stichting
mathematisch
centrum

MC

NUMERIEKE WISKUNDE NW 53/77 NOVEMBER
(DEPARTMENT OF NUMERICAL MATHEMATICS)

P.H.M. WOLKENFELT

BACKWARD DIFFERENTIATION FORMULAS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 65R05

Backward differentiation formulas for Volterra integro-differential equations

by

P.H.M. Wolkenfelt

ABSTRACT

Backward differentiation formulas combined with Gregory quadrature formulas are studied for the numerical solution of Volterra integro-differential equations. In particular, the stability is investigated for a large class of model equations and stability regions are presented. An application to Volterra integral equations of the second kind is given. Numerical experiments support the theory.

KEY WORDS & PHRASES: *Volterra integro-differential equations, backward differentiation formulas, stability*

CONTENTS

1. INTRODUCTION

This report deals with the numerical solution of Volterra integro-differential equations (VIDE) by means of backward differentiation formulas combined with Gregory quadrature formulas.

Systems of Volterra integro-differential equations have the following form

$$(1.1)\qquad \begin{cases} \vec{y}\,'(x) = \vec{F}(x,\vec{y}(x),\vec{z}(x)), & x_0 \le x \le b, \\ \vec{z}(x) = \displaystyle\int_{x_0}^{x} \vec{K}(x,t,\vec{y}(t))dt, & \end{cases}$$

with initial condition

$$(1.2)\qquad \vec{y}(x_0) = \vec{y}_0.$$

Here, $\vec{F}(x,\vec{y},\vec{z})$ and $\vec{K}(x,t,\vec{y})$ are prescribed vector functions which are assumed to satisfy the following conditions:

i) $\vec{F}$ and $\vec{K}$ are uniformly continuous in all variables,

ii) $\|\vec{F}(x,\vec{y}_1,\vec{z}) - \vec{F}(x,\vec{y}_2,\vec{z})\| \le L_1\|\vec{y}_1 - \vec{y}_2\|$,
$\|\vec{F}(x,\vec{y},\vec{z}_1) - \vec{F}(x,\vec{y},\vec{z}_2)\| \le L_2\|\vec{z}_1 - \vec{z}_2\|$,

iii) $\|\vec{K}(x,t,\vec{y}_1) - \vec{K}(x,t,\vec{y}_2)\| \le L_3\|\vec{y}_1 - \vec{y}_2\|$.

with respect to some norm $\|\cdot\|$.

Under these conditions, equation (1.1) with initial condition (1.2) has a unique solution.

A survey of numerical methods for the solution of (1.1) is given by CRYER in [8] and more recently by BAKER in [3, chapter 21]. It turns out that much attention is paid to the construction of methods together with proofs of convergence, including the concept of zero-stability (i.e. stability in the limit as $h \to 0$). However, only few authors are concerned with stability for fixed non-zero h. BRUNNER & LAMBERT [1] give a theory of weak stability; their analysis is based on the scalar model equation

$$(1.3) \qquad y'(x) = (\lambda + \mu)y(x) - \lambda\mu \int_0^x y(t)dt$$

where $(\lambda + \mu)$ and $(-\lambda\mu)$ are negative real constants. Very recently, MATTHYS [7] developed a theory based on the model equation (1.3) but λ and μ now being complex constants with $Re(\lambda) < 0$ and $Re(\mu) < 0$. He also gives an extension to systems of VIDES. In both papers the model equation reduces, in fact, to a system of ordinary differential equations; the numerical scheme then reduces to a composite linear multistep method.

We present a stability analysis for a much wider class of model equations.

Further we have tested the *convergence* properties of the methods and *partly* the stability properties. In a forthcoming report, however, stability will be tested more extensively.

In section 2 a derivation of the computational scheme is given. Section 3 deals with results concerning convergence and consistency. Our stability analysis is displayed in section 4. An application to Volterra integral equations can be found in section 5. Numerical experiments are reported in section 6, and finally in section 7 some concluding remarks are given.

2. DEFINITION OF THE CLASS OF METHODS

Let $x_n = nh$, $n = 0,1,\dots,N$ $(b = Nh)$ denote reference points on the x-axis and let $\vec{y}_n$, $\vec{z}_n$ denote approximations to the exact values $\vec{y}(x_n)$, $\vec{z}(x_n)$ respectively.

Equation (1.1) can be considered as a system of ordinary differential equations of the form

$$(2.1) \qquad \vec{y}\,'(x) = \vec{f}(x,\vec{y}(x),\vec{z}(x))$$

where $\vec{z}(x)$ is assumed to be a known vector function of x. This observation suggests the use of ODE methods for the solution of (1.1). In this report we will consider a special class of linear multistep methods, viz. the *backward differentiation formulas* (see e.g. [2]). These formulas have the following form

$$(2.2) \qquad \vec{y}_{n+1} = \sum_{\ell=1}^{k} a_\ell \vec{y}_{n+1-\ell} + hb_0\vec{f}(x_{n+1},\vec{y}_{n+1},\vec{z}(x_{n+1}))$$

where the coefficients a_ℓ and b_0 define the differentiation formula used. At the end of this section, in table 2.1, the coefficients a_ℓ and b_0 are reproduced for $k = 1,2,\ldots,6$ (cf. [5, p. 242]). Note that in (1.1) the vector function $\vec{z}(x)$ defined by

$$\vec{z}(x) = \int_{x_0}^{x} \vec{K}(x,t,\vec{y}(t))dt,$$

depends on the "whole history" of the solution $\vec{y}(s)$ for $0 \le s \le x$; Application of (2.2) to equation (1.1) yields the following (formal) scheme

$$(2.3) \qquad \begin{cases} \vec{y}_{n+1} = \sum_{\ell=1}^{k} a_\ell \vec{y}_{n+1-\ell} + hb_0\vec{F}(x_{n+1},\vec{y}_{n+1},\vec{z}(x_{n+1})), \\ \vec{z}(x_{n+1}) = \int_{x_0}^{x_{n+1}} \vec{K}(x_{n+1},t,\vec{y}(t))dt. \end{cases}$$

We observe that the computability of scheme (2.3) actually depends on the computability of $\vec{z}(x_{n+1})$. Therefore, we have to *specify a quadrature formula* for approximating $\vec{z}(x_{n+1})$ of the form

$$(2.4) \qquad \vec{z}(x_{n+1}) \simeq \vec{z}_{n+1} = \sum_{j=0}^{n+1} w_{n+1\,j}\vec{K}(x_{n+1},x_j,\vec{y}_j).$$

From (2.3) and (2.4) we have the following computational scheme

$$(2.5) \qquad \begin{cases} \vec{y}_{n+1} = \sum_{\ell=1}^{k} a_\ell \vec{y}_{n+1-\ell} + hb_0\vec{F}(x_{n+1},\vec{y}_{n+1},\vec{z}_{n+1}), \\ \vec{z}_{n+1} = \sum_{j=0}^{n+1} w_{n+1\,j}\vec{K}(x_{n+1},x_j,\vec{y}_j). \end{cases} \qquad n = k-1,\ldots$$

This scheme requires in the n-th step the evaluation of $\vec{K}(x_{n+1},x_0,\vec{y}_0),\ldots,$ $\vec{K}(x_{n+1},x_n,\vec{y}_n)$, the solution of a system of nonlinear equations for $\vec{y}_{n+1}$ and the evaluation of $\vec{K}(x_{n+1},x_{n+1},\vec{y}_{n+1})$. Moreover, k starting vectors $\vec{y}_0,\vec{y}_1,\ldots,\vec{y}_{k-1}$ must be given. In this report we assume that these vectors have been computed to adequate accuracy by some starting procedure. In the sequel we will denote this combined scheme (2.5) by $\{\rho,\sigma;W\}$ where $\{\rho,\sigma\}$ and $\{W\}$ refer to the backward differentiation and quadrature formula, respectively.

For the quadrature formula (2.4) we have chosen the *Gregory formula* (see e.g. [9]), which has the property that

$$\text{(2.6)}\qquad w_{n+1\,j} = w_{nj}, \qquad j = 0,1,\ldots,n-q+1, \qquad n \geq q-1,$$

if q is the order of the quadrature formula used. In order to be more precise we give the following definition:

<u>DEFINITION</u>. A quadrature formula is said to be of *order* q if q is the largest integer such that

$$\int_{x_0}^{x_n} \phi(t)dt - \sum_{j=0}^{n} w_{nj}\phi(x_j) = C_n(h)h^q$$

and $\lim_{\substack{h\to 0\\ nh\to x_n}} |C_n(h)| < \infty$

for all sufficiently smooth integrands ϕ.

To conclude this section we give the weights of the quadrature formulas (2.4) in matrix-form $W_k = (w_{ij})$, (cf. [4])

Second order formula (q = 2)

$$\text{(2.7)}\qquad W_2 = \frac{h}{2}\begin{bmatrix} 1 & 1 & & & & & \\ 1 & 2 & 1 & & \bigcirc & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & & \cdot & & & \\ \cdot & \cdot & & & \cdot & & \\ 1 & 2 & \cdot & \cdot & \cdot & 2 & 1 \end{bmatrix} ;$$

Third order formula (q = 3)

$$
(2.8) \qquad W_3 = \frac{h}{12}\begin{bmatrix}
6 & 6 & & & & & & \\
5 & 14 & 5 & & & \bigcirc & & \\
5 & 13 & 13 & 5 & & & & \\
5 & 13 & 12 & 13 & 5 & & & \\
\cdot & \cdot & \cdot\;\cdot & & \cdot & & & \\
\cdot & \cdot & \cdot & & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & & & \cdot & \cdot & \\
5 & 13 & 12 & \cdots & & 12 & 13 & 5
\end{bmatrix} ;
$$

Fourth order formula (q = 4)

$$
(2.9) \qquad W_4 = \frac{h}{24}\begin{bmatrix}
8 & 32 & 8 & & & & & & \\
9 & 27 & 27 & 9 & & & \bigcirc & & \\
9 & 28 & 22 & 28 & 9 & & & & \\
9 & 28 & 23 & 23 & 28 & 9 & & & \\
9 & 28 & 23 & 24 & 23 & 28 & 9 & & \\
\cdot & \cdot & \cdot & \cdot\;\cdot & & & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & \\
\cdot & \cdot & \cdot & \cdot & & \cdot & & & \cdot \\
9 & 28 & 23 & 24 & \cdots & 24 & 23 & 28 & 9
\end{bmatrix} ;
$$

Fifth order formula (q = 5)

$$
(2.10) \qquad W_5 = \frac{h}{720}\begin{bmatrix}
270 & 810 & 810 & 270 & & & & & & & \\
251 & 916 & 546 & 916 & 251 & & & & \bigcirc & & \\
251 & 897 & 652 & 652 & 897 & 251 & & & & & \\
251 & 897 & 633 & 758 & 633 & 897 & 251 & & & & \\
251 & 897 & 633 & 739 & 739 & 633 & 897 & 251 & & & \\
251 & 897 & 633 & 739 & 720 & 739 & 633 & 897 & 251 & & \\
\cdot & \cdot & \cdot & \cdot & \cdot\;\cdot & & & & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & & & & \cdot \\
251 & 897 & 633 & 739 & 720 & \cdots & 720 & 739 & 633 & 897 & 251
\end{bmatrix} ;
$$

Sixth order formula (q = 6)

(2.11) $\frac{1440}{h} W_6 =$

$$\begin{bmatrix} 448 & 2048 & 768 & 2048 & 448 & & & & & & \\ 475 & 1875 & 1250 & 1250 & 1875 & 475 & & & & & \\ 475 & 1902 & 1077 & 1732 & 1077 & 1902 & 475 & & & & \\ 475 & 1902 & 1104 & 1559 & 1559 & 1104 & 1902 & 475 & & & \\ 475 & 1902 & 1104 & 1586 & 1386 & 1586 & 1104 & 1902 & 475 & & \\ 475 & 1902 & 1104 & 1586 & 1413 & 1413 & 1586 & 1104 & 1902 & 475 & \\ 475 & 1902 & 1104 & 1586 & 1413 & 1440 & 1413 & 1586 & 1104 & 1902 & 475 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot & & & & \cdot \\ 475 & 1902 & 1104 & 1586 & 1413 & 1440 & \cdots & 1440 & 1413 & 1586 & 1104 & 1902 & 475 \end{bmatrix}$$

k	c_k	$b_0{*}c_k$	$a_1{*}c_k$	$a_2{*}c_k$	$a_3{*}c_k$	$a_4{*}c_k$	$a_5{*}c_k$	$a_6{*}c_k$
1	1	1	1					
2	3	2	4	-1				
3	11	6	18	-9	2			
4	25	12	48	-36	16	-3		
5	137	60	300	-300	200	-75	12	
6	147	60	360	-450	400	-225	72	-10

Table 2.1. Coefficients of the backward differentiation formulas for k = 1(1)6

3. CONSISTENCY AND CONVERGENCE

Theorems concerning convergence of general linear multi-step methods for solving VIDEs are well known and can be found in [6] and [1]. Although their treatment concerns scalar VIDEs, their results are applicable to systems of VIDEs. We only give the convergence theorem.

THEOREM. *Assume*

i) $\vec{F}$ *and* $\vec{K}$ *satisfy* (1.i - 1.iii),

ii) $\{\rho,\sigma\}$ *is convergent (i.e. zero-stable and consistent), and*

iii) W *is of least order one.*

then $\{\rho;\sigma;W\}$ *is convergent.*

The proof is omitted. Instead we will derive an estimate for the local error at the point $x = x_{n+1}$. To this end we define the vectors

$$\vec{Z}(x_{n+1}) = \int_{x_0}^{x_{n+1}} \vec{K}(x_{n+1},t,\vec{y}(t))dt;$$

$$\vec{z}(x_{n+1}) = \sum_{j=0}^{n+1} w_{n+1j} \vec{K}(x_{n+1},x_j,\vec{y}(x_j));$$

$$\vec{z}_{n+1} = \sum_{j=0}^{n+1} w_{n+1j} \vec{K}(x_{n+1},x_j,\vec{y}_j).$$

Further, we assume that

(3.a) $\{\rho,\sigma\}$ is of order p,

(3.b) W is of order q, and

(3.c) no previous errors have been made, that is

$$\vec{y}_j = \vec{y}(x_j), \qquad j = 0,\dots,n.$$

(the so-called "localizing assumption").

Now, let $\vec{y}(x)$ be the exact solution of (1.1) then (3.a) and (3.b) yield

$$\vec{y}(x_{n+1}) = \sum_{\ell=1}^{k} a_\ell \vec{y}(x_{n+1-\ell}) + b_0 h\vec{F}(x_{n+1},\vec{y}(x_{n+1}),\vec{Z}(x_{n+1})) + + \vec{C}_1(x_{n+1},h)h^{p+1} \tag{3.1}$$

and

$$(3.2) \qquad \vec{Z}(x_{n+1}) = \vec{z}(x_{n+1}) + \vec{C}_2(x_{n+1},h)h^q ,$$

where $\vec{C}_1(x_{n+1},h)$ and $\vec{C}_2(x_{n+1},h)$ are bounded functions as $h \to 0$. That is, for these functions we assume

$$\|\vec{C}_1(x_{n+1},h)\| \le c_1 \quad \text{and} \quad \|\vec{C}_2(x_{n+1},h)\| \le c_2 \quad \text{as} \quad h \to 0.$$

By virtue of (3.c) we have

$$(3.3) \qquad \vec{y}_{n+1} = \sum_{\ell=1}^{k} a_\ell \, \vec{y}(x_{n+1-\ell}) + b_0 h\vec{F}(x_{n+1},\vec{y}_{n+1},\vec{z}_{n+1})$$

$$(3.4) \qquad \vec{z}_{n+1} = \vec{z}(x_{n+1}) - w_{n+1n+1}\{\vec{K}(x_{n+1},x_{n+1},\vec{y}(x_{n+1})) - \vec{K}(x_{n+1},x_{n+1},\vec{y}_{n+1})\}$$

Hence, for the local error we have the relation

$$\begin{aligned}(3.5) \qquad \vec{y}(x_{n+1}) - \vec{y}_{n+1} = {} & b_0 h\vec{F}(x_{n+1},\vec{y}(x_{n+1}),\vec{Z}(x_{n+1})) \\ & - b_0 h\vec{F}(x_{n+1},\vec{y}(x_{n+1}),\vec{z}(x_{n+1})) + b_0 h\vec{F}(x_{n+1},\vec{y}(x_{n+1}),\vec{z}(x_{n+1})) \\ & - b_0 h\vec{F}(x_{n+1},\vec{y}_{n+1},\vec{z}(x_{n+1})) + b_0 h\vec{F}(x_{n+1},\vec{y}_{n+1},\vec{z}(x_{n+1})) \\ & - b_0 h\vec{F}(x_{n+1},\vec{y}_{n+1},\vec{z}_{n+1}) + \vec{C}_1(x_{n+1},h)h^{p+1} .\end{aligned}$$

Using the Lipschitz conditions on $\vec{F}$ we have

$$\begin{aligned}(3.6) \qquad \|\vec{y}(x_{n+1}) - \vec{y}_{n+1}\| \le {} & b_0 hL_2\|\vec{Z}(x_{n+1}) - \vec{z}(x_{n+1})\| + \\ & + b_0 hL_1\|\vec{y}(x_{n+1}) - \vec{y}_{n+1}\| + b_0 hL_2\|\vec{z}(x_{n+1}) - \vec{z}_{n+1}\| \\ & + c_1 h^{p+1} .\end{aligned}$$

Using (3.2) we have

(3.7) $$\|\vec{Z}(x_{n+1}) - \vec{z}(x_{n+1})\| \le c_2 h^q.$$

The Lipschitz condition for $\vec{K}$ together with (3.4) yields

(3.8) $$\|\vec{z}(x_{n+1}) - \vec{z}_{n+1}\| \le |w_{n+1n+1}| L_3 \|\vec{y}(x_{n+1}) - \vec{y}_{n+1}\|.$$

Piecing the bits together, we have the following estimate for the local error.

(3.9) $$\|\vec{y}(x_{n+1}) - \vec{y}_{n+1}\| \le \frac{1}{1-L} [b_0 L_2 c_2 h^{q+1} + c_1 h^{p+1}]$$

where

$$L = b_0 h[L_1 + L_2 L_3 |w_{n+1n+1}|].$$

For $h \to 0$, we have $L = \mathcal{O}(h)$ so (3.9) yields

(3.10) $$\|\vec{y}(x_{n+1}) - \vec{y}_{n+1}\| = \mathcal{O}(h^{q+1}) + \mathcal{O}(h^{p+1}) \qquad \text{as } h \to 0.$$

By virtue of (3.10) we define the order of a method $\{\rho,\sigma;W\}$ as follows.

DEFINITION:

let $\{\rho,\sigma\}$ be of order p and let W be of order q then the order r of $\{\rho,\sigma;W\}$ is defined by

$$r = \min(p,q).$$

It should be observed that the coefficient of h^{q+1} in (3.9) depends on L_2. Hence, for problems with larger Lipschitz constant L_2, the local error due to quadrature may become dominant.

In our case, a k-step *backward differentiation formula* (which is known to have order k) *combined with a k-th order Gregory formula* yields a convergent method $\{\rho,\sigma;W\}$ of order k. However, in order to ensure a good and reliable computational scheme, we have to investigate its sensitivity for (small) perturbations. This stability analysis is carried out in the next section.

4. STABILITY

The stability analysis of integration formulas for VIDEs is usually based on the linear equation (see [1])

$$(4.1)\qquad y'(x) = \xi y(x) + \eta \int_0^x y(t)dt.$$

Scheme (2.5) applied to (4.1) yields

$$(4.2)\qquad y_{n+1} = \sum_{\ell=1}^{k} a_\ell y_{n+1-\ell} + hb_0(\xi y_{n+1} + \eta z_{n+1})$$

where

$$(4.3)\qquad z_{n+1} = \sum_{j=0}^{n+1} w_{n+1j}\, y_j\ .$$

We may write (4.3) as

$$(4.4)\qquad z_{n+1} = z_n + \sum_{j=0}^{n+1} (w_{n+1j} - w_{nj})y_j$$

Using (2.6) with $q = k$ and denoting

$$(4.5)\qquad \nabla w_{n+1j} = w_{n+1j} - w_{nj}$$

we have

$$(4.6)\qquad z_{n+1} = z_n + \sum_{j=n-k+2}^{n+1} \nabla w_{n+1j}\, y_j.$$

Because of the fact that ∇w_{n+1j} does not depend on n we may write (4.2) and (4.6) as

(4.2') $$y_{n+1} = \sum_{\ell=1}^{k} a_\ell y_{n+1-\ell} + hb_0(\xi y_{n+1} + \eta z_{n+1})$$

(4.3') $$z_{n+1} = z_n + h \sum_{\ell=0}^{k-1} b_\ell^* \, y_{n+1-\ell}$$

For the values b_ℓ^* we have (using (2.7 - 2.11)).

k	c_k	$b_0^* {*} c_k$	$b_1^* {*} c_k$	$b_2^* {*} c_k$	$b_3^* {*} c_k$	$b_4^* {*} c_k$	$b_5^* {*} c_k$
2	2	1	1				
3	12	5	8	-1			
4	24	9	19	-5	1		
5	720	251	646	-264	106	-19	
6	1440	475	1427	-798	482	-173	27

These are the coefficients of the well-known Adams-Moulton formulas. In fact, equation (4.1) can be written as a system of ODEs

(4.1') $$\begin{cases} y' = \xi y + \eta z \\ z' = y \end{cases}$$

and scheme (4.2') - (4.3') can be looked at as a combination of a k-th order, k-step backward differentiation formula and a k-th order, (k-1)-step Adams-Moulton formula for solving the first and second differential equation in (4.1'), respectively. In [1] and [7] this idea is employed for general linear multistep formulas and quadrature formulas which reduce to linear multistep methods when applied to the model equation. Actually, in case the kernel function does not depend on x the stability problem for VIDEs is reduced to the stability problem of a composite linear multistep method, for solving ODEs. The stability regions of such methods are only known for some low order formulas (see [1]). Following [4], in this report we are able, due to the special form of our formulas, to investigate a more general class of equations by only imposing the following restriction on the kernel function:

(4.7) $$\vec{K}(x,t,\vec{y}) = \vec{G}(x,t) + H(x)\vec{y} \; ,$$

where $\vec{G}$ is an arbitrary vector function and H(x) a matrix function. Scheme (2.5) then yields

$$(4.8)\qquad \begin{cases} \vec{y}_{n+1} = \sum_{\ell=1}^{k} a_\ell \vec{y}_{n+1-\ell} + b_0 h\vec{F}(x_{n+1},\vec{y}_{n+1},\vec{z}_{n+1}) \\ \vec{z}_{n+1} = \sum_{j=0}^{n+1} w_{n+1j}\, \vec{G}(x_{n+1},x_j) + H(x_{n+1}) \sum_{j=0}^{n+1} w_{n+1j}\, y_j \, . \end{cases}$$

If the vectors $\vec{y}_j$, $j = 0,\dots,n$ are perturbed by $\Delta\vec{y}_j$ then (4.8) yields the error equation for $\Delta\vec{y}_{n+1}$ (provided that $\Delta\vec{y}_j$ is sufficiently small)

$$(4.9)\qquad \Delta\vec{y}_{n+1} = \sum_{\ell=1}^{k} a_\ell \Delta\vec{y}_{n+1-\ell} + b_0 h\left[J^{(1)}_{n+1}\Delta\vec{y}_{n+1} + J^{(2)}_{n+1}H_{n+1} \sum_{j=0}^{n+1} w_{n+1j}\Delta\vec{y}_j\right]$$

where

$$J^{(1)}_{n+1} = \frac{\partial\vec{F}}{\partial\vec{y}}(x_{n+1},\vec{y}_{n+1},\vec{z}_{n+1});\; J^{(2)}_{n+1} = \frac{\partial\vec{F}}{\partial\vec{z}}(x_{n+1},\vec{y}_{n+1},\vec{z}_{n+1});$$

$$H_{n+1} = H(x_{n+1}).$$

Defining the quantity $\Delta\vec{s}_{n+1}$ by

$$\Delta\vec{s}_{n+1} = \sum_{j=0}^{n+1} w_{n+1j}\Delta\vec{y}_j$$

and recalling (4.4) - (4.6) we obtain

$$(4.10)\qquad \Delta\vec{s}_{n+1} = \Delta\vec{s}_n + h\sum_{\ell=0}^{k-1} b^*_\ell\, \Delta\vec{y}_{n+1-\ell} \, .$$

Denoting the vector of perturbations $\Delta\vec{v}_n$ by

$$\Delta\vec{v}_n = (\Delta\vec{s}_n,\Delta\vec{y}_n,\dots,\Delta\vec{y}_{n-k+1})^T$$

we can write (4.9) in the form

$$(4.9') \qquad A_n \Delta\vec{v}_{n+1} = B_n \Delta\vec{v}_n,$$

where A_n and B_n are the matrices

$$A_n = \begin{bmatrix} I & -b_0^* hI & 0 & \cdots & 0 \\ -b_0 hJ_{n+1}^{(2)} H_{n+1} & I - b_0 hJ_{n+1}^{(1)} & 0 & \cdots & 0 \\ 0 & 0 & I & & \bigcirc \\ & \bigcirc & & \ddots & \\ & & & & I \end{bmatrix},$$

$$B_n = \begin{bmatrix} I & b_1^* hI & \cdots & \cdots & b_{k-1}^* hI & 0 \\ 0 & a_1 I & a_2 I & & a_{k-1} I & a_k I \\ 0 & I & 0 & \cdots & \cdots & 0 \\ & & \ddots & & \bigcirc & \vdots \\ & \bigcirc & & \ddots & & \vdots \\ & & & & I & 0 \end{bmatrix}.$$

Now, we define our scheme as *locally stable* if all eigenvalues ζ of $A_n^{-1}B_n$ are within the unit circle. This yields the characteristic equation

$$(4.10) \qquad \det(\zeta A_n - B_n) = 0.$$

Although (4.10) can be used for analyzing general systems of VIDEs we will only consider scalar VIDEs. We remark that by taking

$$J_{n+1}^{(1)} = \xi_{n+1} = \text{constant}; \quad J_{n+1}^{(2)}\, H_{n+1} = \eta_{n+1} = \text{constant}$$

we can use a result of BRUNNER & LAMBERT [1], yielding the characteristic equation (now with non-constant ξ_{n+1} and η_{n+1}),

$$(4.11) \qquad \tilde{\rho}(\zeta)[\rho(\zeta) - h\xi_{n+1}\sigma(\zeta)] - h^2\eta_{n+1}\tilde{\sigma}(\zeta)\sigma(\zeta) = 0$$

where $\{\rho,\sigma\}$ and $\{\tilde{\rho},\tilde{\sigma}\}$ are the defining polynomials of a k-th order backward differentiation formula and a k-th order Adams-Moulton formula, respectively. To be more precise we have

$$\begin{cases} \rho(\zeta) = \zeta^k - a_1\zeta^{k-1} \cdots a_k \\ \sigma(\zeta) = b_0\zeta^k \end{cases}$$

$$\begin{cases} \rho(\zeta) = \zeta^{k-1} - \zeta^{k-2} \\ \tilde{\sigma}(\zeta) = b_0^*\zeta^{k-1} + b_1^*\zeta^{k-2} + \ldots + b^*_{k-1}. \end{cases}$$

Because the characteristic equations (4.11) are equivalent to those obtained in [4], the stability regions displayed in that work can readily be used for our purposes. Therefore, we reproduce these regions in the $(h\xi,h^2\eta)$ - plane in the figures 4.1 - 4.5. The regions in the first quadrant of the $(h\xi,h^2\eta)$-plane are omitted because the VIDE is inherently unstable in that part of the plane (see [1]). It should be noted that the second order formula is A_0-stable, i.e. it contains the whole third quadrant.

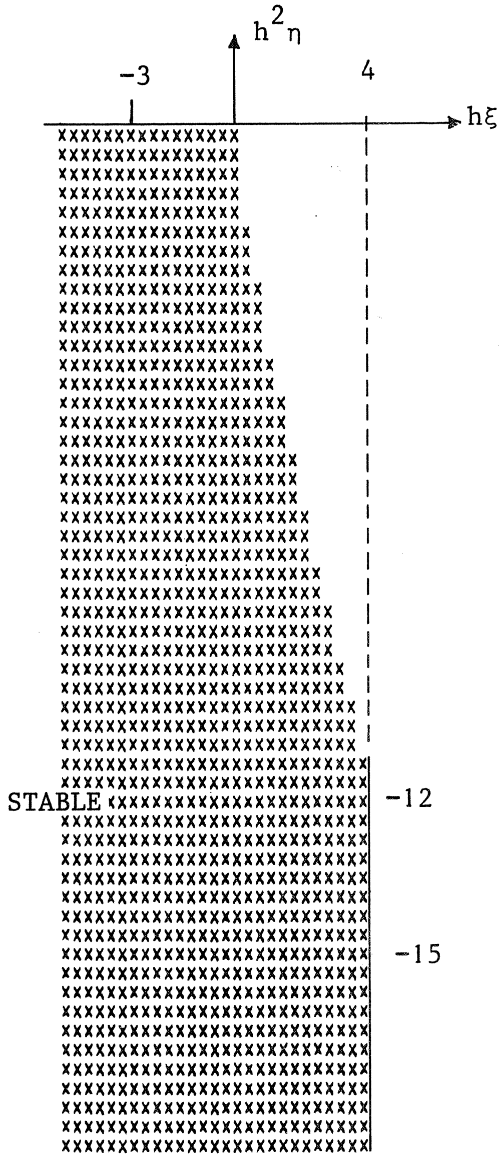
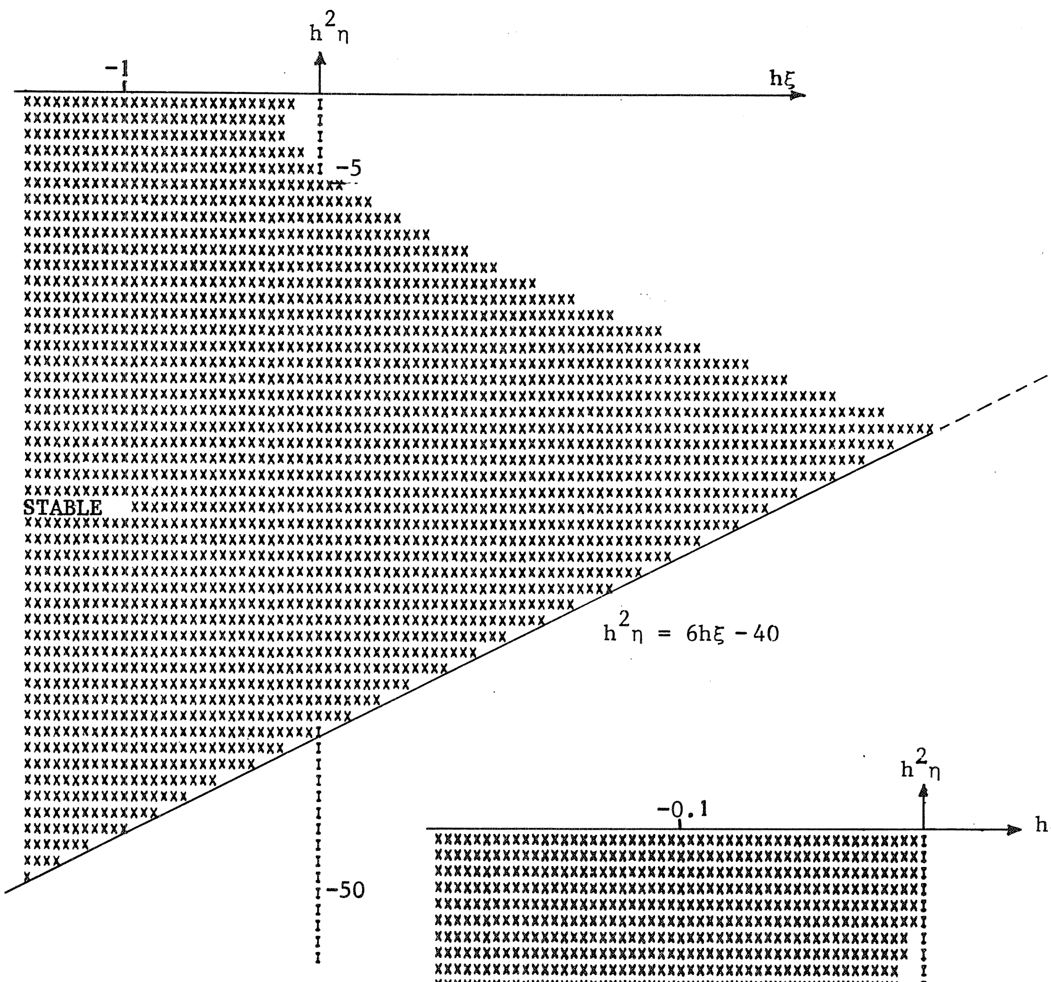
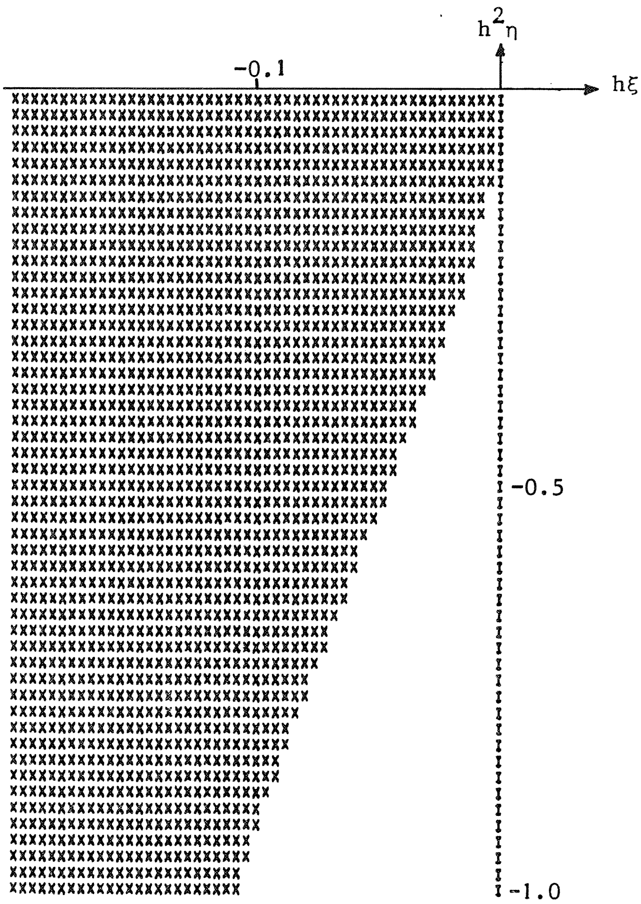


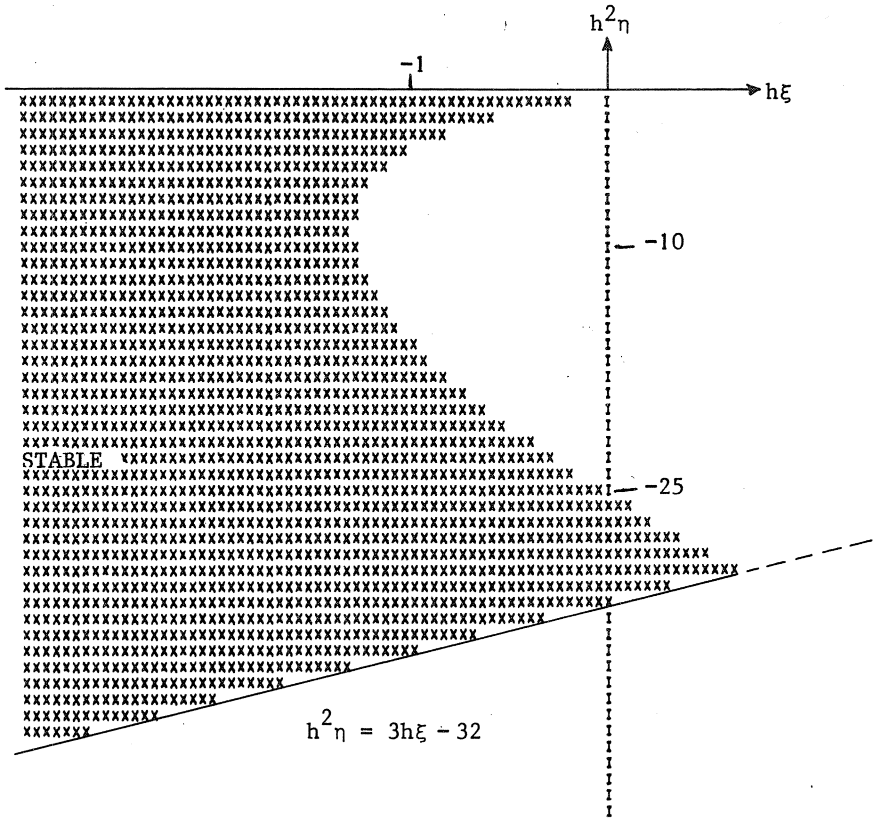
Figure 4.1 k=2





Detail
near
the
origin

Figure 4.2 k=3



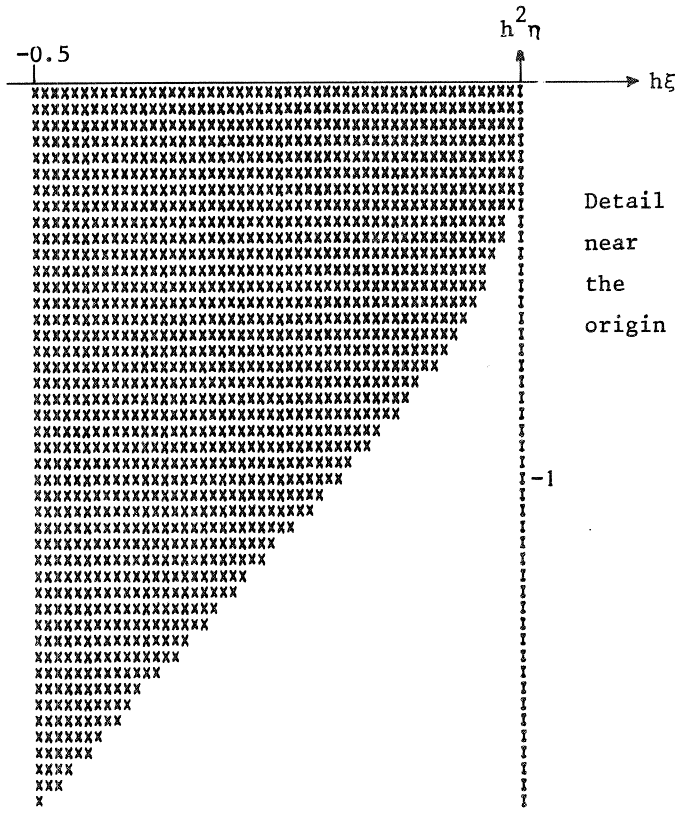
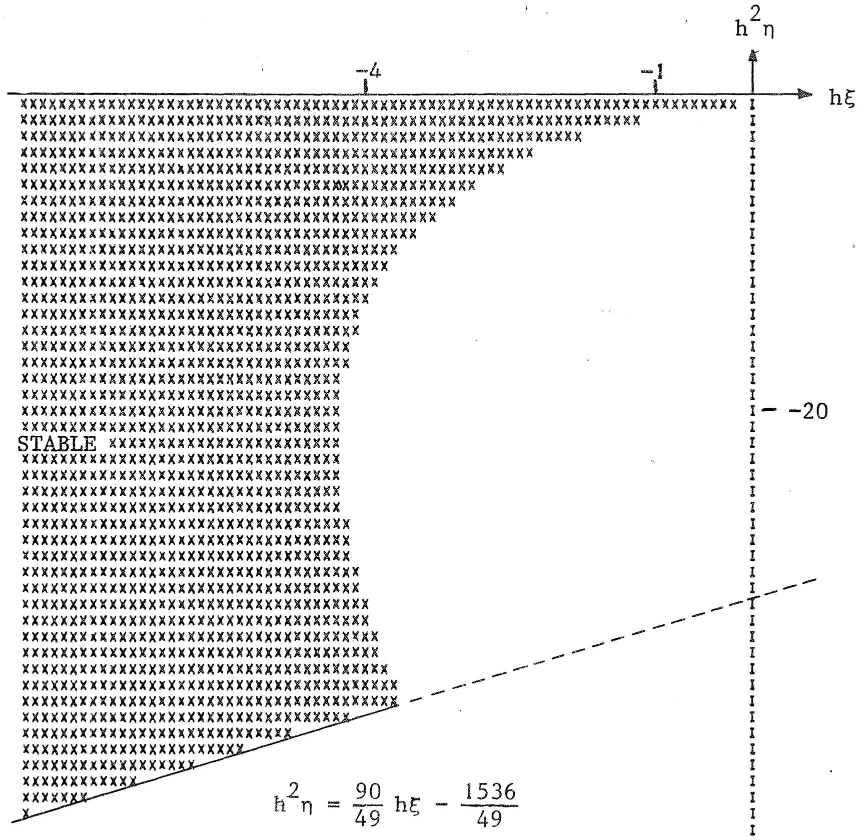
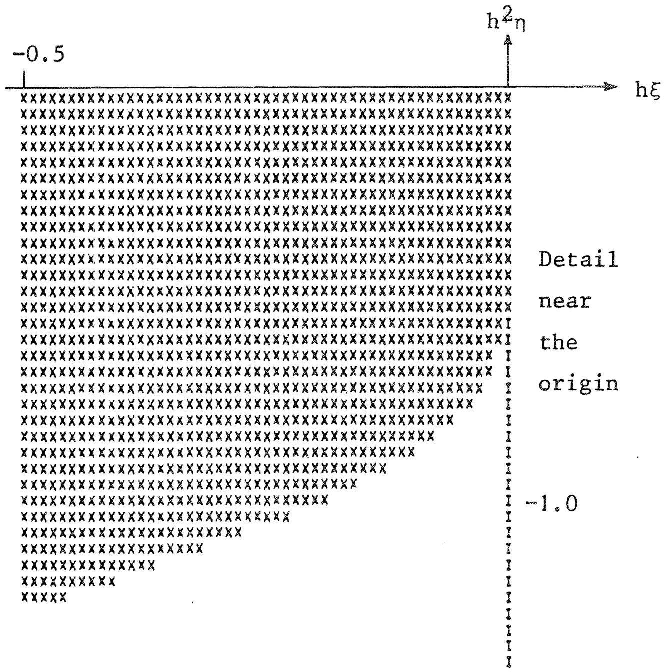


Figure 4.3 <u>k=4</u>

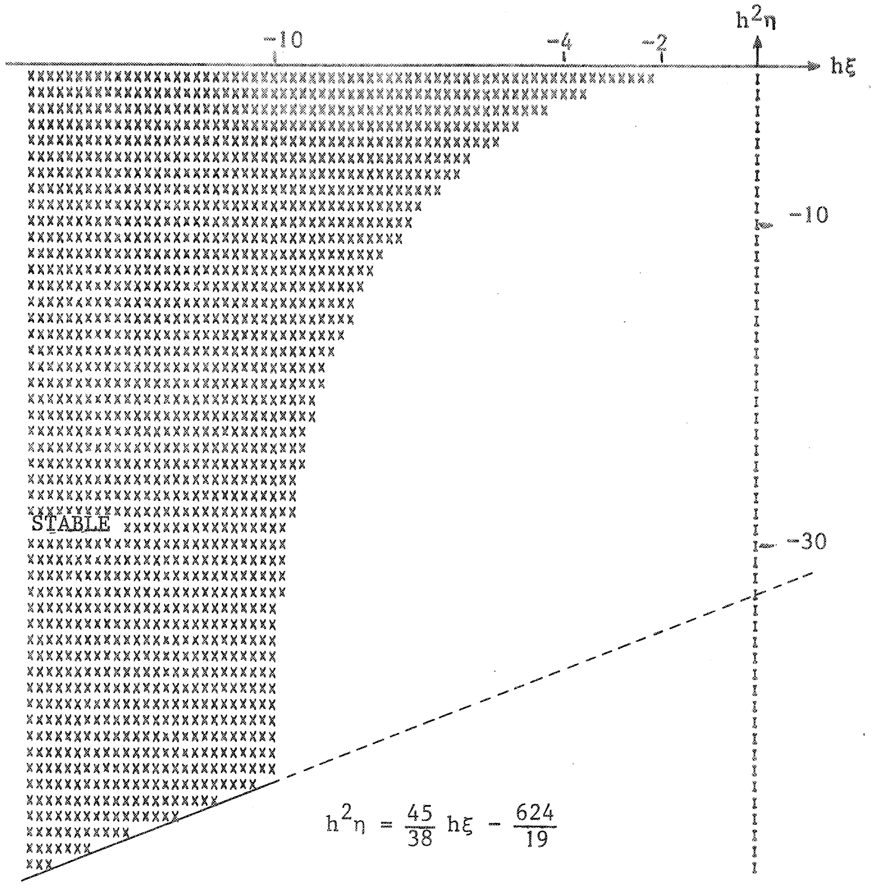


h²η
-4
-1
hξ
-20
STABLE
$h^2\eta = \frac{90}{49} h\xi - \frac{1536}{49}$



h²η
-0.5
hξ
Detail
near
the
origin
-1.0

Figure 4.4 k=5



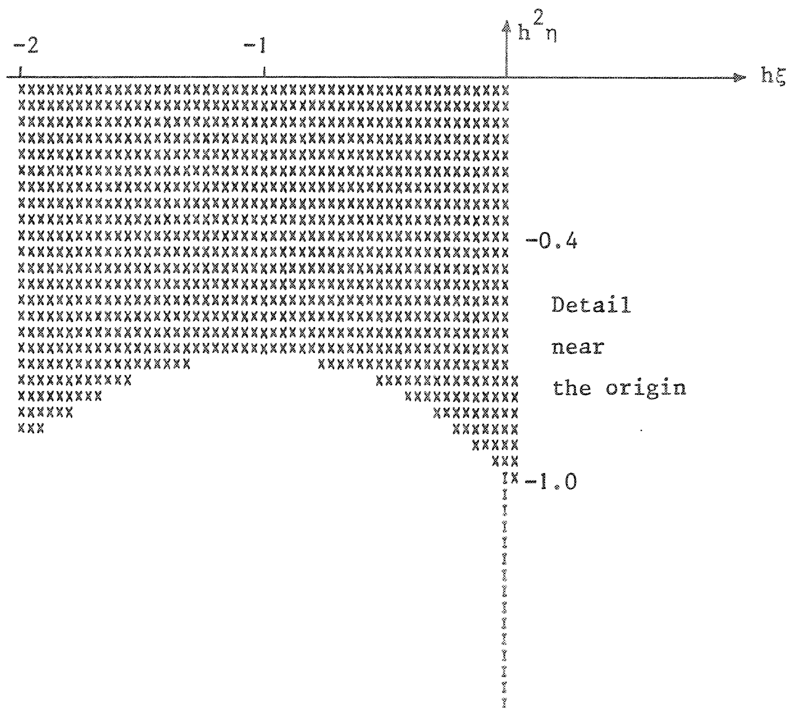


Figure 4.5 <u>k=6</u>

5. APPLICATION TO VOLTERRA INTEGRAL EQUATIONS.

In this section we derive a computational scheme for the solution of Volterra integral equations of the second kind of the following form

$$f(x) = g(x) + \int_{x_0}^{x} K(x,t,f(t))dt, \tag{5.1}$$

where g and K are prescribed functions. Therefore, we transform equation (5.1) into the integro-differential equation

$$f'(x) = g'(x) + \int_{x_0}^{x} \frac{\partial K}{\partial x}(x,t,f(t))dt + K(x,x,f(x)), \tag{5.2}$$

with initial condition $f(x_0) = g(x_0)$.

In case the differentiations of g and K are carried out explicitly, scheme (2.5) is readily applicable and yields

$$\begin{cases} f_{n+1} = \sum_{\ell=1}^{k} a_\ell f_{n+1-\ell} + b_0 h[g'(x_{n+1}) + K(x_{n+1},x_{n+1},f_{n+1}) + z_{n+1}] \\ z_{n+1} = \sum_{j=0}^{n+1} w_{n+1j} \frac{\partial K}{\partial x}(x_{n+1},x_j,f_j) \end{cases} \tag{5.3}$$

However, in realistic problems, such an explicit differentiation may be undesirable, very tedious or perhaps impossible. In that case numerical approximations of these derivatives are needed; one may take the class of formulas

$$\begin{cases} b_0 h \frac{\partial K}{\partial x}(x_{n+1},t,f) = \sum_{\ell=-k_1}^{k_2} d_\ell K(x_{n+1+\ell},t,f) \quad \text{and} \\ b_0 h g'(x_{n+1}) = \sum_{\ell=-k_1}^{k_2} d_\ell g(x_{n+1+\ell}) \end{cases} \tag{5.4}$$

i.e., differentiation formulas derived from Newton-backward ($k_2 = 0$), Newton-forward ($k_1 = 0$) or other interpolation formulas. Requiring that these approximations are at least of first order (i.e., exact for constant and linear functions in x) yields the relations

$$\text{(5.5)} \qquad \sum_{\ell=-k_1}^{k_2} d_\ell = 0; \qquad \sum_{\ell=-k_1}^{k_2} d_\ell x_{n+1+\ell} = b_0 h.$$

Substituting (5.4) in (5.3) yields the scheme

$$\text{(5.6)} \qquad f_{n+1} = \sum_{\ell=1}^{k} a_\ell f_{n+1-\ell} + b_0 hK(x_{n+1}, x_{n+1}, f_{n+1}) + \sum_{\ell=-k_1}^{k_2} d_\ell [g(x_{n+1+\ell}) + \sum_{j=0}^{n+1} w_{n+1j} K(x_{n+1+\ell}, x_j, f_j)]$$

Defining the function

$$\text{(5.7)} \qquad Q_{n+1}(x) = g(x) + \sum_{j=0}^{n+1} w_{n+1j} K(x, x_j, f_j)$$

we may write (5.6) as

$$\text{(5.8)} \qquad f_{n+1} = \sum_{\ell=1}^{k} a_\ell f_{n+1-\ell} + b_0 hK(x_{n+1}, x_{n+1}, f_{n+1}) + \sum_{\ell=-k_1}^{k_2} d_\ell Q_{n+1}(x_{n+1+\ell})$$

Due to our special choice of quadrature formulas we have (cf. (4.4'))

$$\text{(5.9)} \qquad Q_{n+1}(x) = Q_n(x) + \sum_{j=n-k+2}^{n} \nabla w_{n+1j} K(x, x_j, f_j) + w_{n+1n+1} K(x, x_{n+1}, f_{n+1})$$

Now we are able to write down the following computational scheme, which is inspired by the scheme given in [4].

$$f_{n+1} = \sum_{\ell=1}^{k} a_\ell f_{n+1-\ell} + b_0 h K(x_{n+1}, x_{n+1}, f_{n+1}) + \\ + \sum_{\ell=-k_1}^{k_2} d_\ell \, Q_{n+1}(x_{n+1+\ell}),$$

$$Q_{n+1}(x_{n+1+\ell}) = Q_n(x_{n+1+\ell}) + \sum_{j=n-k+2}^{n} \nabla w_{n+1j} K(x_{n+1+\ell}, x_j, f_j) + \\ + w_{n+1n+1} K(x_{n+1+\ell}, x_{n+1}, f_{n+1}) \\ \ell = -k_1, \ldots, 0, \ldots, k_2 - 1, \tag{5.10}$$

$$Q_{n+1}(x_{n+1+k_2}) = g(x_{n+1+k_2}) + \sum_{j=0}^{n} w_{n+1j} K(x_{n+1+k_2}, x_j, f_j) + \\ + w_{n+1n+1} K(x_{n+1+k_2}, x_{n+1}, f_{n+1}).$$

In stepping from x_n to x_{n+1} this scheme requires the evaluation of $g(x_{n+1+k_2})$, the evaluation of $K(x_{n+1+k_2}, x_0, f_0), \ldots, K(x_{n+1+k_2}, x_n, f_n)$, the solution of a system of equations for f_{n+1} and finally the evaluation of $K(x_{n+2-k_1}, x_{n+1}, f_{n+1}), \ldots, K(x_{n+1+k_2}, x_{n+1}, f_{n+1})$. As in section 2, starting values must be given. We remark that in case $k_1 \neq 0$ values of $K(x,t,f)$ are needed for $x < t$. This may cause problems if the kernel function is not defined there; however, this problem is eliminated by taking $k_1 = 0$. Notice that by taking $k_2 \neq 0$ we use values of the kernel function both in "past" and in "future" and one might ask whether this has any effect on the numerical stability of scheme (5.10). This we analyse by considering the same class of model equations as in [4] i.e., we assume the kernel function to be of the form

$$K(x,t,f) = A(t,f) + xHf \; .$$

Applying scheme (5.6) to this model equation yields

$$(5.11)\qquad f_{n+1} = \sum_{\ell=1}^{k} a_\ell f_{n+1-\ell} + b_0 h[A(x_{n+1},f_{n+1}) + x_{n+1}Hf_{n+1}] + + \sum_{\ell=-k_1}^{k_2} d_\ell\{g(x_{n+1+\ell}) + \sum_{j=0}^{n+1} w_{n+1j}[A(x_j,f_j) + x_{n+1+\ell}Hf_j]\},$$

and for the error equation we have

$$\Delta f_{n+1} = \sum_{\ell=1}^{k} a_\ell \Delta f_{n+1-\ell} + b_0 h \frac{\partial K}{\partial f}(x_{n+1},x_{n+1},f_{n+1})\Delta f_{n+1} + + \sum_{j=0}^{n+1} w_{n+1j} \frac{\partial A}{\partial f}(x_j,f_j)\Delta f_j \left(\sum_{\ell=-k_1}^{k_2} d_\ell\right) + H \sum_{j=0}^{n+1} w_{n+1j}\Delta f_j \left(\sum_{\ell=-k_1}^{k_2} d_\ell x_{n+1-\ell}\right)$$

which gives, in view of (5.5)

$$(5.12)\qquad \Delta f_{n+1} = \sum_{\ell=1}^{k} a_\ell \Delta f_{n+1-\ell} + b_0 h \frac{\partial K}{\partial f}(x_{n+1},x_{n+1},f_{n+1})\Delta f_{n+1} + + b_0 hH \sum_{j=0}^{n+1} w_{n+1j}\,\Delta f_j .$$

This error equation is the same as in [4], where the derivatives were approximated by means of a k-th order (backward) differentiation formula (i.e., $k_1 = k$ and $k_2 = 0$). *An important conclusion is that the stability analysis does not depend on the numerical differentiation formula (5.4) used.* Of course, in order to have a global error of order k the coefficients d_ℓ must be chosen such that

$$\| b_0 h g'(x_{n+1}) - \sum_{\ell=-k_1}^{k_2} d_\ell g(x_{n+1+\ell}) \| = O(h^{k+1}) \qquad \text{as} \qquad h \to 0.$$

6. NUMERICAL EXPERIMENTS

In this section we present numerical experiments with backward differentiation formulas combined with Gregory quadrature formulas of orders 2 up to and including 6. To this end, we have constructed a number of test problems with known exact solutions. Since it is our purpose to *test the convergence and partly the stability*, rather than to test an automatic VIDE-solver, a minimum of strategies has been incorporated in our implementation. That is:

i) integration was performed using a constant stepsize;

ii) the required starting values were computed from the exact solution;

iii) Newton-iteration has been used for the solution of the non-linear equations.

With respect to iii) we remark the following: in order to preserve the *implicitness* of our formulas the number of iterations was not fixed. When the number of iterations, however, exceeded 6, divergence was assumed and the integration process was terminated. As a first approximation of y_{n+1} the value of y_n was used. The stop criterion was based on a Newton correction less than $_{10}{-12}$. All calculations have been performed on a CDC CYBER 173/73-28 using 14 significant digits.

In the tables of results we give the relative error at the endpoint of integration.

Problem 6.1

$$(6.1) \qquad \begin{cases} y'(x) = -3y(x) - 2\int_0^x y(t)dt, & 0 \le x \le 6 \\ y(0) = 1. \end{cases}$$

with exact solution y(x) = 2 exp(-2x) - exp(-x). This problem is linear in y and z. The results are listed in table 6.1.

h	rel. error at x = 6 k=2	k=3	k=4	k=5	k=6
1/2	$6.6_{10}{-1}$	$9.4_{10}{-1}$	$7.0_{10}{-1}$	$9.7_{10}{-1}$	$4.0_{10}{-2}$
1/4	$1.9_{10}{-1}$	$1.1_{10}{-1}$	$3.7_{10}{-2}$	$2.0_{10}{-2}$	$3.6_{10}{-4}$
1/8	$5.1_{10}{-2}$	$1.4_{10}{-2}$	$2.4_{10}{-3}$	$6.1_{10}{-4}$	$1.1_{10}{-4}$
1/16	$1.3_{10}{-2}$	$1.7_{10}{-3}$	$1.5_{10}{-4}$	$1.9_{10}{-5}$	$1.8_{10}{-6}$

Table 6.1. Results for problem 6.1.

Remarks. For all values of h the points $(h\xi, h^2\eta)$ were inside the stability regions. The asymptotical convergence factor of $(1/2)^k$ is obvious, even for these relatively large stepsizes, but, of course, this is a very mild problem.

Problem 6.2.

$$(6.2)\qquad \begin{cases} y'(x) = e^x - y(x) - \int_0^x e^{x-t} y(t)dt, & 0 \le x \le 2 \\ y(0) = 1 \end{cases}$$

with exact solution $y(x) \equiv 1$. This is an example of a linear VIDE which is not a member of our class of model equations. In all cases the point $(h\xi_{n+1}, h^2\eta_{n+1})$ is within the stability region. Table 6.2 shows the results.

h	rel. error at x = 2				
	k=2	k=3	k=4	k=5	k=6
1/4	$1.1_{10}{-2}$	$1.8_{10}{-3}$	$1.7_{10}{-4}$	$5.0_{10}{-5}$	$3.5_{10}{-6}$
1/8	$2.6_{10}{-3}$	$2.0_{10}{-4}$	$1.2_{10}{-5}$	$1.5_{10}{-6}$	$8.5_{10}{-8}$
1/16	$6.5_{10}{-4}$	$2.3_{10}{-5}$	$7.7_{10}{-7}$	$4.2_{10}{-8}$	$1.5_{10}{-9}$
1/32	$1.6_{10}{-4}$	$2.7_{10}{-6}$	$4.9_{10}{-8}$	$1.2_{10}{-9}$	$2.5_{10}{-11}$
1/64	$4.1_{10}{-5}$	$3.2_{10}{-7}$	$3.1_{10}{-9}$	$3.7_{10}{-11}$	$5.2_{10}{-13}$
1/128	$1.0_{10}{-5}$	$4.0_{10}{-8}$	$2.0_{10}{-10}$	$8.4_{10}{-13}$	0

Table 6.2. Results for problem 6.2

Remarks. Again it is apparent that the global error is $\mathcal{O}(h^k)$ as $h \to 0$.

Problem 6.3.

$$(6.3)\qquad \begin{cases} y'(x) = 50 - 50.75\, e^{-x} - 0.25\, y(x) - 50 \int_0^x y(t)dt, & 0 \le x \le 10 \\ y(0) = 1 \end{cases}$$

with solution $y(x) = \exp(-x)$. This linear problem depends strongly on the "Volterra" part of the equation. In the next table we indicate by S and I, respectively, whether or not the point $(h\xi, h^2\eta)$ belongs to the stability region.

h	k=2	k=3	k=4	k=5	k=6
1/4	S	I	I	I	I
1/8	S	I	I	S/I	S
1/16	S	S	S	S	S
1/32	S	S	S	S	S

Experiments yield the following results.

h	rel. error at x = 10				
	k=2	k=3	k=4	k=5	k=6
1/4	$1.9_{10}-1$	$1.4_{10}+2$	$4.1_{10}+2$	$6.6_{10}+2$	$3.0_{10}+3$
1/8	$1.5_{10}+0$	$7.2_{10}+1$	$4.0_{10}+0$	$1.4_{10}-1$	$7.1_{10}-5$
1/16	$1.6_{10}+0$	$2.5_{10}-1$	$5.8_{10}-4$	$3.2_{10}-4$	$1.1_{10}-5$
1/32	$1.0_{10}-1$	$1.0_{10}-1$	$9.7_{10}-5$	$2.6_{10}-5$	$1.2_{10}-9$

Table 6.3.1. Results for problem 6.3

The relatively bad results for the second order method are *not* due to instability but are a consequence of the low order of the formula used. In order to test the convergence for $h \to 0$ we have integrated problem 6.3 on the interval $0 \le x \le 1$. The results are given in table 6.3.2.

h	rel. error at x = 1				
	k=2	k=3	k=4	k=5	k=6
1/16	$3.6_{10}{-4}$	$1.1_{10}{-4}$	$1.4_{10}{-6}$	$4.7_{10}{-7}$	$3.6_{10}{-9}$
1/32	$7.5_{10}{-6}$	$1.0_{10}{-5}$	$2.4_{10}{-8}$	$2.8_{10}{-9}$	$3.4_{10}{-11}$
1/64	$6.0_{10}{-6}$	$2.6_{10}{-6}$	$5.3_{10}{-10}$	$1.7_{10}{-10}$	$7.0_{10}{-13}$
1/128	$2.2_{10}{-6}$	$3.9_{10}{-7}$	$9.0_{10}{-11}$	$9.4_{10}{-12}$	$2.3_{10}{-12}$
1/256	$6.2_{10}{-7}$	$5.3_{10}{-8}$	$7.1_{10}{-12}$	$9.6_{10}{-13}$	$3.5_{10}{-12}$

Table 6.3.2. Convergence test for problem 6.3

As may be seen, the results for small h and k ≥ 4 are of order $_{10}{-12}$ i.e. the tolerance for the Newton iteration process.

Problem 6.4.

$$(6.4) \qquad \begin{cases} y'(x) = 25 - 51\, y(x) + 25\, y^2(x) - 25 \left[\int_0^x y(t)dt \right]^2; & 0 \le x \le 2 \\ y(0) = 1. \end{cases}$$

with solution $y(x) = e^{-x}$. This problem is non-linear in y and z. Here, we have

$$\frac{\partial F}{\partial y} = 50y - 51; \qquad \frac{\partial F}{\partial z} \cdot \frac{\partial K}{\partial y} = -50z.$$

Thus, for $h\xi$ and $h^2\eta$ we have (computed along the exact solution of (6.4))

$$\xi = (50e^{-x} - 51) \qquad \text{and} \qquad \eta = -50(1 - e^{-x}).$$

The stability theory indicates that there will be no instabilities. Indeed, this is verified by the experiments (see table 6.4).

h	rel. error at x = 2				
	k=2	k=3	k=4	k=5	k=6
1/4	$1.0_{10}{-2}$	$1.0_{10}{-3}$	$2.2_{10}{-4}$	$4.2_{10}{-5}$	$6.3_{10}{-6}$
1/8	$2.5_{10}{-3}$	$7.6_{10}{-5}$	$1.2_{10}{-5}$	$7.3_{10}{-7}$	$8.5_{10}{-8}$
1/16	$6.0_{10}{-4}$	$7.7_{10}{-6}$	$6.9_{10}{-7}$	$1.7_{10}{-8}$	$1.3_{10}{-9}$
1/32	$1.5_{10}{-4}$	$9.1_{10}{-7}$	$4.2_{10}{-8}$	$4.8_{10}{-10}$	$2.1_{10}{-11}$
1/64	$3.7_{10}{-5}$	$1.2_{10}{-7}$	$2.5_{10}{-9}$	$1.5_{10}{-11}$	$3.2_{10}{-13}$

Table 6.4. Results for problem 6.4.

Problem 6.5. (from [1])

$$(6.5)\qquad \begin{cases} y'(x) = -x - \dfrac{1}{(1+x)^2} + \dfrac{1}{y(x)} \ell n \dfrac{2+2x}{2+x} + \displaystyle\int_0^x \dfrac{dt}{1+(1+x)y(t)} ; \\ \qquad\qquad\qquad\qquad\qquad 0 \le x \le 10 \\ y(0) = 1 \end{cases}$$

with solution $y(x) = \frac{1}{1+x}$. This problem is non-linear in y and linear in z; however, the kernel function is non-linear in y. For ξ and η we have (computed along the exact solution of (6.5)).

$$\xi = -(1+x)^2 \ell n\left(\frac{2+2x}{2+x}\right); \qquad \eta = -\frac{1}{4}(1+x).$$

Therefore, these values monotonically increase with x.

Our (local) stability theory predicts no instabilities. In table 6.5 we give the results.

Table 6.5. Results for problem 6.5

h	rel. error at x = 10				
	k=2	k=3	k=4	k=5	k=6
1/2	$6.4_{10}{-4}$	$2.9_{10}{-4}$	$1.5_{10}{-4}$	$8.8_{10}{-5}$	$6.8_{10}{-5}$
1/4	$2.5_{10}{-4}$	$8.0_{10}{-5}$	$3.3_{10}{-5}$	$1.5_{10}{-5}$	$1.0_{10}{-5}$
1/8	$8.0_{10}{-5}$	$1.6_{10}{-5}$	$4.5_{10}{-6}$	$1.4_{10}{-6}$	$5.3_{10}{-7}$
1/16	$2.3_{10}{-5}$	$2.7_{10}{-6}$	$4.4_{10}{-7}$	$8.8_{10}{-8}$	$2.0_{10}{-8}$
1/32	$6.0_{10}{-6}$	$4.0_{10}{-7}$	$3.6_{10}{-8}$	$4.0_{10}{-9}$	$5.4_{10}{-10}$

Problem 6.6.

$$(6.6)\qquad \begin{cases} y'(x) = [g(x) - \alpha y(x) - \beta z(x)]^3 - 1, & 0 \le x \le 4. \\ z(x) = \displaystyle\int_0^x (x+\gamma t)^\delta y^3(t)\,dt & \\ y(0) = 1. & \end{cases}$$

With

$$g(x) = 1 + \alpha + \frac{\beta x^{\delta+1}}{\gamma(\delta+1)} \left[(1+\gamma)^{\delta+1} - 1\right]$$

we have the solution $y(x) \equiv 1$.

This non-linear problem was chosen in order to illustrate the stability theory. For ξ and η we have

$$\xi = -3\alpha; \qquad \eta = -9\beta x^\delta (1+\gamma)^\delta.$$

We have chosen the values

$$\alpha = 1; \quad \beta = 15; \quad \gamma = 2 \quad \text{and} \quad \delta = 3/2.$$

Thus $|\eta|$ increases monotonically, whereas ξ remains constant. This is a problem for which the point $(h\xi, h^2\eta)$ goes to infinity along the line $h\xi$ = constant as $x \to \infty$. From the stability regions we have computed with $h = 1/8$ the values of x for which stability is expected, theoretically. These values are listed in table 6.6.1. In table 6.6.2 we give the actual error for different values of x resulting from an integration with stepsize $h = 1/8$.

k	stable if
2	$x > 0$
3	$0 < x < 2.5$
4	$0 < x < 0.26$ and $1.6 < x < 2.1$
5	$0 < x < 0.23$
6	$0 < x < 0.18$

Table 6.6.1. Stability intervals for problem 6.6

x	k=2	k=3	k=4	k=5	k=6
1.00	$+2.8_{10}{-4}$	$+1.1_{10}{-4}$	$-3.1_{10}{-5}$	$-1.1_{10}{-5}$	$+2.2_{10}{-6}$
1.50	$+1.1_{10}{-4}$	$+1.4_{10}{-4}$	$+4.1_{10}{-5}$	.	.
1.75	$+7.1_{10}{-5}$	$+1.7_{10}{-4}$	$+2.6_{10}{-5}$	.	.
2.00	$+5.2_{10}{-5}$	$+4.3_{10}{-5}$	$-6.5_{10}{-6}$	$-9.0_{10}{-5}$	$-1.9_{10}{-5}$
2.25	.	$-9.2_{10}{-6}$	$-3.0_{10}{-5}$	.	.
2.50	.	$-1.8_{10}{-5}$	.	.	.
.	.	.	.	.	.
3.00	$+3.4_{10}{-5}$	$-2.6_{10}{-5}$	$-2.7_{10}{-4}$	unstable	$-9.4_{10}{-4}$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
4.00	$+2.4_{10}{-5}$	$-1.9_{10}{-4}$	$-2.2_{10}{-3}$	unstable	unstable

Table 6.6.2. Results for problem 6.6 with h = 1/8.

Remark. The experiments confirm the theoretical results listed in table 6.6.1. It should be noted that the problems for which $|h^2\eta| >> |h\xi|$ stability imposes a severe restriction on the stepsize when using the methods for k = 3,4,5 and 6. Because of the A_0-stability of the method for k = 2, however, we have stability for all stepsizes.

7. CONCLUDING REMARKS

The numerical experiments confirm the convergence theory stated in section 3. There is little difference in the computational efficiency of the method, because the major part of the computational effort lies in the repeated evaluation of the kernel K(x,t,y) which is approximately $\frac{1}{2}N^2$ (N is the total number of steps). In the case that $|h^2\eta| >> |h\xi|$ high order methods ($k \geq 3$) may cause instability, which is illustrated by problem 6.6.

Finally we remark that we intend to test extensively the stability of the methods considered in this report. These results will appear in a forthcoming report.

ACKNOWLEDGEMENTS

The author is grateful to P.J. van der Houwen and H.J.J. te Riele for their helpful discussions.

REFERENCES

[1] BRUNNER, H. & J.D. LAMBERT, *Stability of numerical methods for Volterra integro-differential eqiations*, Computing 12, pp. 75-89 (1974).

[2] GEAR, C.W., *Numerical integration of stiff ordinary differential equations*, University of Illinois, Dept. of Computer Science Report no. 221 (1967).

[3] HALL, G. & J.M. WATT (editors), *Modern numerical methods for ordinary differential equations*, Clarendon Press, 1976.

[4] HOUWEN, P.J. VAN DER & H.J.J. TE RIELE, *Backward differentiation formulas for Volterra integral equations of the second kind I convergence and stability*, report NW 48/77, Mathematisch Centrum, Amsterdam (1977).

[5] LAMBERT, J.D., *Computational methods in ordinary differential equations*, Wiley, 1973.

[6] LINZ, P., *Linear multistep methods for Volterra integro-differential equations*, J. ACM 16, pp. 295-301 (1969).

[7] MATTHYS, J., *A-stable linear multistep methods for Volterra integro-differential equations*, Numer. Math. 27, pp. 85-94 (1976).

[8] SCHMITT, K. (editor), *Delay and functional differential equations and their applications*, Academic Press, 1972.

[9] STEINBERG, J., *Numerical solution of Volterra integral equations*, Numer. Math. 19, pp. 212-217 (1972).