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(DEPARTMENT OF NUMERICAL MATHEMATICS)

NW 63/78 OKTOBER

P.J. VAN DER HOUWEN, H.J.J. TE RIELE &
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ON THE STABILITY OF MULTISTEP FORMULAS FOR SYSTEMS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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On the stability of multistep formulas for systems of Volterra integro-differential equations*)

by

P.J. van der Houwen, H.J.J. te Riele & P.H.M. Wolkenfelt

ABSTRACT

The stability behaviour of linear multistep methods is analyzed for systems of integro-differential equations of Volterra type which are only restricted by the condition that the kernel function is finitely decomposable. Stability regions are derived for backward differentiation type schemes of orders 2 up to 6 and for the trapezoidal rule. In particular, attention is paid to these schemes when applied to the special integro-differential equation, which arises from differentiating a Volterra integral equation of the second kind. Numerical experiments are reported, which confirm the theoretical stability results.

KEY WORDS & PHRASES: *Numerical analysis, Volterra integro-differential equations, Volterra integral equations of the second kind, stability*

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*) This paper will be submitted for publication elsewhere.

1. INTRODUCTION

We shall consider systems of Volterra integro-differential equations of the form

$$\frac{df(x)}{dx} = \Phi(x,f(x),z(x)), \qquad x_0 \le x \le b, \tag{1.1a}$$

$$z(x) = \int_{x_0}^{x} K(x,y,f(y))dy, \tag{1.1b}$$

with the initial condition

$$f(x_0) = f_0. \tag{1.2}$$

Here, Φ and K are prescribed vector functions and $f(x)$ is the unknown vector function. We assume that this problem has a unique solution.

The main purpose of this paper is to analyze the stability behaviour of a large class of numerical methods for the solution of this problem.

In section 2 a general numerical scheme is given based on a linear multistep formula for the integration of (1.1a) and a quadrature rule for the approximation of the integral in (1.1b). Two special choices of this general scheme, based on the well-known Curtiss-Hirschfelder formulas for (1.1a), are specified. In the sequel of this paper they will serve as a means of illustrating the stability analysis. One of these two schemes is further elaborated when it is applied to the class of "integro-differential equations" obtained by differentiating a Volterra integral equation of the second kind. This rather unconventional treatment of Volterra integral equations of the second kind was motivated by the wish to exploit the excellent stability properties of the Curtiss-Hirschfelder formulas when applied to ordinary differential equations.

In section 3.1 the order of convergence of the general scheme is proved, along the lines indicated by TAVERNINI [11]. In section 3.2, the "kernel" of this paper, a stability analysis is carried out for this scheme, when applied to (1.1), where we have tried to restrict the kernel function $K(x,y,f)$ as little as possible. Following the approach presented in [6] it turned out

that *local stability conditions* can be derived for the kernel functions which are *finitely decomposable*, i.e. which can be written as a sum of terms of the form $Q(x)R(y,f)$, $Q(x)$ and $R(y,f)$ being arbitrary matrices. With local stability we mean that numerical errors in a point x are not accumulated in a step-by-step process, provided that $\partial\Phi/\partial f$, $\partial\Phi/\partial z$ and $\partial K/\partial f$ are sufficiently slowly varying. With respect to the local theory three observations can be made. Firstly, the local stability conditions are at least necessary for global stability. Secondly, when compared with the conditions derived by BRUNNER & LAMBERT [1] and MATTHYS [8], which are based on the test equation

$$\frac{df(x)}{dx} = \xi f(x) + \eta \int_0^x f(y)dy, \tag{1.3}$$

the present analysis shows that one should be prepared for much stronger conditions. Thirdly, for *systems* of equations, verifying the stability conditions will require a lot of computational effort. An exception form the systems where the matrices $\partial\Phi/\partial f$, and $\partial\Phi/\partial z * \partial K/\partial f$ possess a common eigensystem. For this special class of integro-differential equations we have plotted the stability regions of the two special schemes given in section 2. In addition, we have derived the stability conditions for the scheme where (1.1a) is integrated by the trapezoidal rule. On the basis of these results, these schemes may be qualified as "highly stable".

In section 4, a starting scheme is described to provide the initial values for the multistep formulas. We chose the rather stable trapezoidal rule and used extrapolation to obtain sufficient accuracy.

In section 5, numerical experiments are reported to test the stability theory and the order of convergence. The results of these experiments are in agreement with the theoretical stability conditions and order of convergence. Finally, a few remarks are devoted to some experiments, carried out in [10], which compare the efficiency of one of our schemes with that of an implicit Runge-Kutta scheme of DE HOOG & WEISS [4].

The contents of this paper are based on three institute reports [5], [10] and [13].

2. THE COMPUTATIONAL SCHEME

We will define a class of linear multistep methods for general Volterra integro-differential equations and, in particular, we will discuss these methods when applied to the special equation which arises from the differentiation of a Volterra integral equation of the second kind.

2.1 Integro-differential equations

Let $x_n = x_0+nh$, $n = 1,\ldots,N$ ($x_N=b$) denote the discretization points of the x-axis. Suppose that approximations f_j to $f(x_j)$ are obtained in the points $x_0,\ldots,x_n$ and let z_j denote an approximation to $z(x_j)$. Then, by applying a linear multistep method (defined by its coefficients $\{a_\ell,b_\ell\}$) to the "differential" equation (1.1a) we arrive at the scheme

$$\sum_{\ell=0}^{k} a_\ell f_{n+1-\ell} = h \sum_{\ell=0}^{k} b_\ell \Phi(x_{n+1-\ell},f_{n+1-\ell},z_{n+1-\ell}), \quad n = k-1,k,\ldots \tag{2.1}$$

In order to give a step-by-step formula for f_{n+1}, we have to specify the formula for $z_{n+1-\ell}$; we will use a quadrature rule of the form

$$z_{n+1-\ell} = \sum_{j=0}^{n+1-\ell} w_{n+1-\ell,j} K(x_{n+1-\ell},x_j,f_j), \tag{2.2}$$

where the weights $w_{i,j}$ satisfy a relation of the form

$$\begin{aligned} &\sum_{\ell=0}^{\tilde{k}} \tilde{a}_\ell w_{n+1-\ell,j} = 0, \quad 0 \le j \le n-\tilde{k}, \quad n \ge 2\tilde{k}-1, \\ &\sum_{\ell=0}^{\tilde{k}} \tilde{a}_\ell = 0, \quad \tilde{a}_0 \ne 0, \end{aligned} \tag{2.3a}$$

and where we define

$$h\tilde{b}_{n+1-j} = \sum_{\ell=0}^{\tilde{k}} \tilde{a}_\ell w_{n+1-\ell,j}, \quad n-\tilde{k}+1 \le j \le n+1, \quad n \ge 2\tilde{k}-1. \tag{2.3b}$$

Quadrature formulas satisfying relation (2.3) can be constructed by applying a linear multistep method with coefficients $\{\tilde{a}_\ell,\tilde{b}_\ell\}$ to a quadrature problem written as a differential equation. Details of such formulas can be found

in [12]. Choosing a linear multistep method $\{a_\ell, b_\ell\}$ for (2.1) and $\{\tilde{a}_\ell, \tilde{b}_\ell\}$ for (2.3) the computational scheme is completely determined (apart from the starting values). We will denote such a combined scheme by $\{a_\ell, b_\ell; \tilde{a}_\ell, \tilde{b}_\ell\}$. It is clear that a large number of combinations is possible. However, as already remarked in the introduction, it is our main purpose to *present a stability analysis* for a large class of kernel functions rather than to investigate and compare a large number of different schemes. Therefore, in order to *illustrate* the analysis, we have concentrated on the following special schemes; we have chosen for

$\{a_\ell, b_\ell\}$: the Curtiss-Hirschfelder (or backward differentiation) formulas (the consideration was to exploit their excellent stability properties). In table 2.1 the coefficients are listed for $k = 1,2,\ldots,6$ (cf. [7, p.242]);

$\{\tilde{a}_\ell, \tilde{b}_\ell\}$:

1. the Adams-Moulton formulas, which correspond to the well-known Gregory quadrature. This combination will be denoted by {CH;AM}. In table 2.2 we give the coefficients for $\tilde{k} = 1,\ldots,5$. The corresponding quadrature weights are given in table 2.3;
2. Again the Curtiss-Hirschfelder formulas in order to illustrate that a much larger stability region can be obtained than for {CH;AM}. We will denote this scheme by {CH;CH}. The weights of the quadrature rule generated by CH-formulas can be found in [12].

In addition, we shortly discuss the stability conditions for the scheme in which both $\{a_\ell, b_\ell\}$ and $\{\tilde{a}_\ell, \tilde{b}_\ell\}$ correspond to the trapezoidal rule.

k	c_k	b_0*c_k	a_1*c_k	a_2*c_k	a_3*c_k	a_4*c_k	a_5*c_k	a_6*c_k
1	1	1	-1					
2	3	2	-4	1				
3	11	6	-18	9	-2			
4	25	12	-48	36	-16	3		
5	137	60	-300	300	-200	75	-12	
6	147	60	-360	450	-400	225	-72	10

$a_0 = 1$, $b_2,\ldots,b_k = 0$

Table 2.1. Coefficients of the Curtiss-Hirschfelder formulas for k = 1(1)6.

$\tilde{k}$	$c_{\tilde{k}}$	$\tilde{b}_0*c_{\tilde{k}}$	$\tilde{b}_1*c_{\tilde{k}}$	$\tilde{b}_2*c_{\tilde{k}}$	$\tilde{b}_3*c_{\tilde{k}}$	$\tilde{b}_4*c_{\tilde{k}}$	$\tilde{b}_5*c_{\tilde{k}}$
1	2	1	1				
2	12	5	8	-1			
3	24	9	19	-5	1		
4	720	251	646	-264	106	-19	
5	1440	475	1427	-798	482	-173	27

$\tilde{a}_0 = 1$, $\tilde{a}_1 = -1$

Table 2.2. Coefficients of the Adams-Moulton formulas for $\tilde{k}$ = 1(1)5.

Second order formula

$$\frac{h}{2}\begin{bmatrix} 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ . & . & . & & \bigcirc & & \\ . & . & & & & & \\ . & . & & & & & \\ 1 & 2 & . & . & . & 2 & 1 \end{bmatrix};$$

Third order formula

$$\frac{h}{12}\begin{bmatrix} 6 & 6 & & & & & \\ 5 & 14 & 5 & & & & \\ 5 & 13 & 13 & 5 & & & \\ 5 & 13 & 12 & 13 & 5 & & \\ . & . & . & . & . & & \\ . & . & . & . & & . & \\ . & . & . & & . & & . \\ 5 & 13 & 12 & \cdots & 12 & 13 & 5 \end{bmatrix};$$

Fourth order formula

$$\frac{h}{24}\begin{bmatrix} 8 & 32 & 8 & & & & & \\ 9 & 27 & 27 & 9 & & & & \\ 9 & 28 & 22 & 28 & 9 & & & \\ 9 & 28 & 23 & 23 & 28 & 9 & & \\ 9 & 28 & 23 & 24 & 23 & 28 & 9 & \\ . & . & . & . & . & & . & \\ . & . & . & . & . & & & . \\ . & . & . & . & & . & & \\ 9 & 28 & 23 & 24 & \cdots & 24 & 23 & 28 & 9 \end{bmatrix};$$

Fifth order formula

$$\frac{h}{720}\begin{bmatrix} 270 & 810 & 810 & 270 & & & & & & & \\ 251 & 916 & 546 & 916 & 251 & & & & & & \\ 251 & 897 & 652 & 652 & 897 & 251 & & & & & \\ 251 & 897 & 633 & 758 & 633 & 897 & 251 & & & & \\ 251 & 897 & 633 & 739 & 739 & 633 & 897 & 251 & & & \\ 251 & 897 & 633 & 739 & 720 & 739 & 633 & 897 & 251 & & \\ . & . & . & . & . & & & & . & & \\ . & . & . & . & . & . & & & & . & \\ . & . & . & . & . & & . & & & & . \\ 251 & 897 & 633 & 739 & 720 & \cdots & 720 & 739 & 633 & 897 & 251 \end{bmatrix};$$

Sixth order formula

$$\frac{h}{1440}\begin{bmatrix}
448 & 2048 & 768 & 2048 & 448 & & & & & & & \\
475 & 1875 & 1250 & 1250 & 1875 & 475 & & & & & & \\
475 & 1902 & 1077 & 1732 & 1077 & 1902 & 475 & & & & & \\
475 & 1902 & 1104 & 1559 & 1559 & 1104 & 1902 & 475 & & & & \\
475 & 1902 & 1104 & 1586 & 1386 & 1586 & 1104 & 1902 & 475 & & & \\
475 & 1902 & 1104 & 1586 & 1413 & 1413 & 1586 & 1104 & 1902 & 475 & & \\
475 & 1902 & 1104 & 1586 & 1413 & 1440 & 1413 & 1586 & 1104 & 1902 & 475 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & & & & \\
475 & 1902 & 1104 & 1586 & 1413 & 1440 & \ldots & 1440 & 1413 & 1586 & 1104 & 1902 & 475
\end{bmatrix}$$

Table 2.3. Gregory quadrature weights

The scheme (2.1)-(2.2) requires in the n-th step the evaluation of the functions

$$K(x_{n+1},x_j,f_j), \qquad j = 0,\ldots,n+1,$$

and the solution of a system of nonlinear equations for f_{n+1}. Moreover, k starting vectors $f_0,\ldots,f_{k-1}$ are needed. In section 4 a starting scheme for computing these vectors is presented which is based on extrapolation of the trapezoidal rule.

2.2 The computational scheme for integral equations

Volterra integral equations of the second kind have the form

$$f(x) = g(x) + \int_{x_0}^{x} K^*(x,y,f(y))dy \tag{2.4}$$

where $g(x)$ and $K^*(x,y,f)$ are given vector functions. Differentiating both sides yields an integro-differential equation of the form

$$(2.5)\qquad \begin{cases} \dfrac{d}{dx} f(x) = \dfrac{d}{dx} g(x) + K^*(x,x,f(x)) + z(x) \\ z(x) = \displaystyle\int_{x_0}^{x} \frac{\partial}{\partial x} K^*(x,y,f(y))dy \\ f(x_0) = g(x_0), \end{cases}$$

and the scheme (2.1)-(2.2) can be applied. However, when the derivatives of g and K^* with respect to x are not available we assume that they are approximated by a numerical differentiation formula of the form

$$(2.6)\qquad \frac{d}{dx} g(x) \simeq \frac{1}{h} \sum_{i=k_1}^{k_2} d_i g(x-ih), \quad k_1 < k_2,$$

where h is the mesh width and the coefficients d_i determine the differentiation formula. For instance, when a numerical differentiation formula, based on Newton-backward interpolation, is used we have

$$(2.6')\qquad \frac{d}{dx} g(x) \simeq \frac{1}{b_0 h} \sum_{i=0}^{k_2} a_i g(x-ih),$$

with a_i and b_0 the coefficients of the Curtiss-Hirschfelder formula (see table 2.1).

Applying the scheme (2.1)-(2.2) to equation (2.5) and using (2.6) yields the following scheme for integral equations:

$$(2.1')\qquad \sum_{\ell=0}^{k} a_\ell f_{n+1-\ell} = \sum_{\ell=0}^{k} b_\ell [hK^*(x_{n+1-\ell}, x_{n+1-\ell}, f_{n+1-\ell}) + \sum_{i=k_1}^{k_2} d_i F_{n+1-\ell}(x_{n+1-\ell-i})], \quad n = k-1,k,\ldots,$$

where

$$(2.7)\qquad F_{n+1-\ell}(x) = g(x) + \sum_{j=0}^{n+1-\ell} w_{n+1-\ell,j} K^*(x,x_j,f_j).$$

In the computation of the $F_{n+1-\ell}(x)$ values we may use the relation (2.3a) between the weights $w_{n+1-\ell,j}$. Taking a linear combination of the

$F_{n+1-\ell}(x)$ with coefficients $\tilde{a}_\ell$ results in the formula

$$(2.8)\qquad F_{n+1}(x) = -\tilde{a}_0^{-1}\Big[\sum_{\ell=1}^{\tilde{k}} \tilde{a}_\ell F_{n+1-\ell}(x) - h \sum_{\ell=0}^{\tilde{k}} \tilde{b}_\ell K^*(x, x_{n+1-\ell}, f_{n+1-\ell})\Big],$$

where we have used (2.3b).

In order to see which F_{n+1}-values in the (n+1)-st step can be computed from F-values which have been evaluated already in preceding steps, we have indicated in figure 2.1 by ○ and ⊕ the index points (r,s) of those values $F_s(x_r)$ which are needed in scheme (2.1') for the calculation of f_n. In addition, the $F_s(x_r)$-values needed for the calculation of f_{n+1} corresponds to index points (r,s) denoted by ⊕ and +.

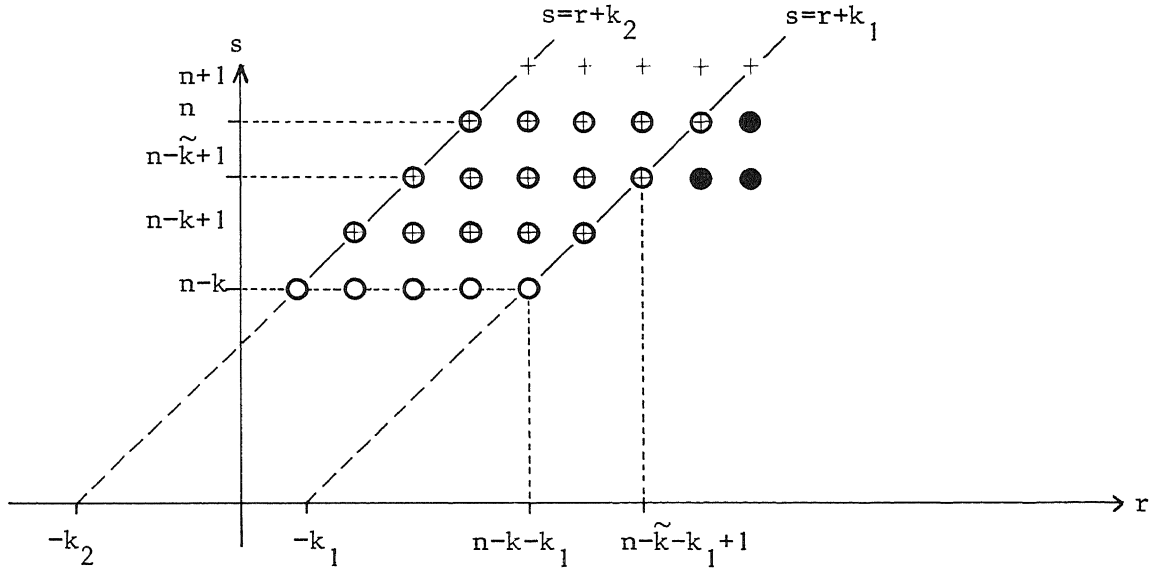

Fig.2.1 ○ index points needed for f_n
+ index points needed for f_{n+1}
⊕ index points needed for both f_n and f_{n+1}
● index points of additional F-values occuring in (2.8).

From (2.8) and this figure it is immediate that the values of $F_{n+1}(x_{n+1-k_2}),\ldots,F_{n+1}(x_{n+1-k_1-\tilde{k}})$ can be computed by (2.8) from known F-values plus a few additional K^*-values. The computation of $F_{n+1}(x_{n+2-k_1-\tilde{k}}),\ldots,F_{n+1}(x_{n+1-k_1})$ by (2.8) requires F-values which are not yet known unless the corresponding coefficient $\tilde{a}_\ell$ equals zero. (In figure 2.1 these unknown values correspond to index points indicated by ●.) As soon as such an

F-value is asked for by (2.8), $F_{n+1}(x)$ should be evaluated by the direct formula (2.7). The use of formula (2.7) is rather expensive in actual computation, even when all K^*-values are available, because of the many multiplications and additions, particularly for large values of n. Thus, the computational effort of the scheme (2.1'), (2.7), (2.8) is largely determined by the number of times formula (2.7) has to be used. This observation suggests to use quadrature rules for which the coefficients $\tilde{a}_\ell$ satisfy (cf. figure 2.1)

$$\tilde{a}_2 = \tilde{a}_3 = \ldots = \tilde{a}_{\tilde{k}} = 0. \tag{2.9}$$

In that case only $F_{n+1}(x_{n+1})$ has to be evaluated by (2.7) and all other F_{n+1}-values can be computed by (2.8) from preceding F-values. Quadrature rules satisfying (2.3a) and (2.9) are the well-known Gregory rules based on the Adams-Moulton multistep methods discussed in section 2.1 (cf. table 2.3).

In order to specify the computational effort per integration step a choice has to be made for the remaining coefficients d_i, a_ℓ and b_ℓ. When we choose the d_i according to (2.6') it is necessary to define the kernel function $K^*(x,y,f)$ in points where $x < y$, otherwise (2.8) cannot be used. An alternative could be to approximate the unknown K^*-values by extrapolation (cf. DE HOOG and WEISS [4]) or to use coefficients d_i for which $k_2 = 0$ (cf. [13]). In the following we have chosen the d_i's according to (2.6'). Finally, we consider the choice of the multistep method. As already observed in the introduction, the reason to construct an integration formula by converting the integral equation into a "quasi" integro-differential equation was the consideration to exploit the highly stable Curtiss-Hirschfelder formulas (or backward differentiation formulas) used for ordinary differential equations (e.g. [7]). Combining these formulas with the Gregory rules leads to a {CH;AM} formula mentioned in section 2.1. From the order of convergence of this scheme (cf. section 3.1) it follows that we have k-th order accuracy as $h \to 0$ if $\tilde{k} = k-1$, $k_1 = 0$, $k_2 = k$. Using (2.6'), (2.9) and (2.3b), the resulting scheme for Volterra integral equations (2.4) then reads

$$\sum_{\ell=0}^{k} a_\ell f_{n+1-\ell} = b_0 h K^*(x_{n+1}, x_{n+1}, f_{n+1}) + \sum_{\ell=0}^{k} a_\ell F_{n+1}(x_{n+1-\ell}), \tag{2.1''}$$

$$n = k-1, k, \ldots,$$

where

$$F_{n+1}(x_{n+1-\ell}) = F_n(x_{n+1-\ell}) + \sum_{j=n+2-k}^{n} (w_{n+1,j}-w_{n,j})K^*(x_{n+1-\ell},x_j,f_j) +$$

$$+ w_{n+1,n+1}K^*(x_{n+1-\ell},x_{n+1},f_{n+1}), \quad \ell = 1,2,\ldots,k,$$

and

$$F_{n+1}(x_{n+1}) = g(x_{n+1}) + \sum_{j=0}^{n+1} w_{n+1,j}K^*(x_{n+1},x_j,f_j).$$

The a_ℓ ($\ell = 0,1,\ldots,k$) and b_0 are given in table 2.1, and the weights $w_{i,j}$ in table 2.3. This scheme requires in the (n+1)-st step the evaluation of the function

$$g(x_{n+1}),$$

$$K(x_{n+1},x_j,f_j), \quad j = 0,1,\ldots,n,$$

$$K(x_j,x_{n+1},f_{n+1}), \quad j = n-k+2,\ldots,n+1,$$

and the solution of a system of equations for f_{n+1}. To start the integration we must precompute (approximations of) the quantities

$$f_j, \quad j = 0,1,\ldots,k-1$$

$$K(x_i,x_j,f_j), \quad i,j = 0,1,\ldots,k-1$$

$$F_{k-1}(x_j), \quad j = 0,1,\ldots,k-1.$$

3. CONVERGENCE AND STABILITY

In this section the order of convergence and the stability properties of scheme (2.1)-(2.2) will be derived. Our convergence analysis is based on the work of TAVERNINI [11] which enables us to confine the analysis to the derivation of a recurrence relation for the error $f(x_j)-f_j$. The stability analysis is based on [6] where the stability properties of direct quadrature rules for Volterra integral equations of the second kind were investigated.

3.1 Convergence

Following the convergence analysis used in ordinary differential equations we first derive a bound for the residual term Λ_{n+1}, when the exact solution f(x) of (1.1) is substituted into the scheme (2.1)-(2.2):

$$\Lambda_{n+1} = \sum_{\ell=0}^{k} a_\ell f(x_{n+1-\ell}) - h \sum_{\ell=0}^{k} b_\ell \Phi(x_{n+1-\ell}, f(x_{n+1-\ell}), \sum_{j=0}^{n+1-\ell} w_{n+1-\ell,j} K(x_{n+1-\ell}, x_j, f(x_j)))$$

$$= \sum_{\ell=0}^{k} a_\ell f(x_{n+1-\ell}) - h \sum_{\ell=0}^{k} b_\ell f'(x_{n+1-\ell})$$

$$+ h \sum_{\ell=0}^{k} b_\ell \Phi(x_{n+1-\ell}, f(x_{n+1-\ell}), \int_{x_0}^{x_{n+1-\ell}} K(x_{n+1-\ell}, y, f(y))dy)$$

$$- h \sum_{\ell=0}^{k} b_\ell \Phi(x_{n+1-\ell}, f(x_{n+1-\ell}), \sum_{j=0}^{n+1-\ell} w_{n+1-\ell,j} K(x_{n+1-\ell}, x_j, f(x_j))).$$

Assuming that the quadrature formula is of order q and that the linear multistep formula is of order p, we have the following bound för Λ_{n+1}:

$$\|\Lambda_{n+1}\| \le C_1 h^{p+1} + hL_2 C_2 h^q B = O(h^{p+1}) + O(h^{q+1}), \quad \text{as } h \to 0,$$

where we have taken $B = \sum_{\ell=0}^{k} |b_\ell|$ and where C_1, L_2 and C_2 are constants.

The next step is to derive a relation between the errors

$$e_j = f_j - f(x_j), \quad j = 0,\ldots,n+1,$$

assuming that the starting errors satisfy

$$e_j = O(h^s) \qquad \text{as } h \to 0, \quad j = 0(1)k-1.$$

Subtracting (3.1) from (2.1)-(2.2) yields the recurrence relation for the global error

$$\sum_{\ell=0}^{k} a_\ell e_{n+1-\ell} - h \sum_{\ell=0}^{k} B_{n+1}^{(\ell)}(e_0,\ldots,e_{n+1-\ell}) = -\Lambda_{n+1},$$

where

$$B_{n+1}^{(\ell)}(e_0,\ldots,e_{n+1-\ell}) = b_\ell\Big[\Phi(x_{n+1-\ell},f(x_{n+1-\ell}) + e_{n+1-\ell},$$
$$\sum_{j=0}^{n+1-\ell} w_{n+1-\ell,j}K(x_{n+1-\ell},x_j,f(x_j)+e_j))$$
$$- \Phi(x_{n+1-\ell},f(x_{n+1-\ell}), \sum_{j=0}^{n+1-\ell} w_{n+1-\ell,j}K(x_{n+1-\ell},x_j,f(x_j)))\Big].$$

This type of difference equations was investigated by TAVERNINI [11] from which it can be deduced that for sufficiently smooth functions Φ and K the order of convergence equals

$$\min\{p,q,s\}.$$

In the case of the differentiated integral equation (2.4) we have for the residual term Λ_{n+1}^* of the scheme (2.1'):

$$\Lambda_{n+1}^* = \sum_{\ell=0}^{k} a_\ell f(x_{n+1-\ell}) - h \sum_{\ell=0}^{k} b_\ell K^*(x_{n+1-\ell},x_{n+1-\ell},f(x_{n+1-\ell}))$$
$$- \sum_{\ell=0}^{k} b_\ell\Big\{\sum_{i=k_1}^{k_2} d_i\Big[g(x_{n+1-i-\ell}) + \sum_{j=0}^{n+1-\ell} w_{n+1-\ell,j}K^*(x_{n+1-i-\ell},x_j,f(x_j))\Big]\Big\}$$
$$= \sum_{\ell=0}^{k} a_\ell f(x_{n+1-\ell}) - h \sum_{\ell=0}^{k} b_\ell f'(x_{n+1-\ell})$$
$$+ \sum_{\ell=0}^{k} b_\ell\Big\{hg'(x_{n+1-\ell}) - \sum_{i=k_1}^{k_2} d_i g(x_{n+1-i-\ell})\Big\} +$$
$$+ h \sum_{\ell=0}^{k} b_\ell\Big\{\int_{x_0}^{x_{n+1-\ell}} \frac{\partial K^*}{\partial x}(x_{n+1-\ell},y,f(y))dy -$$
$$- \sum_{j=0}^{n+1-\ell} w_{n+1-\ell,j}\frac{\partial K^*}{\partial x}(x_{n+1-\ell},x_j,f(x_j))\Big\}$$
$$+ \sum_{\ell=0}^{k} b_\ell\Big\{\sum_{j=0}^{n+1-\ell} w_{n+1-\ell,j}\Big[h\frac{\partial K^*}{\partial x}(x_{n+1-\ell},x_j,f(x_j)) -$$
$$- \sum_{i=k_1}^{k_2} d_i K^*(x_{n+1-i-\ell},x_j,f(x_j))\Big]\Big\}.$$

Assuming that the numerical differentiation formula defined by the coefficients d_i has order r, i.e.

$$|g'(x_n) - \frac{1}{h} \sum_{i=k_1}^{k_2} d_i g(x_{n-i})| = O(h^r) \quad \text{as } h \to 0,$$

we have

$$\|\Lambda^*_{n+1}\| = O(h^{p+1}) + O(h^{r+1}) + O(h^{q+1}) + O(h^{r+1}) \quad \text{as } h \to 0.$$

Thus, the order of the residual term Λ^*_{n+1} equals $\min(p+1,q+1,r+1)$. From this we can deduce along the same lines as above that the order of convergence of scheme (2.1') equals

$$\min\{p,q,r,s\}.$$

This result suggests to combine a multistep method, a quadrature rule and a differentiation formula of the same order k. The resulting scheme is then of the same order. Thus, the {CH;AM} scheme specified in section 2.2 is of order k.

3.2 Stability

The variational equation of (2.1) with $z_{n+1-\ell}$ defined by (2.2) is of the form

$$(3.2) \qquad \sum_{\ell=0}^{k} a_\ell \Delta f_{n+1-\ell} = h \sum_{\ell=0}^{k} b_\ell \Big[\frac{\partial \Phi}{\partial f}(x_{n+1-\ell}, f_{n+1-\ell}, z_{n+1-\ell}) \Delta f_{n+1-\ell} + \frac{\partial \Phi}{\partial z}(x_{n+1-\ell}, f_{n+1-\ell}, z_{n+1-\ell}) \sum_{j=0}^{n+1-\ell} w_{n+1-\ell,j} \frac{\partial K}{\partial f}(x_{n+1-\ell}, x_j, f_j) \Delta f_j \Big].$$

In order to convert this relation into a fixed-term recurrence relation we assume that the derivative of the kernel function with respect to f satisfies the relation

$$(3.3) \qquad \frac{\partial K}{\partial f}(x,y,f) = \sum_{m=1}^{r} Q_m(x) R_m(y,f),$$

where Q_m and R_m are arbitrary matrices depending on x and (y,f), respectively. Substitution into (3.2) and writing

$$(3.4)\qquad \Delta G^{(m)}_{n+1-\ell} = \sum_{j=0}^{n+1-\ell} w_{n+1-\ell,j} R_m(x_j,f_j)\Delta f_j, \quad m = 1,\dots,r, \quad \ell = 0,1,\dots,k$$

yields the (r+1)(k+1)-terms relation

$$(3.5)\qquad \sum_{\ell=0}^{k} a_\ell \Delta f_{n+1-\ell} = h \sum_{\ell=0}^{k} b_\ell \Big[\frac{\partial \Phi}{\partial f}(x_{n+1-\ell}, f_{n+1-\ell}, z_{n+1-\ell}) \Delta f_{n+1-\ell} + \frac{\partial \Phi}{\partial z}(x_{n+1-\ell}, f_{n+1-\ell}, z_{n+1-\ell}) \sum_{m=1}^{r} Q_m(x_{n+1-\ell}) \Delta G^{(m)}_{n+1-\ell} \Big].$$

In addition, we have from (2.3a) for the perturbations $\Delta G^{(m)}_{n+1-\ell}$ the $2(\tilde{k}+1)$-terms recurrence relation

$$(3.6)\qquad \sum_{\ell=0}^{\tilde{k}} \tilde{a}_\ell \Delta G^{(m)}_{n+1-\ell} = h \sum_{j=n-\tilde{k}+1}^{n+1} \tilde{b}_{n+1-j} R_m(x_j,f_j)\Delta f_j, \quad m = 1,\dots,r$$

where the coefficients $\tilde{b}_\ell$ are defined by (2.3b).

Introducing the abbreviations

$$(3.7)\qquad \begin{aligned} L_\ell &= a_\ell I - b_\ell h \frac{\partial \Phi}{\partial f}(x_{n+1-\ell}, f_{n+1-\ell}, z_{n+1-\ell}), \\ M^{(m)}_\ell &= -b_\ell h \frac{\partial \Phi}{\partial z}(x_{n+1-\ell}, f_{n+1-\ell}, z_{n+1-\ell}) Q_m(x_{n+1-\ell}), \\ N^{(m)}_\ell &= -\tilde{b}_\ell h R_m(x_{n+1-\ell}, f_{n+1-\ell}), \end{aligned}$$

we may write the recurrence relations (3.5) and (3.6) in the form

$$(3.8)\qquad \begin{aligned} &\sum_{\ell=0}^{k} \Big[L_\ell \Delta f_{n+1-\ell} + \sum_{m=1}^{r} M^{(m)}_\ell \Delta G^{(m)}_{n+1-\ell} \Big] = 0, \\ &\sum_{\ell=0}^{\tilde{k}} \Big[N^{(m)}_\ell \Delta f_{n+1-\ell} + \tilde{a}_\ell \Delta G^{(m)}_{n+1-\ell} \Big] = 0, \quad m = 1,\dots,r. \end{aligned}$$

Without loss of generality we assume (in this section only) that $\tilde{k} = k$. For, if $\tilde{k} < k$ we may define the coefficients $\tilde{a}_\ell$ and $\tilde{b}_\ell$ to be zero for $\ell = \tilde{k}+1,\dots,k$ without affecting the method and the analysis; in the case $\tilde{k} > k$ the coefficients a_ℓ and b_ℓ are defined to be zero for $\ell = k+1,\dots,\tilde{k}$. With this assumption we may write (3.8) in the form

$$(3.9)\qquad \sum_{\ell=0}^{k} \mathcal{B}_\ell \Delta\vec{G}_{n+1-\ell} = 0,$$

where

$$\Delta\vec{G}_{n+1-\ell} = (\Delta G^{(0)}_{n+1-\ell},\ldots,\Delta G^{(r)}_{n+1-\ell})^T, \quad \Delta G^{(0)}_{n+1-\ell} = \Delta f_{n+1-\ell}$$

and

$$(3.10)\qquad \mathcal{B}_\ell = \begin{bmatrix} L_\ell & M_\ell^{(1)} & \cdots & M_\ell^{(r)} \\ N_\ell^{(1)} & \tilde{a}_\ell I & & \\ \vdots & & \ddots & \\ N_\ell^{(r)} & & & \tilde{a}_\ell I \end{bmatrix} .$$

Assuming that $\mathcal{B}_0^{-1}$ exists, we write (3.9) as

$$\Delta\vec{V}_{n+1} = A_n \, \Delta\vec{V}_n,$$

where

$$\Delta\vec{V}_{n+1} = (\Delta\vec{G}_{n+1},\ldots,\Delta\vec{G}_{n+2-k})^T$$

and

$$(3.11)\qquad A_n = \begin{bmatrix} -\mathcal{B}_0^{-1}\mathcal{B}_1 & -\mathcal{B}_0^{-1}\mathcal{B}_2 & \cdots & -\mathcal{B}_0^{-1}\mathcal{B}_k \\ I & 0 & & \\ & \ddots & \ddots & \\ & & I & 0 \end{bmatrix} .$$

Let us first consider the special case where A_n does not depend on n, i.e. $A_n = A$ = matrix with constant elements. For any subordinate matrix norm we have the inequality

$$\|\Delta\vec{V}_{n+1}\| \le \|A\| \, \|\Delta\vec{V}_n\| .$$

Hence, the vectors $\Delta\vec{V}_n$ are uniformly bounded if $\|A\| \le 1$. The well-known norm equivalence theorem (see ORTEGA [9, p.18]) states among others that there exists a constant $c \ge 0$, such that

$$\| \Delta\vec{V}_n \|_\infty \leq c \| \Delta\vec{V}_n \|, \quad \forall\ \Delta\vec{V}_n .$$

From this it follows that every component of $\Delta\vec{V}_n$, in particular the component Δf_n, is uniformly bounded. The conditions for the existence of a vector norm for which $\|A\| \leq 1$, are related to the spectralradius $R(A)$ of the matrix A. The necessary condition is $R(A) \leq 1$ i.e. all eigenvalues are within or on the unit circle. This condition is also sufficient provided that in the case $R(A) = 1$ the Jordan blocks (in the Jordan normal form of the matrix A) corresponding to the eigenvalues which are on the unit circle are matrices of order 1 (see ORTEGA [9, p.24]). It is convenient to divide the eigenvalues of A into two classes according to the following definition.

DEFINITION 3.1. An *eigenvalue* ζ of a matrix A is called:

a) *strongly stable* if $|\zeta| < 1$,

b) *weakly stable* if $|\zeta| = 1$ and the Jordan block corresponding to ζ in the Jordan normal form of A is of order 1.

Thus for matrices A with only strongly or weakly stable eigenvalues the existence of a norm with $\|A\| \leq 1$ is guaranteed.

It should be noted that the uniform boundedness of $\Delta\vec{V}_n$ can be proved only if A_n is a constant matrix. In our case, however, L_ℓ, $M_\ell^{(i)}$, and $N_\ell^{(i)}$, and therefore the matrix A_n depend on n, and consequently the above requirement on the eigenvalues of the matrix A_n only has a *local* meaning.

DEFINITION 3.2. For kernel functions of the class (3.3), the scheme (2.1)-(2.2) satisfying (2.3a) will be called *locally stable* at the point x_{n+1} with respect to the perturbations $\Delta G_{n+1}^{(i)}$ when all eigenvalues of the matrix A_n are strongly or weakly stable.

It is to be expected that local stability in a sequence of points $x_n, x_{n+1}, \ldots, x_{n+m}$ implies global stability in the range $[x_n, x_{n+m}]$ provided that A_n is slowly varying (with respect to the stepsize h) in this interval.

In the following, we will concentrate on the derivation of *local* stability criteria. To this end we need the characteristic equation of (3.11). It is easily verified (assuming $\det \mathcal{B}_0 \neq 0$) that

$$\det(A_n-\zeta I) = \det(B_0^{-1})\det\Big(\sum_{\ell=0}^{k} B_\ell \zeta^{k-\ell}\Big)[\det(-I)]^k. \tag{3.12}$$

Substitution of (3.10) into (3.12) gives the characteristic equation

$$\det\begin{pmatrix} \sum_0^k L_\ell\zeta^{k-\ell} & \sum_0^k M_\ell^{(1)}\zeta^{k-\ell} & \cdots & \sum_0^k M_\ell^{(r)}\zeta^{k-\ell} \\ \sum_0^k N_\ell^{(1)}\zeta^{k-\ell} & \sum_0^k \tilde{a}_\ell I\zeta^{k-\ell} & & \\ \vdots & & \ddots & \\ \sum_0^k N_\ell^{(r)}\zeta^{k-\ell} & & & \sum_0^k \tilde{a}_\ell I\zeta^{k-\ell} \end{pmatrix} = 0. \tag{3.13}$$

In order to simplify this equation we prove the following lemma.

LEMMA 3.1. *Suppose* B_m, C_m, $m = 1,\ldots,r$, A *and* D *are square matrices of the same order. If*

$$DC_m = C_m D \quad \textit{for } m = 1,\ldots,r, \tag{3.14}$$

then

$$\det\begin{pmatrix} A & B_1 & \cdots & B_r \\ C_1 & D & & \\ \vdots & & \ddots & \\ C_r & & & D \end{pmatrix} = \det\Big(AD - \sum_{m=1}^{r} B_m C_m\Big)(\det D)^{r-1}. \tag{3.15}$$

PROOF. First we prove the lemma for regular matrices D. Since

$$\begin{bmatrix} A & B_1 & \cdots & B_r \\ C_1 & D & & \\ \vdots & & \ddots & \\ C_r & & & D \end{bmatrix}\begin{bmatrix} I & & & \\ -D^{-1}C_1 & D^{-1} & & \\ \vdots & & \ddots & \\ -D^{-1}C_r & & & D^{-1} \end{bmatrix} = \begin{bmatrix} A - \sum_{m=1}^{r} B_m D^{-1} C_m, & B_1 D^{-1} & \cdots & B_r D^{-1} \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix},$$

we find

$$(3.15')\quad \det\begin{bmatrix} A & B_1 & \cdots & B_r \\ C_1 & D & & \\ \vdots & & \ddots & \\ C_r & & & D \end{bmatrix} (\det D^{-1})^r = \det(A - \sum_{m=1}^{r} B_m D^{-1} C_m).$$

Multiplying both sides with $(\det D)^r$, we have for the right-hand side of (3.15')

$$\det(A - \sum_{m=1}^{r} B_m D^{-1} C_m)(\det D)^r = \det(AD - \sum_{m=1}^{r} B_m D^{-1} C_m D)(\det D)^{r-1}.$$

Using (3.14) gives the right-hand side in (3.15), which completes the proof for regular matrices D. In the case that the matrix D is singular, we reason as follows. For all sufficiently (in modulus) small scalars Δ the matrices $D^*(\Delta) = D + \Delta I$ are regular. Moreover, $D^*(\Delta)C_m = (D+\Delta I)C_m = C_m D + C_m \Delta I = C_m D^*(\Delta)$. Applying the lemma with the regular matrix $D^*(\Delta)$ yields (3.15) with D replaced by $D^*(\Delta)$. Since both sides are continuous functions of the parameter Δ, (3.15) holds for $D^*(0)$. However, $D^*(0) = \lim_{\Delta \to 0} D^*(\Delta) = D$ and hence we have proved the lemma for singular matrices D. □

As in the theory of linear multistep methods for ordinary differential equations we need the following definition (see e.g. HENRICI [3,p.16]).

DEFINITION 3.3. The polynomial $\sum_{\ell=0}^{k} a_\ell \zeta^{k-\ell}$ with $a_0 \neq 0$ is said to satisfy the *root condition* if its k roots ζ_i satisfy $|\zeta_i| \leq 1$ (i = 1,2,...,k), and if the roots satisfying $|\zeta_i| = 1$ have multiplicity 1.

Next we give the following theorem

THEOREM 3.1.

i) *The eigenvalues of the matrix* (3.11) *satisfy*

$$(3.16a)\quad \sum_{\ell=0}^{k} \tilde{a}_\ell \zeta^{k-\ell} = 0$$

or

$$(3.16b)\quad \det\left\{ \sum_{\ell=0}^{k} \sum_{i=0}^{k} \left[\tilde{a}_\ell L_i - \sum_{m=1}^{r} M_i^{(m)} N_\ell^{(m)} \right] \zeta^{2k-\ell-i} \right\} = 0.$$

ii) *Let the polynomial* $\sum_{\ell=0}^{k} \tilde{a}_\ell \zeta^{k-\ell}$ *satisfy the root condition, then the scheme* (2.1)-(2.2) *is locally stable in the sense of definition* (3.2) *when the eigenvalues satisfying* (3.16b) *are strongly stable.*

PROOF. We already showed that the eigenvalues of the matrix (3.11) satisfy the characteristic equation (3.13). By observing that the matrix $\Sigma_0^k \tilde{a}_\ell \zeta^{k-\ell} I$ in (3.13) is a multiple of the unit matrix, and therefore commutes with all matrices of the same order, we obtain by virtue of lemma 3.1 that (3.13) may be reduced to

$$\det\left\{ \sum_{\ell=0}^{k} \sum_{i=0}^{k} \left[\tilde{a}_\ell L_i - \sum_{m=1}^{r} M_i^{(m)} N_\ell^{(m)} \right] \zeta^{2k-\ell-i} \right\} x$$

(3.17)

$$\left(\det\left\{ \sum_{\ell=0}^{k} \tilde{a}_\ell I \, \zeta^{k-\ell} \right\} \right)^{r-1} = 0,$$

from which (3.16a) and (3.16b) immediately follow.

In order to prove the second part of the theorem, we have to prove that the eigenvalues which satisfy (3.16a) are strongly or weakly stable. From the root condition it follows that all eigenvalues which satisfy (3.16a) are within or on the unit circle. Hence, it remains to show that the eigenvalues on the unit circle are weakly stable. Let $\tilde{\zeta}$ be a root of $\sum_{\ell=0}^{k} \tilde{a}_\ell \zeta^{k-\ell}$ with $|\tilde{\zeta}| = 1$. From the root condition it follows that $\tilde{\zeta}$ is simple. If the matrices in (3.10) have order s then $\tilde{\zeta}$ is a root of (3.17) with multiplicity s(r-1). The next step is to prove that there exist s(r-1) independent eigenvectors of A_n corresponding to the eigenvalue $\tilde{\zeta}$. Let $\vec{e}(\tilde{\zeta})$ be such an eigenvector with component vectors $(\vec{e}_1, \ldots, \vec{e}_k)$. Solving

$$A_n \vec{e}(\tilde{\zeta}) = \tilde{\zeta} \, \vec{e}(\tilde{\zeta})$$

we find

$$\vec{e}_1 = \tilde{\zeta}^{k-1} \vec{e}_k, \ldots, \vec{e}_{k-1} = \tilde{\zeta} \, \vec{e}_k$$

and

$$\left(\sum_{\ell=0}^{k} B_\ell \tilde{\zeta}^{k-\ell} \right) \vec{e}_k = \vec{0}. \tag{3.18}$$

Thus, the number of independent eigenvectors $\vec{e}(\tilde{\zeta})$ equals the number of independent vectors $\vec{e}_k$ which satisfy (3.18). Denoting the components of $\vec{e}_k$ by $(\vec{e}_k^{(0)},\ldots,\vec{e}_k^{(r)})$ and solving (3.18) we find the system of equations

$$(3.19a)\qquad \Big(\sum_0^k L_\ell \tilde{\zeta}^{k-\ell}\Big)\vec{e}_k^{(0)} + \Big(\sum_0^k M_\ell^{(1)}\tilde{\zeta}^{k-\ell}\Big)\vec{e}_k^{(1)} + \ldots + \Big(\sum_0^k M_\ell^{(r)}\tilde{\zeta}^{k-\ell}\Big)\vec{e}_k^{(r)} = \vec{0},$$

$$(3.19b)\qquad \Big(\sum_0^k N_\ell^{(m)}\tilde{\zeta}^{k-\ell}\Big)\vec{e}_k^{(0)} = \vec{0}, \qquad m = 1,\ldots,r.$$

Since each vector $\vec{e}_k^{(i)}$ has s components, $i = 0,\ldots,r$, we have $s(r+1)$ unknowns which must satisfy (3.19a)-(3.19b). Recall that we have to prove that $s(r-1)$ unknowns can be chosen independently. This is achieved by chosing the components of $\vec{e}_k^{(0)}$ equal to zero, i.e. $\vec{e}_k^{(0)} = \vec{0}$, in which case (3.19b) is satisfied. The number of unknowns is then reduced to sr, and we are left with the s equations

$$(3.20)\qquad \Big(\sum_0^k M_\ell^{(1)}\tilde{\zeta}^{k-\ell}\Big)\vec{e}_k^{(1)} + \ldots + \Big(\sum_0^k M_\ell^{(r)}\tilde{\zeta}^{k-\ell}\Big)\vec{e}_k^{(r)} = \vec{0}.$$

From (3.20) it is immediate that $s(r-1)$ components can be chosen arbitrarily, and therefore one can find $s(r-1)$ independent vectors $\vec{e}_k$. □

Equation (3.16b) can completely be expressed in terms of the Jacobian matrices $\partial\Phi/\partial f$, $\partial\Phi/\partial z$ and $\partial K/\partial f$ in a number of points close to (x_n,f_n,z_n) and (x_n,x_n,f_n), respectively. A straightforward calculation yields by substitution of L_i, $M_i^{(m)}$ and $N_\ell^{(m)}$

$$(3.16b')\qquad \det\Big\{\sum_{\ell=0}^{\tilde{k}}\sum_{i=0}^{k}\Big[\tilde{a}_\ell a_i I - \tilde{a}_\ell b_i h\,\frac{\partial\Phi}{\partial f}(x_{n+1-i},f_{n+1-i},z_{n+1-i}) - \tilde{b}_\ell b_i h^2\,\frac{\partial\Phi}{\partial z}(x_{n+1-i},f_{n+1-i},z_{n+1-i})\,\frac{\partial K}{\partial f}(x_{n+1-i},x_{n+1-\ell},f_{n+1-\ell})\Big]\zeta^{k+\tilde{k}-\ell-i}\Big\}=0.$$

In the case where the matrices $\partial\Phi/\partial f$, $\partial\Phi/\partial z$ and $\partial K/\partial f$ can reasonably be approximated by constant matrices, equation (3.16b') reduces to

$$(3.21)\qquad \det\Big\{\tilde{\rho}(\zeta)[\rho(\zeta)I - \sigma(\zeta)h\,\frac{\partial\Phi}{\partial f}] - \sigma(\zeta)\tilde{\sigma}(\zeta)h^2\,\frac{\partial\Phi}{\partial z}\,\frac{\partial K}{\partial f}\Big\} = 0,$$

where $\{\rho,\sigma\}$ and $\{\tilde{\rho},\tilde{\sigma}\}$ are the characteristic polynomials associated to the linear multistep methods $\{a_\ell,b_\ell\}$ and $\{\tilde{a}_\ell,\tilde{b}_\ell\}$, respectively, that is

$$\rho(\zeta) = \sum_{\ell=0}^{k} a_\ell \zeta^{k-\ell}, \quad \sigma(\zeta) = \sum_{\ell=0}^{k} b_\ell \zeta^{k-\ell},$$

and similar expressions for $\{\tilde{\rho},\tilde{\sigma}\}$. For scalar integro-differential equations the determinant operator in (3.21) may be omitted and the equation reduces to the characteristic equation given by BRUNNER & LAMBERT [1] and MATTHYS [8]. From the *stability region* defined as the set of points $(h\frac{\partial\Phi}{\partial f}, h^2\frac{\partial\Phi}{\partial z}\frac{\partial K}{\partial f})$ where equation (3.21) has its roots within the unit circle, an indication is obtained for which step sizes and values of the derivatives, scheme (2.1)-(2.2) has a stable behaviour.

In this paper it is our aim to get stability regions for *systems* of equations with *not* necessarily *constant* $\partial\Phi/\partial f$, $\partial\Phi/\partial z$ and $\partial K/\partial f$. Within the rather large class of kernel functions satisfying condition (3.3), equation (3.16b') may serve as a starting point in the derivation of stability regions. Such derivations may often be simplified by using the following lemma (for a proof we refer to e.g. [6]):

LEMMA 3.2. *Let* $A(\zeta)$ *be an* $(s*s)$ *matrix-valued function of the scalar* ζ *with eigenvalues* $\alpha_j(\zeta)$, $j = 1,2,\ldots,s$. *Then the roots of the equation* $\det\{A(\zeta)\} = 0$ *are within the unit circle when the roots of the equations* $\alpha_j(\zeta) = 0$, $j = 1,\ldots,s$ *are within the unit circle.*

By virtue of this lemma equation (3.16b') can be simplified in all cases where the matrices $\partial\Phi/\partial f$, $\partial\Phi/\partial z$ and $\partial K/\partial f$ have a common eigensystem $\{e_j\}$: the double sum of matrices in (3.16b') then also has eigenvectors e_j with eigenvalues $\alpha_j(\zeta)$, say, so that (3.16b') can be replaced by $\alpha_j(\zeta) = 0$, $j = 1,2,\ldots,s$.

Before analyzing some special formulas, the stability of the integro-differential equation itself will be investigated.

3.2.1 Inherent stability

We apply the same type of stability analysis to the variational equation of (1.1), i.e. the equation

$$(3.22)\qquad \frac{d}{dx}\,\Delta f(x) = \frac{\partial \Phi}{\partial f}\,\Delta f(x) + \frac{\partial \Phi}{\partial z}\int_{x_0}^{x} \frac{\partial K}{\partial f}\,(x,y,f(y))\Delta f(y)dy,$$

as we did for the variational equation (3.2). Again, restricting the kernel functions to the class satisfying (3.3) and introducing the additional perturbations

$$\Delta G^{(m)}(x) = \int_{x_0}^{x} R_m(y,f)\Delta f(y)dy,$$

we obtain a system of ordinary differential equations of the form

$$\frac{d}{dx}\,\Delta f(x) = \frac{\partial \Phi}{\partial f}\,\Delta f(x) + \frac{\partial \Phi}{\partial z}\sum_{m=1}^{r} Q_m(x)\Delta G^{(m)}(x),$$

$$\frac{d}{dx}\,\Delta G^{(m)}(x) = R_m(x,f(x))\Delta f(x), \quad m = 1,2,\ldots,r.$$

This system is inherently stable when all eigenvalues of its Jacobian matrix are in the left half plane. Since the characteristic equation of the Jacobian matrix is, by virtue of lemma 3.1,

$$[\det(\zeta I)]^{r-1}\,\det\{\zeta[\frac{\partial \Phi}{\partial f}\,(x,f(x),z(x)) - \zeta I] +$$

$$+ \frac{\partial \Phi}{\partial z}\,(x,f(x),z(x))\,\frac{\partial K}{\partial f}\,(x,x,f(x))\} = 0,$$

we may conclude that the integro-differential equation is *unstable* at the point x when the roots of the equation

$$(3.23)\qquad \det\{\zeta^2 I - \zeta\,\frac{\partial \Phi}{\partial f}\,(x,f(x),z(x)) - \frac{\partial \Phi}{\partial z}\,(x,f(x),z(x))\,\frac{\partial K}{\partial f}\,(x,x,f(x))\} = 0$$

are in the *right half plane*. In order to get more practical conditions from this equation we have to restrict the function Φ to special classes of functions. Such a class is the set of functions Φ which satisfy the condition that

$$(3.24)\qquad \frac{\partial \Phi}{\partial f}(x,f(x),z(x)) \textit{ and } \frac{\partial \Phi}{\partial z}\,(x,f(x),z(x))\,\frac{\partial K}{\partial f}\,(x,x,f(x)) \textit{ share the same eigensystem.}$$

When Φ belongs to this class, equation (3.23) can be reduced by using lemma 3.2 to the equation

$$\zeta^2 - \zeta\xi - \eta = 0, \tag{3.25}$$

where ξ and η are the eigenvalues of $\frac{\partial\Phi}{\partial f}$ and $\frac{\partial\Phi}{\partial z}\frac{\partial K}{\partial f}$, respectively, corresponding to a common eigenvector. Thus, a *necessary condition* for *inherent stability* of the integro-differential equation (1.1) satisfying (3.3) and (3.24) is

$$\text{Re } \xi < 0, \quad \text{Re } \eta < 0. \tag{3.26}$$

In the following we will assume that this condition is satisfied at the point x_{n+1}.

3.2.2 Stability regions of the {CH;CH} and {CH;AM} formulas

Let $\{a_\ell, b_\ell\}$ be defined by table 2.1, then equation (3.16b') reduces to

$$\det\{\tilde{\rho}(\zeta)\rho(\zeta)I - b_0 h\zeta^k\tilde{\rho}(\zeta)\frac{\partial\Phi}{\partial f}(x_{n+1},f_{n+1},z_{n+1}) - b_0h^2\zeta^k\tilde{\sigma}(\zeta)\frac{\partial\Phi}{\partial z}(x_{n+1},f_{n+1},z_{n+1})\frac{\partial K}{\partial f}(x_{n+1},x_{n+1},f_{n+1})\}=0. \tag{3.27}$$

Furthermore, let $\{\tilde{a}_\ell, \tilde{b}_\ell\}$ correspond either to the Curtiss-Hirschfelder or the Adams-Moulton formulas and let Φ and K satisfy the conditions (3.3) and (3.24). Then, by theorem 3.1 part ii) and lemma 3.2 we have local stability when the roots of the equation

$$\tilde{\rho}(\zeta)\rho(\zeta) - b_0 h\zeta^k\tilde{\rho}(\zeta)\xi_{n+1} - b_0h^2\zeta^k\tilde{\sigma}(\zeta)\eta_{n+1} = 0 \tag{3.27'}$$

are within the unit circle for all eigenvalues pairs $\{\xi_{n+1},\eta_{n+1}\}$ of the matrices $\{\frac{\partial\Phi}{\partial f}, \frac{\partial\Phi}{\partial z}\frac{\partial K}{\partial f}\}_{n+1}$ corresponding to their common eigenvectors. For negative values of ξ_{n+1} and η_{n+1} the stability regions of the {CH;CH} and {CH;AM} formulas are given in [12] and [13], respectively, and are reproduced in the appendix.

To the particular case of second kind Volterra integral equations these stability regions do also apply when the eigenvalues ξ_{n+1} and η_{n+1}

are understood to be the eigenvalues of the matrices

$$\frac{\partial \Phi}{\partial f}\,(x_{n+1},f_{n+1},z_{n+1}) = \frac{\partial K^*}{\partial f}\,(x_{n+1},x_{n+1},f_{n+1})$$

and

$$\frac{\partial \Phi}{\partial z}(x_{n+1},f_{n+1},z_{n+1})\frac{\partial K}{\partial f}(x_{n+1},x_{n+1},f_{n+1}) = \begin{cases} \dfrac{\partial^2 K^*}{\partial x \partial f}(x_{n+1},x_{n+1},f_{n+1}) \\ \\ h^{-1}\displaystyle\sum_{i=k_1}^{k_2} d_i \frac{\partial K^*}{\partial f}(x_{n+1-i},x_{n+1},f_{n+1}) \end{cases}$$

and provided that these matrices have a common eigensystem.

3.2.3 Stability conditions for the trapezoidal rule

A frequently used method is based on the application of the trapezoidal rule both for the approximation of the integral and the derivative in equation (1.1). This results in the coefficients ($k = \tilde{k} = 1$)

$$a_0 = \tilde{a}_0 = 1, \quad a_1 = \tilde{a}_1 = -1, \quad b_0 = \tilde{b}_0 = b_1 = \tilde{b}_1 = \tfrac{1}{2}.$$

Equation (3.16b') assumes the form

$$\begin{aligned} (3.28) \quad \det\{&[I - \tfrac{1}{2}hX_{n+1} - \tfrac{1}{4}h^2Y_{n+1,n+1}]\zeta^2 - \\ &[2I - \tfrac{1}{2}hX_{n+1} + \tfrac{1}{2}hX_n + \tfrac{1}{4}h^2Y_{n+1,n} + \tfrac{1}{4}h^2Y_{n,n+1}]\zeta + \\ &[I + \tfrac{1}{2}hX_n - \tfrac{1}{4}h^2Y_{n,n}]\} = 0, \end{aligned}$$

where we have written

$$X_n = \frac{\partial \Phi}{\partial f}(x_n,f_n,z_n), \quad Y_{n,m} = \frac{\partial \Phi}{\partial z}(x_n,f_n,z_n)\frac{\partial K}{\partial f}(x_n,x_m,f_m).$$

In order to continue the analysis we restrict the functions Φ and K to the class of functions for which all matrices X_n and $Y_{n,m}$ occurring in (3.28) possess the same eigensystem with eigenvalues ξ_n and $\eta_{n,m}$. By theorem 3.1

and lemma 3.2 equation (3.28) can be reduced to

$$(3.28') \quad [1-\tfrac{1}{2}h\xi_{n+1} - \tfrac{1}{4}h^2\eta_{n+1,n+1}]\zeta^2 - [2-\tfrac{1}{2}h(\xi_{n+1}-\xi_n) + \tfrac{1}{4}h^2(\eta_{n+1,n}+\eta_{n,n+1})]\zeta + [1 + \tfrac{1}{2}h\xi_n - \tfrac{1}{4}h^2\eta_{n,n}] = 0.$$

For real values of the eigenvalues ξ and η the stability region easily follows from Hurwitz' criterion. A straightforward calculation yields the conditions

$$(3.29) \quad \begin{aligned} &\xi_{n+1} < 0, \quad \eta_{n+1,n+1} < 0 \\ &h(\xi_{n+1}+\xi_n) + \tfrac{1}{2}h^2(\eta_{n+1,n+1}-\eta_{n,n}) < 0, \\ &h^2(\eta_{n+1,n+1}+\eta_{n+1,n}+\eta_{n,n+1}+\eta_{n,n}) < 0, \\ &h(\xi_{n+1}-\xi_n) + \tfrac{1}{4}h^2(\eta_{n+1,n+1}-\eta_{n+1,n}-\eta_{n,n+1}+\eta_{n,n}) < 4, \end{aligned}$$

where we have assumed that the equation is inherently stable at the point x_{n+1} (cf. condition (3.26)). Note that these conditions are more restrictive than those given by BRUNNER and LAMBERT [1] ($\xi_{n+1} < 0$, $\eta_{n+1,n+1} < 0$). But it is also evident that (3.29) only restricts the integration step h substantially when ξ and η, and therefore the matrices $\partial\Phi/\partial f$ and $\partial\Phi/\partial z\ \partial K/\partial f$, are rapidly varying with their arguments. In such cases, however, the integration step should be suffiently small in order to get enough accuracy. Hence, it is expected that the conditions (3.29) yield quite acceptable integration steps in actual computation.

We will illustrate the use of conditions (3.29) by applying them to the equation

$$(3.30) \quad \begin{aligned} &\frac{d}{dx} f(x) = -b + (b+1)\cos x + ((b+c)x-a)(f(x) - \sin x) + bz(x) \\ &z(x) = \int_0^x f(y)dy \end{aligned},$$

where a, b and c are given parameters. This equation is, in fact, the differentiated form of the Volterra integral equation

$$(3.30') \qquad f(x) = (1-c)\sin x + a-bx+((b+c)x-a)\cos x + \int_0^x (-a+bx+cy)f(y)dy.$$

Evidently, we have

$$\xi_n = (b+c)x_n - a, \qquad \eta_{n,m} = b,$$

so that (3.29) leads to the stability conditions (at the point x_{n+1})

$$(3.31) \qquad \begin{aligned} &(b+c)(x_n+h) < a, \quad (b+c)(x_n+\tfrac{1}{2}h) < a, \quad b < 0, \\ &(b+c)h^2 < 4. \end{aligned}$$

The first three conditions are automatically fulfilled when we restrict the class of equations to inherently stable equations. The fourth condition limits the stepsize in cases where $b+c > 0$. It may be interesting to compare the conditions (3.31) with those obtained for the trapezoidal rule when directly applied to the integral equation (3.30'). A stability analysis carried out in [6] results in the same conditions as (3.31) except for the fourth one which becomes $ch^2 < 4$. Thus, we may conclude that differentiation of the integral equation relaxes the stability conditions if $c \gg 1$ and $b+c \cong 0$.

4. A STARTING SCHEME, BASED ON EXTRAPOLATION OF THE TRAPEZOIDAL RULE

In order to start the two schemes (2.1)-(2.2) and (2.1") we need, in addition to the given initial vector f_0, the vectors $f_1, f_2, \ldots, f_{k-1}$. These vectors should be computed with an error of magnitude $O(h^k)$ as $h \to 0$. We will use the following starting schemes which are based on extrapolation of the trapezoidal rule. Starting vectors for (2.1)-(2.2) are computed with the scheme (we assume $x_0 = 0$)

$$f_n^h = f_{n-1}^h + \frac{h}{2}[\Phi((n-1)h, f_{n-1}^h, z_{n-1}^h) + \Phi(nh, f_n^h, z_n^h)] \qquad n = 1,2,\ldots,k-1;$$

$$z_n^h = h \sum_{i=0}^{n}{}'' K(nh, x_i^h, f_i^h)$$

$$(4.1) \qquad f_n^{h/2} = f_{n-1}^{h/2} + \frac{h}{4}[\Phi((n-1)h/2, f_{n-1}^{h/2}, z_{n-1}^{h/2}) + \Phi(nh/2, f_n^{h/2}, z_n^{h/2})] \qquad n = 1,2,\ldots,2(k-1);$$

$$z_n^{h/2} = \frac{h}{2} \sum_{i=0}^{n}{}'' K(nh/2, x_i, f_i^{h/2})$$

$$f_n = \frac{4}{3} f_{2n}^{h/2} - \frac{1}{3} f_n^h, \qquad n = 1,2,\ldots,k-1.$$

Starting vectors for (2.1") are computed with the scheme (for $x_0 = 0$)

$$f_n^h = g(nh) + h \sum_{j=0}^{n}{}'' K^*(nh, x_j, f_j^h), \qquad n = 1,2,\ldots,k-1;$$

$$(4.2) \qquad f_n^{h/2} = g(nh/2) + \frac{h}{2} \sum_{j=0}^{n}{}'' K^*(nh/2, jh/2, f_j^{h/2}), \qquad n = 1,2,\ldots,2(k-1);$$

$$f_n = \frac{4}{3} f_{2n}^{h/2} - \frac{1}{3} f_n^h, \qquad n = 1,2,\ldots,k-1.$$

In [10] it is proved that for $k \geq 2$ the scheme (4.2) has an error of magnitude $O(h^5)$. This also holds for the scheme (4.1), which can be seen by writing the integro-differential equation (1.1) as an integral equation

$$\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} f_0 \\ 0 \end{pmatrix} + \int_{x_0}^{x} \begin{pmatrix} \Phi(y, f_1(y), f_2(y)) \\ K(x, y, f_1(y)) \end{pmatrix} dy,$$

and by verifying that application of the scheme (4.2) to this "integral equation" is equivalent with the scheme (4.1).

Since the trapezoidal rule has a local error of order $O(h^3)$ as $h \to 0$, it is sufficient to apply only the first line of the scheme (4.2) when $k = 2$ or $k = 3$. However, when $k = 4$ or $k = 5$, we have to apply the full scheme, in order to get the required order k. For $k = 6$, we may either continue the extrapolation process or try to improve the starting vectors by other methods. When extrapolation is used, we first compute the vectors $f_n^{h/4}$, $n = 1,2,..,4(k-1)$,

and then we form the combinations

$$\frac{16}{15}\left[\frac{4}{3}f_{4n}^{h/4} - \frac{1}{3}f_{2n}^{h/2}\right] - \frac{1}{15}\left[\frac{4}{3}f_{2n}^{h/2} - \frac{1}{3}f_{n}^{h}\right], \qquad n = 1,2,\ldots,k-1.$$

One easily proves that this expression approximates f(nh) at least within $O(h^6)$, as $h \to 0$. The stability conditions for this starting scheme are given by (3.29).

In [10] another $O(h^6)$ starting scheme is given, based on a high order Newton-Cotes quadrature formula, but special attention must be paid to the solution of the nonlinear equations involved, in order to assure the convergence of the scheme. Therefore we have preferred the schemes based on the trapezoidal rule.

5. NUMERICAL EXPERIMENTS

In this section we present the results of numerical experiments with the scheme {CH;AM}, given in section 2.1, both for Volterra integro-differential equations (section 5.1) and for Volterra integral equations (section 5.2). Moreover, we present results of numerical experiments with the scheme {CH;CH} for Volterra integro-differential equations (section 5.1), in order to illustrate the improved stability properties of this scheme, compared with the {CH;AM} scheme. Finally, in section 5.3 we pay attention to the performance of the {CH;AM} scheme, in comparison with an implicit Runge-Kutta scheme of DE HOOG and WEISS ([4]).

The numerical experiments have been carried out with test problems with known exact solution. Our purpose was to test the convergence and stability properties of our schemes, rather than design and implement an automatic integro-differential (cq. integral) equation solver. No special strategies have been incorporated in our implementations. Integration was performed with constant stepsize. The nonlinear equations, arising in the case of test problems which are nonlinear in f, were "exactly" solved (i.e., almost within machine precision) by the Newton-Raphson method. As a first approximation of f_{n+1} the value of f_n was used. The stopping criterion of the iteration process was based on a Newton correction of 10^{-12}. The starting values $f_1, f_2, \ldots, f_{k-1}$ were computed with the starting schemes given in

section 4. All calculations have been performed on a CDC CYBER 73-28/173-8 installation, using 14 significant digits.

5.1 Integro-differential equations

In this section we give results of numerical experiments with the {CH;AM} and the {CH;CH} scheme, respectively.

Problem 5.1.1. (linear)

$$\begin{cases} f'(x) = \exp(x) - f(x) - \int_0^x \exp(x-y) f(y)\,dy, & 0 \le x \le 2, \\ f(0) = 1, \end{cases}$$

with exact solution $f(x) \equiv 1$. Our purpose is to demonstrate that the global error is $O(h^k)$ as $h \to 0$. In table 5.1.1a and 5.1.1b we give the relative error at $x = 2$ for the {CH;AM} and the {CH;CH} scheme.

	rel. error at x = 2				
h	k = 2	k = 3	k = 4	k = 5	k = 6
1/4	$1.0_{10}{-2}$	$1.1_{10}{-3}$	$1.7_{10}{-4}$	$4.9_{10}{-5}$	$3.5_{10}{-6}$
1/8	$2.6_{10}{-3}$	$1.5_{10}{-4}$	$1.2_{10}{-5}$	$1.5_{10}{-6}$	$8.5_{10}{-8}$
1/16	$6.5_{10}{-4}$	$1.9_{10}{-5}$	$7.7_{10}{-7}$	$4.1_{10}{-8}$	$1.5_{10}{-9}$
1/32	$1.6_{10}{-4}$	$2.5_{10}{-6}$	$4.9_{10}{-8}$	$1.2_{10}{-9}$	$2.5_{10}{-11}$
1/64	$4.1_{10}{-5}$	$3.1_{10}{-7}$	$3.1_{10}{-9}$	$3.6_{10}{-11}$	$3.4_{10}{-13}$
1/128	$1.0_{10}{-5}$	$3.9_{10}{-8}$	$1.9_{10}{-10}$	$6.2_{10}{-13}$	$9.2_{10}{-14}$

Table 5.1.1.a. Results with the {CH;AM} scheme for problem 5.1.1.

h	rel. error at x = 2				
	k = 2	k = 3	k = 4	k = 5	k = 6
1/4	$3.6_{10}{}^{-2}$	$6.0_{10}{}^{-3}$	$9.1_{10}{}^{-4}$	$1.3_{10}{}^{-4}$	$1.9_{10}{}^{-5}$
1/8	$9.8_{10}{}^{-3}$	$8.9_{10}{}^{-4}$	$7.9_{10}{}^{-5}$	$7.3_{10}{}^{-6}$	$7.1_{10}{}^{-7}$
1/16	$2.5_{10}{}^{-3}$	$1.2_{10}{}^{-4}$	$5.5_{10}{}^{-6}$	$2.7_{10}{}^{-7}$	$1.4_{10}{}^{-8}$
1/32	$6.4_{10}{}^{-4}$	$1.5_{10}{}^{-5}$	$3.6_{10}{}^{-7}$	$9.3_{10}{}^{-9}$	$2.4_{10}{}^{-10}$
1/64	$1.6_{10}{}^{-4}$	$1.9_{10}{}^{-6}$	$2.3_{10}{}^{-8}$	$3.1_{10}{}^{-10}$	$6.5_{10}{}^{-12}$
1/128	$4.1_{10}{}^{-5}$	$2.4_{10}{}^{-7}$	$1.5_{10}{}^{-9}$	$1.9_{10}{}^{-11}$	$2.1_{10}{}^{-11}$

Table 5.1.1.b. Results with the {CH;CH} scheme for problem 5.1.1.

From these tables the asymptotic convergence factor $(\frac{1}{2})^k$ is obvious; further the {CH;AM} scheme is more accurate than the {CH;CH} scheme. This is due to the fact that the quadrature rule in the {CH;AM} scheme corresponds to a more accurate linear multistep method, viz. Adams-Moulton, than the quadrature rule in the {CH;CH} scheme.

Problem 5.1.2. (linear)

$$\begin{cases} f'(x) = 50-50.75\ \exp(-x) - 0.25\ f(x) - 50 \int_0^x f(y)dy, & 0 \le x \le x_e \\ f(0) = 1, \end{cases}$$

with exact solution $f(x) = \exp(-x)$. This linear problem depends strongly on the "Volterra" part of the equation. Our aim was to verify the stability theory for the {CH;AM} and the {CH;CH} scheme. Since the values

$$\xi = \frac{\partial \Phi}{\partial f} = -0.25 \qquad \text{and} \qquad \eta = \frac{\partial \Phi}{\partial z} \cdot \frac{\partial K}{\partial f} = -50$$

do not depend on x,y or f, our local stability conditions may be used as conditions for global stability. In table 5.1.2a and 5.1.2b we indicate by S and U respectively, whether or not the point $(h\xi, h^2\eta)$ belongs to the

stability region (see appendix) of the corresponding scheme. The second entry in these tables is the absolute error at the endpoint of integration $x_e = 128h$.

h	k = 2	k = 3	k = 4	k = 5	k = 6
1/2	S,8.0$_{10}$-15	S,3.5$_{10}$-9	U,6.4$_{10}$+2	U,2.8$_{10}$+5	U,3.0$_{10}$+18
1/4	S,1.5$_{10}$-12	U,1.2$_{10}$-1	U,9.4$_{10}$+4	U,3.4$_{10}$+10	U,2.2$_{10}$+14
1/8	S,5.1$_{10}$-6	U,7.3$_{10}$-3	U,9.6$_{10}$-4	U,3.8$_{10}$-6	S,1.1$_{10}$-9
1/16	S,6.6$_{10}$-6	S,8.9$_{10}$-7	S,4.8$_{10}$-7	S,4.8$_{10}$-7	S,9.7$_{10}$-10
1/32	S,5.8$_{10}$-5	S,5.9$_{10}$-6	S,8.2$_{10}$-9	S,4.1$_{10}$-8	S,9.3$_{10}$-12

Table 5.1.2a. Stability test of the {CH;AM} scheme with respect to problem 5.1.2

h	k = 2	k = 3	k = 4	k = 5	k = 6
1/2	S,2.5$_{10}$-14	S,7.1$_{10}$-12	U,7.5$_{10}$+1	U,7.9$_{10}$+11	U,7.8$_{10}$+20
1/4	S,3.9$_{10}$-14	U,9.1$_{10}$-4	U,3.4$_{10}$+5	U,4.6$_{10}$+12	U,2.3$_{10}$+17
1/8	S,6.1$_{10}$-7	U,6.2$_{10}$-2	U,1.2$_{10}$-1	U,1.8$_{10}$-2	S,1.2$_{10}$-5
1/16	S,2.2$_{10}$-4	S,6.4$_{10}$-5	S,5.2$_{10}$-9	S,5.9$_{10}$-7	S,2.4$_{10}$-9
1/32	S,1.7$_{10}$-4	S,1.6$_{10}$-5	S,7.6$_{10}$-8	S,4.7$_{10}$-8	S,2.4$_{10}$-11

Table 5.1.2b. Stability test of the {CH;CH} scheme with respect to problem 5.1.2

The behaviour of the numerical solution agrees with the predicted behaviour; for the case h = 1/8, k = 3,4,5 this may be less apparent, but this is explained by the fact that the point $(h\xi, h^2\eta)$ is close to the boundary of the stability region resulting in an amplification factor $|\zeta|$ close to 1.

While this test showed the presence of a region of instability near the $h^2\eta$ - axis, the following test will show that the stability regions of the {CH;CH} scheme are much larger than those of the {CH;AM} scheme.

Problem 5.1.3. (non-linear)

$$\begin{cases} f'(x) = [d(x) - \alpha f(x) - \beta z(x)]^3 - 1, & 0 \le x \le 16 \\ z(x) = \int_0^x (x + \gamma y)^\delta f^3(y)dy, \\ f(0) = 1. \end{cases}$$

With

$$d(x) = 1 + \alpha + \frac{\beta x^{\delta+1}}{\gamma(\delta+1)} [(1+\gamma)^{\delta+1} - 1],$$

the exact solution becomes $f(x) \equiv 1$. For ξ_{n+1} and η_{n+1} we have (computed along the exact solution)

$$\xi_{n+1} = -3\alpha, \quad \eta_{n+1} = -9\beta x_{n+1}^\delta (1+\gamma)^\delta.$$

Hence $|\eta_{n+1}|$ increases monotonically, whereas ξ_{n+1} remains constant. We have considered the case

$$\alpha = 40, \quad \beta = 15, \quad \gamma = 2, \quad \delta = 3/2, \quad h = 1/8.$$

The values of α and h were chosen such that the vertical line $h\xi = -3h\alpha$ is within the stability regions of the {CH;CH} schemes. Hence it is expected that the {CH;CH} schemes will integrate this problem in a stable manner. From the stability regions of the {CH;AM} schemes one can predict, theoretically, the values of $h^2\eta$, and therefore the values of x for which the {CH;AM} scheme will be stable. Approximate x-values are listed in table 5.1.3a.

k	2	3	4	5	6
x<	∞	5.20	3.67	3.07	2.77

Table 5.1.3a. Predicted stability ranges for x for the {CH;AM} scheme with respect to problem 5.1.3

Having stated our prediction, a final numerical experiment must give a decisive answer. In table 5.1.3b we have listed for the {CH;AM} scheme the true error $f(x_n)-f_n$ at the point x = 1 and at some other relevant points. The points were chosen as follows: for a k-th order {CH;AM} scheme we have listed one point close to the x-value for which instability is predicted by the theory (cf. table 5.1.3a) and another point at which the integration process was (mostly prematurely) terminated. In the table of results the prediction point is marked with a dotted line. In table 5.1.3c we list the results of the {CH;CH} scheme at the same x-points. Inspection of these tables reveals that:

(i) the {CH;CH} scheme is indeed stable; in fact, the true error is damped out.

(ii) the {CH;AM} scheme indeed becomes unstable for $k \geq 3$; however, for this problem the theory predicts instability for smaller values of x than shown in actual computation.

x	k = 2	k = 3	k = 4	k = 5	k = 6
1.000	$+5.7_{10}{}^{-5}$	$-5.8_{10}{}^{-5}$	$-2.1_{10}{}^{-6}$	$-2.1_{10}{}^{-6}$	$+1.3_{10}{}^{-7}$
2.750	$-2.2_{10}{}^{-5}$	$-8.2_{10}{}^{-6}$	$-2.3_{10}{}^{-7}$	$-1.2_{10}{}^{-7}$	$+1.3_{10}{}^{-7}$
3.000	$-2.0_{10}{}^{-5}$	$-7.1_{10}{}^{-6}$	$-2.0_{10}{}^{-7}$	$-1.1_{10}{}^{-7}$	$+1.5_{10}{}^{-7}$
3.625	$-1.6_{10}{}^{-5}$	$-5.3_{10}{}^{-6}$	$-1.5_{10}{}^{-7}$	$-1.2_{10}{}^{-7}$	$-6.2_{10}{}^{-7}$
5.125	$-1.1_{10}{}^{-5}$	$-3.3_{10}{}^{-6}$	$-9.0_{10}{}^{-8}$	$-5.5_{10}{}^{-6}$	$-1.8_{10}{}^{-3}$
5.250	$-1.0_{10}{}^{-5}$	$-3.2_{10}{}^{-6}$	$-8.7_{10}{}^{-8}$	$+9.9_{10}{}^{-6}$	*
6.375	$-8.2_{10}{}^{-6}$	$-2.4_{10}{}^{-6}$	$-7.3_{10}{}^{-8}$	$-4.3_{10}{}^{-3}$	*
9.375	$-5.2_{10}{}^{-6}$	$-1.5_{10}{}^{-6}$	$-5.6_{10}{}^{-3}$	*	*
14.250	$-3.1_{10}{}^{-6}$	$-5.3_{10}{}^{-4}$	*	*	*
16.000	$-2.7_{10}{}^{-6}$	*	*	*	*

Table 5.1.3b. Results of the {CH;AM} scheme for problem 5.1.3

x	k = 2	k = 3	k = 4	k = 5	k = 6
1.000	$+4.4_{10}{}^{-4}$	$-4.0_{10}{}^{-5}$	$-2.5_{10}{}^{-6}$	$-2.2_{10}{}^{-6}$	$+3.7_{10}{}^{-7}$
2.750	$-5.9_{10}{}^{-5}$	$-8.3_{10}{}^{-7}$	$-5.5_{10}{}^{-7}$	$-2.2_{10}{}^{-7}$	$+3.9_{10}{}^{-8}$
3.000	$-5.5_{10}{}^{-5}$	$-8.8_{10}{}^{-7}$	$-4.5_{10}{}^{-7}$	$-1.4_{10}{}^{-7}$	$-9.5_{10}{}^{-9}$
3.625	$-4.6_{10}{}^{-5}$	$-8.9_{10}{}^{-7}$	$-3.1_{10}{}^{-7}$	$-1.4_{10}{}^{-7}$	$+7.4_{10}{}^{-9}$
5.125	$-3.2_{10}{}^{-5}$	$-7.6_{10}{}^{-7}$	$-1.7_{10}{}^{-7}$	$-9.2_{10}{}^{-8}$	$+9.8_{10}{}^{-9}$
5.250	$-3.1_{10}{}^{-5}$	$-7.4_{10}{}^{-7}$	$-1.6_{10}{}^{-7}$	$-9.0_{10}{}^{-8}$	$-1.2_{10}{}^{-8}$
6.375	$-2.5_{10}{}^{-5}$	$-6.4_{10}{}^{-7}$	$-1.2_{10}{}^{-7}$	$-7.1_{10}{}^{-8}$	$-3.9_{10}{}^{-9}$
9.375	$-1.6_{10}{}^{-5}$	$-4.4_{10}{}^{-7}$	$-7.1_{10}{}^{-8}$	$-4.4_{10}{}^{-8}$	$+1.2_{10}{}^{-9}$
14.250	$-9.9_{10}{}^{-6}$	$-2.8_{10}{}^{-7}$	$-4.1_{10}{}^{-8}$	$-2.7_{10}{}^{-8}$	$+7.4_{10}{}^{-10}$
16.000	$-8.6_{10}{}^{-6}$	$-2.5_{10}{}^{-7}$	$-3.6_{10}{}^{-8}$	$-2.3_{10}{}^{-8}$	$+6.6_{10}{}^{-10}$

Table 5.1.3c. Results of the {CH;CH} scheme for problem 5.1.3

5.2 Integral equation

Results are presented of numerical experiments with the {CH;AM} scheme (2.1") for Volterra integral equations of the second kind. These experiments closely resemble a number of tests carried out in [10]. However, in order to treat the starting procedure, the Newton-Raphson iteration, etc. in the {CH;AM} scheme in the same manner as done in the experiments for integro-differential equations in the preceding section, we have repeated these tests. The results are only slightly different from those reported in [10].

Problem 5.2.1. (Renewal equation from FELLER [2])

$$f(x) = \tfrac{1}{2}x^2\exp(-x) + \tfrac{1}{2}\int_0^x (x-y)^2\exp(-x+y)f(y)dy, \qquad 0 \le x \le 2,$$

with exact solution $f(x) = \frac{1}{3} - \frac{1}{3}\exp(-3x/2)\{\cos(\frac{1}{2}x\sqrt{3}) + \sqrt{3}\sin(\frac{1}{2}x\sqrt{3})\}$. In table 5.2.1 we give the relative error at x = 2, for various choices of h

and k. One easily verifies the asymptotic convergence factor of $(1/2)^k$.

h	k = 2	k = 3	k = 4	k = 5	k = 6
1/4	$3.1_{10}{-2}$	$1.8_{10}{-2}$	$4.5_{10}{-3}$	$1.8_{10}{-3}$	$5.2_{10}{-4}$
1/8	$5.9_{10}{-3}$	$1.8_{10}{-3}$	$1.8_{10}{-4}$	$5.8_{10}{-5}$	$9.7_{10}{-6}$
1/16	$1.3_{10}{-3}$	$2.1_{10}{-4}$	$8.3_{10}{-6}$	$2.0_{10}{-6}$	$1.9_{10}{-7}$
1/32	$3.0_{10}{-4}$	$2.5_{10}{-5}$	$4.4_{10}{-7}$	$6.8_{10}{-8}$	$3.4_{10}{-9}$
1/64	$7.3_{10}{-5}$	$3.1_{10}{-6}$	$2.6_{10}{-8}$	$2.3_{10}{-9}$	$5.7_{10}{-11}$

Table 5.2.1. Results with the {CH;AM} scheme for problem 5.2.1

Problem 5.2.2 (non-linear)

$$f(x) = -15x+17(\exp(x)-1) + \int_0^x \{16(y-x)-1\}\exp(f(y))dy, \quad 0 \le x \le x_e,$$

with exact solution $f(x) = x$. The purpose of this nonlinear problem is to verify the stability theory for the {CH;AM} scheme for Volterra integral equations of the second kind. We have

$$\frac{\partial K^*}{\partial f}(x_{n+1},x_{n+1},f_{n+1}) = -\exp(f_{n+1}), \qquad \text{and}$$

$$\frac{\partial^2 K^*}{\partial x \partial f}(x_{n+1},x_{n+1},f_{n+1}) = -16\exp(f_{n+1}).$$

In table 5.2.2 we have indicated, by S for stable and U for unstable, the stability behaviour of the {CH;AM} scheme, as predicted by the stability regions given in the appendix. The second entry is the absolute error at the endpoint of integration $x_e = 128h$. In case the integration had to be stopped prematurely because of the development of instabilities in the numerical solution, the index of the last computed function value has been added between parentheses. Clearly, the observed behaviour agrees with the predicted behaviour.

h	k = 2	k = 3	k = 4	k = 5	k = 6
1/2	S,7.9_{10}-2	U,2.2_{10}-2	U,6.9_{10}-3	U,1.6_{10}-1(23)	U,2.1_{10}-1(18)
1/4	S,2.2_{10}-2	S,3.5_{10}-3	U,6.0_{10}-4	U,4.6_{10}-1(65)	U,3.4_{10}-1(37)
1/8	S,6.0_{10}-3	S,4.9_{10}-4	S,4.5_{10}-5	S,4.4_{10}-6	U,2.0_{10}-3
1/16	S,1.5_{10}-4	S,6.6_{10}-5	S,3.1_{10}-6	S,1.5_{10}-7	S,8.1_{10}-9
1/32	S,3.9_{10}-4	S,8.5_{10}-7	S,2.0_{10}-7	S,5.2_{10}-9	S,1.4_{10}-10

Table 5.2.2. Stability of the {CH;AM} scheme with respect to problem 5.2.2.

5.3. Comparison with a method of de Hoog and Weiss

In [4] de Hoog and Weiss have proposed implicit Runge-Kutta schemes which have stability properties, comparable with ours. In [10] we have compared the performance of a $O(h^7)$ scheme of de Hoog and Weiss with the $O(h^6)$ - variant of our {CH;AM} scheme. This $O(h^7)$ scheme of de Hoog and Weiss is optimal in the sense that its order of convergence is achieved with a minimum number (namely four) of points used in one Runge-Kutta step. It turned out that the order of convergence of the scheme of de Hoog and Weiss decreases from the expected value $O(h^7)$ to about $O(h^4)$, as the value of $|\partial K/\partial f|$ increases, whereas the expected order of convergence of our scheme is *not* affected by the value of $|\partial K/\partial f|$. As a consequence, our scheme turned out to be more efficient than that of de Hoog and Weiss, for problems for which $|\partial K/\partial f|$ is large. An explanation of the order decrease of the scheme of de Hoog and Weiss may be given by the fact that the error "constant" in the expansion of the error, committed when applying the scheme of de Hoog and Weiss, *itself* satisfies a linear Volterra integral equation of the second kind, *with kernel* $\partial K/\partial f$ (see [4], theorem 4.1). So this error "constant" depends heavily on the value of $|\partial K/\partial f|$.

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APPENDIX. Stability regions of the {CH;AM} and the {CH;CH} schemes

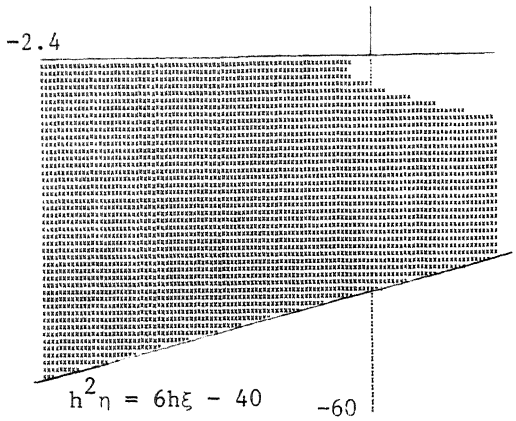
We present here the stability regions in the $(h\xi, h^2\eta)$-plane of the {CH;AM} and the {CH;CH} schemes, determined by equation (3.27'). The shaded areas indicate stability. Since the integral equation *itself* is stable only in the third quadrant (cf. section 3.2.1), we have confined ourselves to this quadrant (and a small strip in the fourth quadrant). The regions are displayed for k = 3, 4, 5 and 6 (for k = 2 the stability region contains the *whole* third quadrant).

{CH;AM}

$h^2\eta$

$h\xi$

{CH;CH}



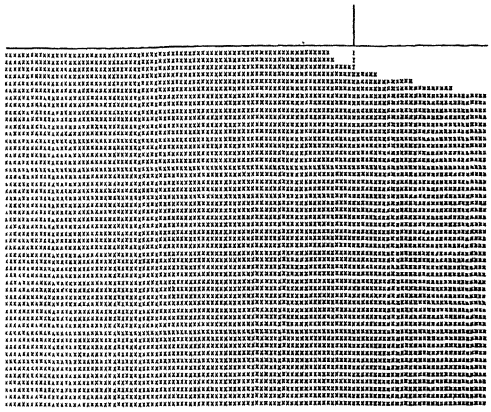
$h^2\eta = 6h\xi - 40$

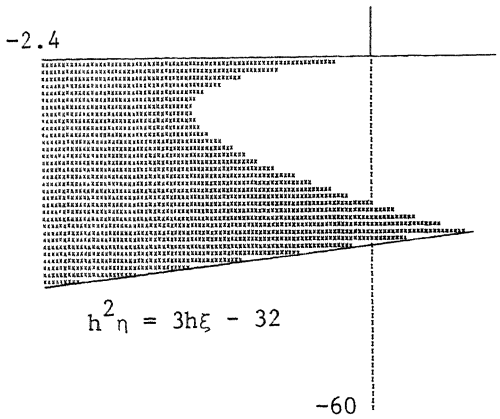
k = 3

units:

→ 0.03

↑ 1.2





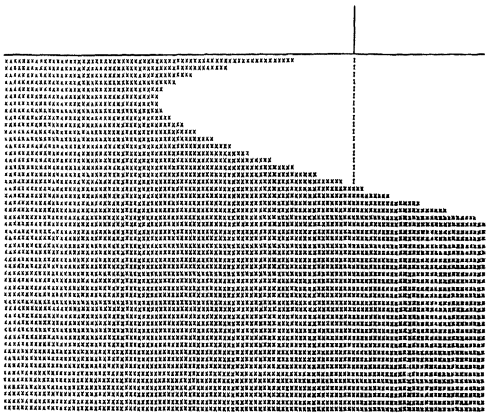
$h^2\eta = 3h\xi - 32$

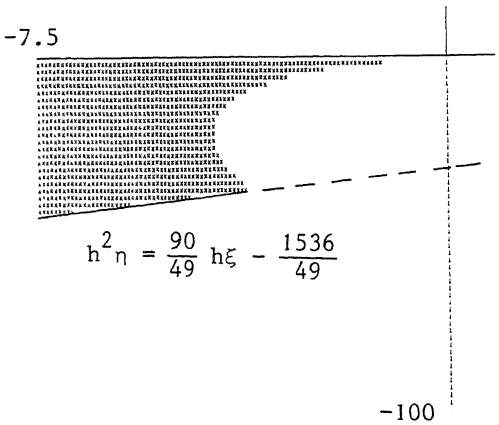
k = 4

units:

→ 0.03

↑ 1.2





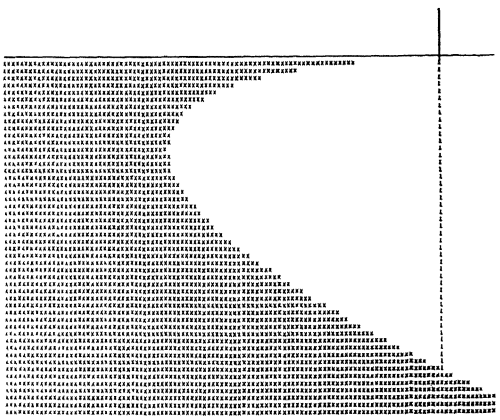
$h^2\eta = \frac{90}{49} h\xi - \frac{1536}{49}$

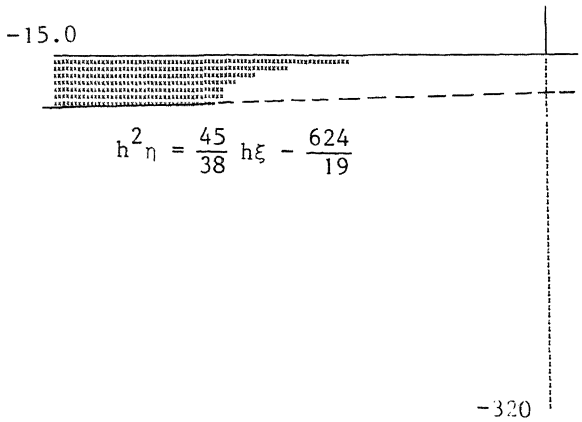
k = 5

units:

→ 0.075

↑ 2.0





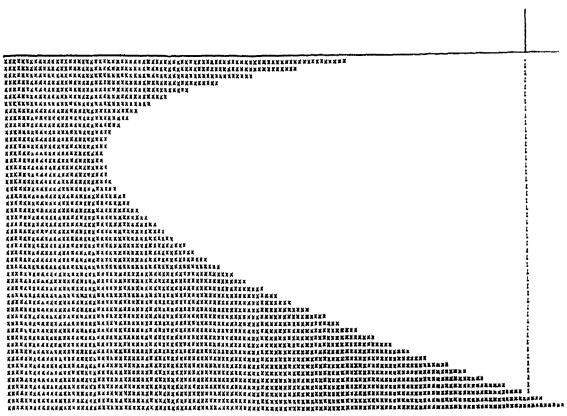
$h^2\eta = \frac{45}{38} h\xi - \frac{624}{19}$

k = 6

units:

→ 0.125

↑ 6.4



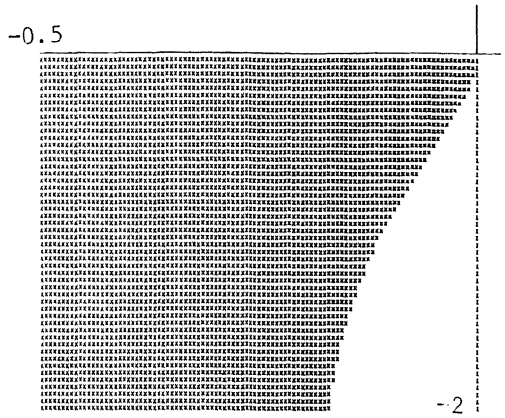
Details near the origin

{CH;AM}

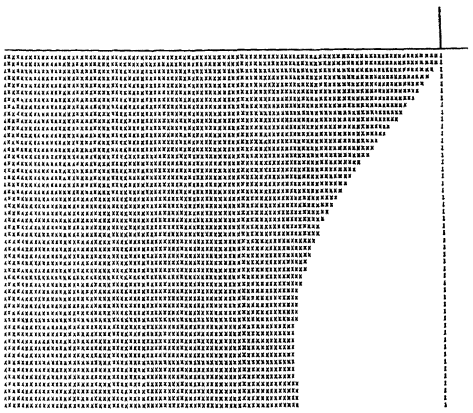
$h^2\eta$

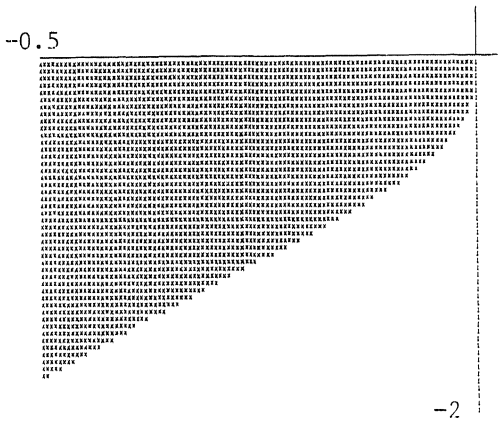
$h\xi$

{CH;CH}

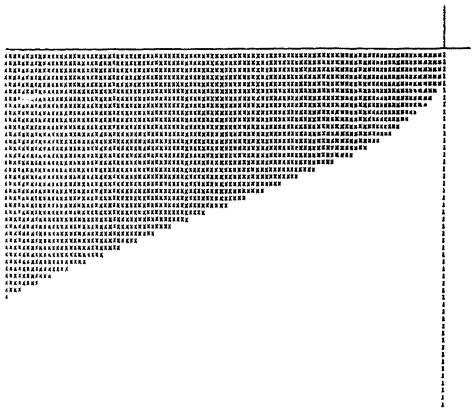


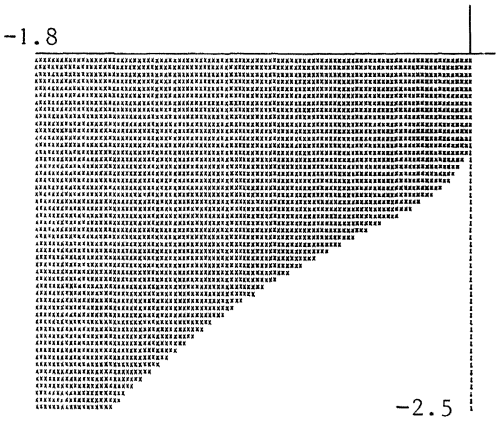
k = 3
units:
→ 0.005
↑ 0.04



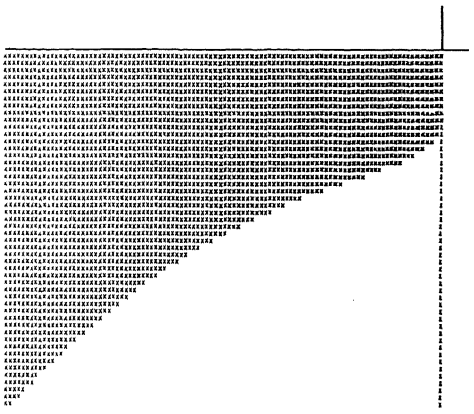


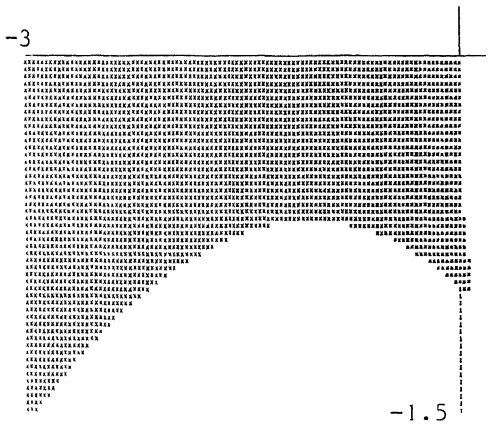
k = 4
units:
→ 0.005
↑ 0.04





k = 5
units:
→ 0.018
↑ 0.05





k = 6
units:
→ 0.03
↑ 0.03