\qquad \begin{aligned} \vec{y}^{(1)}_{n+1} &= \vec{y}_n + h_n\vec{F}(\vec{y}^{(1)}_{n+1},\vec{y}_n), \\ \vec{y}_{n+1} &= \vec{y}^{(1)}_{n+1} + h_n\vec{F}(\vec{y}^{(1)}_{n+1},\vec{y}_{n+1}), \end{aligned}$$

we obtain second order accuracy. This simple second order scheme represents

A) the odd-even hopscotch scheme ([2]) if F is defined by (3.3),

B) the line hopscotch scheme ([3]) if $\vec{F}$ is defined by (3.8),

C) the alternating direction scheme of PEACEMAN and RACHFORD [6] if $\vec{F}$ is defined by (3.8).

With the exception of the line hopscotch method, these methods only apply to 5-point coupled equations $\vec{f}$. As can be seen in section 3.1, to define splittings for 9-point coupled functions $\vec{f}$, we generally need non-identical splitting functions. Let us consider the functions (3.11)-(3.12) defining an alternating direction splitting in case of a 9-point coupling. It is obvious to substitute these functions into scheme (4.1), to obtain an alternating direction scheme for 9-point coupled equations. Unfortunately, the resulting scheme is not unconditionally stable for linear parabolic problems with a mixed derivative (problem (4.3)), and thus of limited use. Therefore, we consider the following formula (computationally expensive)

$$
(4.2) \qquad
\begin{aligned}
\vec{y}^{(1)}_{n+1} &= \vec{y}_n + h_n\vec{F}_1(\vec{y}^{(1)}_{n+1},\vec{y}_n) + h_n\vec{F}_2(\vec{y}_n,\vec{y}_n),\\
\vec{y}_{n+1} &= \vec{y}^{(1)}_{n+1} - h_n\vec{F}_1(\vec{y}_n,\vec{y}_n) + h_n\vec{F}_2(\vec{y}_n,\vec{y}_{n+1}),
\end{aligned}
$$

which not belongs to class (2.1). It can be shown that (4.2) is of first order, provided $\vec{F}_1(\vec{y},\vec{y}) = \vec{F}_2(\vec{y},\vec{y}) = \frac{1}{2}\vec{f}(\vec{y})$. If $\vec{F}_1$ and $\vec{F}_2$ are defined by (3.11) and (3.12), respectively, formula (4.2) represents a Douglas-Rachford alternating direction scheme given by MCKEE and MITCHELL [5]. To illustrate this, we consider the parabolic equation

$$
(4.3) \qquad
\begin{aligned}
&U_t = aU_{x_1x_1} + 2bU_{x_1x_2} + aU_{x_2x_2}\\
&a,\ b > 0,\ b^2 < a^2,\ a \text{ and } b \text{ constant},
\end{aligned}
$$

and assume that appropriate boundary and initial conditions are given. Further, it is assumed that this equation is semi-discretized on a rectilinear grid using standard finite differences to obtain the linear system

$$
(4.4) \qquad \frac{d\vec{y}}{dt} = (M_1 + M_{12} + M_{22})\vec{y}.
$$

The meaning of the matrices M_1, M_{12} and M_{22} shall be clear from the foregoing. For equation (4.4) the splitting functions (3.11) and (3.12) reduce to

$$(4.5)\qquad \vec{F}_1(\vec{u},\vec{v}) = \tfrac{1}{2}M_1\vec{u} + \tfrac{1}{2}M_2\vec{v} + M_{12}\vec{v},$$

$$(4.6)\qquad \vec{F}_2(\vec{u},\vec{v}) = \tfrac{1}{2}M_1\vec{u} + \tfrac{1}{2}M_2\vec{v} + M_{12}\vec{u},$$

and scheme (4.2) then reads (τ_n the steplength)

$$(4.7)\qquad \begin{aligned} \vec{y}^{(1)}_{n+1} &= \vec{y}_n + \tfrac{1}{2}\tau_n M_1 \vec{y}^{(1)}_{n+1} + \tau_n[\tfrac{1}{2}M_1 + 2M_{12} + M_2]\vec{y}_n, \\ \vec{y}_{n+1} &= \vec{y}^{(1)}_{n+1} + \tfrac{1}{2}\tau_n M_2 \vec{y}_{n+1} - \tfrac{1}{2}\tau_n M_2 \vec{y}_n. \end{aligned}$$

The Douglas-Rachford splitting given by Mckee and Mitchell is now easily recognized. They give an analysis from which unconditional stability results.

<u>REMARK 4.1</u> The coefficients of $U_{x_1x_1}$ and $U_{x_2x_2}$ in (4.3) are chosen equal to simplify the expressions (4.5)-(4.6). In case of unequal coefficients scheme (4.2) gives a splitting scheme which slightly differs from the scheme of McKee and Mitchell.

4.2 The three-dimensional Douglas scheme

Assume that $\vec{f}$ satisfies (3.13) and consider the scheme

$$(4.8)\qquad \begin{aligned} \vec{y}^{(1)}_{n+1} &= \vec{y}_n + \tfrac{1}{2}h_n[\vec{H}(\vec{y}^{(1)}_{n+1},\vec{y}_n,\vec{y}_n) + \vec{H}(\vec{y}_n,\vec{y}_n,\vec{y}_n)], \\ \vec{y}^{(2)}_{n+1} &= \vec{y}^{(1)}_{n+1} + \tfrac{1}{2}h_n[\vec{H}(\vec{y}_n,\vec{y}^{(2)}_{n+1},\vec{y}_n) - \vec{H}(\vec{y}_n,\vec{y}_n,\vec{y}_n)], \\ \vec{y}_{n+1} &= \vec{y}^{(2)}_{n+1} + \tfrac{1}{2}h_n[\vec{H}(\vec{y}_n,\vec{y}_n,\vec{y}_{n+1}) - \vec{H}(\vec{y}_n,\vec{y}_n,\vec{y}_n)]. \end{aligned}$$

By defining the functions

$$\vec{F}_1(\vec{u},\vec{v}) = \tfrac{1}{2}\vec{H}(\vec{u},\vec{v},\vec{v}) + \tfrac{1}{2}\vec{H}(\vec{v},\vec{v},\vec{v}),$$

$$(4.9) \qquad \vec{F}_2(\vec{u},\vec{v}) = \tfrac{1}{2}\vec{H}(\vec{v},\vec{u},\vec{v}) - \tfrac{1}{2}\vec{H}(\vec{v},\vec{v},\vec{v}),$$

$$\vec{F}_3(\vec{u},\vec{v}) = \tfrac{1}{2}\vec{H}(\vec{v},\vec{v},\vec{u}) - \tfrac{1}{2}\vec{H}(\vec{v},\vec{v},\vec{v}),$$

scheme (4.8) can be rewritten as

$$\vec{y}^{(1)}_{n+1} = \vec{y}_n + h_n\vec{F}_1(\vec{y}^{(1)}_{n+1},\vec{y}_n),$$

$$(4.8') \qquad \vec{y}^{(2)}_{n+1} = \vec{y}^{(1)}_{n+1} + h_n\vec{F}_2(\vec{y}^{(2)}_{n+1},\vec{y}_n),$$

$$\vec{y}_{n+1} = \vec{y}^{(2)}_{n+1} + h_n\vec{F}_3(\vec{y}_{n+1},\vec{y}_n).$$

It is easily shown that this scheme is of second order and belongs to class (2.1), while its amplification matrix is factorized. Further, if $\vec{H}$ is defined by the alternating direction splitting function (3.14), scheme (4.8) represents a method which goes back to DOUGLAS [1].

4.3 The generalized Douglas scheme

Let $\vec{f}$ satisfy the relation

$$(4.10) \qquad \vec{f}(\vec{y}) = \vec{H}(\vec{y},\vec{y},\ldots,\vec{y}),$$

$\vec{H}$ being an m-argument function still to be prescribed. Analogous to (4.8) we then define

$$\vec{y}^{(1)}_{n+1} = \vec{y}_n + \tfrac{1}{2}h_n\,\vec{H}(\vec{y}^{(1)}_{n+1},\vec{y}_n,\ldots,\vec{y}_n) + \tfrac{1}{2}h_n\,\vec{H}(\vec{y}_n,\vec{y}_n,\ldots,\vec{y}_n),$$

$$(4.11) \qquad \vec{y}^{(j)}_{n+1} = \vec{y}^{(j-1)}_{n+1} + \tfrac{1}{2}h_n\,\vec{H}(\vec{y}_n,\ldots,\vec{y}^{(j)}_{n+1},\ldots,\vec{y}_n) - \tfrac{1}{2}h_n\,\vec{H}(\vec{y}_n,\vec{y}_n,\ldots,\vec{y}_n), \quad j = 2(1)m,$$

$$\vec{y}_{n+1} = \vec{y}^{(m)}_{n+1}.$$

In a similar way as in the preceding section, it can be shown that this scheme is of second order and belongs to class (2.1),(2.5). According to [8], it is called a generalized Douglas scheme if H represents the alternating direction splitting for a 2m+1-point coupled function $\vec{f}$ which originates from semi-discretization of an m-dimensional parabolic equation (compare section 3.2).

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