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A NOTE ON C^0 GALERKIN METHODS FOR TWO-POINT BOUNDARY PROBLEMS

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A note on C^0 Galerkin methods for two-point boundary problems*)

by

M. Bakker

ABSTRACT

As is known [4], the C^0 Galerkin solution of a two-point boundary problem using piecewise polynomial functions, has $O(h^{2k})$ convergence at the knots, where k is the degree of the finite element space. In this note, it is proved that on any segment there are k-1 interior points where the Galerkin solution is of $O(h^{k+2})$, one point better than the global order of convergence. These points are the Lobatto points.

KEY WORDS & PHRASES: *two-point boundary problems, finite element method, superconvergence, Lobatto points.*

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*) This paper will be submitted for publication elsewhere.

1. INTRODUCTION

We consider the two-point boundary problem

$$(1)\quad \begin{aligned} &Lu \equiv -(p(x)u')' + q(x)u = f(x), \qquad x \in [0,1] = I;\\ &u(0) = u(1) = 0. \end{aligned}$$

We suppose that p, q and f are such that (1) has a unique and sufficiently smooth solution.

Let, for a constant integer N, Δ: $0 = x_0 < x_1 < \ldots < x_N = 1$ be a partition of I with

$$h = N^{-1}; \quad x_j = jh; \quad I_j = [x_{j-1}, x_j]$$

and let for a constant integer $k \geq 2$ and for any interval $E \subset I$, $P_k(E)$ be the class of polynomials of degree at most k restricted to E.

We define for $m \geq 0$ and $s \geq 1$

$$(2)\quad \begin{aligned} &W^{m,s}(E) = \{v \mid D^j v \in L^s(E),\ j = 0,\ldots,m\};\\ &H^m(E) = W^{m,2}(E);\\ &H_0^1(I) = \{v \mid v \in H^1(I);\ v(0) = v(1) = 0\};\\ &M_0^k(\Delta) = \{v \mid v \in H_0^1(I);\ v \in P_k(I_j),\ j = 1,\ldots,N\};\\ &\|v\|_{W^{m,s}(E)} = [\sum_{j=0}^{m} \|D^j v\|^2_{L^s(E)}]^{\frac{1}{2}};\\ &\|v\|_{H^m(E)} = [\sum_{j=0}^{m} (D^j v, D^j v)_{L^2(E)}]^{\frac{1}{2}}, \end{aligned}$$

where D^j denotes d^j/dx^j. If E = I, we write (α,β) instead of $(\alpha,\beta)_{L^2(I)}$ and $\|\alpha\|_m$ instead of $\|\alpha\|_{H^m(I)}$.

Let $U \in M_0^k(\Delta)$ be the unique solution of

(3) $$B(U,V) = (f,V), \quad V \in M_0^k(\Delta),$$

where $B: H_0^1(I) \times H_0^1(I) \to \mathbb{R}$ is defined by

(4) $$B(u,v) = (pu',v') + (qu,v); \quad u,v \in H_0^1(I).$$

We assume that B is strongly coercive, i.e. there exists a $C > 0$ such that

(5) $$B(v,v) \geq C\|v\|_1^2, \qquad v \in H_0^1(I).$$

In the sequel, we assume that C, C_1, C_2, etc., are generic positive constants not necessarily the same.

LEMMA 1. *Let* $u \in H_0^1(I) \cap H^{k+1}(I)$ *be the solution of* (1) *and let* $U \in M_0^k(\Delta)$ *be the solution of* (3). *Then the error function* $e(x) = u(x) - U(x)$ *has the bounds*

(6) $$\begin{aligned} \|e\|_\ell &\leq Ch^{k+1-\ell}\|u\|_{k+1}, & \ell &= 0,1; \\ |e(x_j)| &\leq Ch^{2k}\|u\|_{k+1}, & j &= 1,\dots,N-1; \\ \|e\|_{L^\infty(I)} &\leq Ch^{k+1}\|u\|_{k+1}. \end{aligned}$$

PROOF. See [5], [4] and [6]. □

In the next §, we prove that the local order of convergence improves slightly at specific points interior on I_j, if u satisfies stricter smoothness requirements on the interior of I_j.

2. ORDER OF CONVERGENCE AT LOBATTO POINTS

On the segment $[-1,+1]$, we define the Lobatto points $\sigma_0,\dots,\sigma_k$ by

(7) $$(1-\sigma_\ell^2) \frac{d}{d\sigma} P_k(\sigma_\ell) = 0, \qquad \ell = 0,\dots,k,$$

where $P_k(\sigma)$ is the k-th degree Legendre polynomial. Associated to this polynomial is the quadrature formula (see [1, formula 2.5.4.32])

(8)
$$\int_{-1}^{+1} f(\sigma)\,d\sigma = \sum_{\ell=0}^{k} w_\ell f(\sigma_\ell) + R_k f^{(2k)}(s \in (-1,+1));$$
$$w_\ell = \frac{2}{k(k+1)[P_k(\sigma_\ell)]^2}, \qquad \ell = 0,\dots,k.$$

From (7) and (8), we define

(9)
$$\xi_{j\ell} = x_{j-1} + \frac{h}{2}(1+\sigma_\ell); \qquad \ell = 0,\dots,k; \quad j = 1,\dots,N;$$
$$(\alpha,\beta)_j^* = \frac{h}{2}\sum_{\ell=0}^{k} w_\ell \alpha(\xi_{j\ell})\beta(\xi_{j\ell}); \qquad \alpha,\beta \in W^{2k,\infty}(I_j); \quad j = 1,\dots,N;$$
$$(\alpha,\beta)_h = \sum_{j=1}^{N} (\alpha,\beta)_j^*.$$

We return to problems (1) and (3). It is known that

(10) $$B(e,V) = 0, \qquad V \in M_0^K(\Delta).$$

For any I_j, we define

(11) $$M_0^k(I_j) = \{V \mid V \in M_0^k(\Delta),\ \mathrm{supp}(V) = I_j\}.$$

We temporarily drop the subscript j from the numbers $\xi_{\ell j}$. We define a natural basis $\{\phi_i\}_{i=1}^{k-1}$ for $M_0^k(I_j)$ by

(12) $$\phi_i(\xi_\ell) = \delta_{i\ell}, \qquad 1 \le i,\ \ell \le k-1,$$

where $\delta_{i\ell}$ is the Kronecker symbol. If we elaborate (10) for $V = \phi_i$, $i = 1,\dots,k-1$, we get

(13) $$(e,L\phi_i) = [p(x)e(x)\phi_i^1(x)]_{\xi_0}^{\xi_k}, \qquad i = 1,\dots,k-1.$$

Approximation of $(e,L\phi_i)$ by Lobatto quadrature yields

$$\sum_{\ell=1}^{k-1} w_\ell L\phi_i(\xi_\ell)e(\xi_\ell) = 2h^{-1}[p(x)e(x)\phi_i^1(x)]_{\xi_0}^{\xi_k} - w_0 e(\xi_0)L\phi_i(\xi_0) -$$

(14)
$$-w_k e(\xi_k)L\phi_i(\xi_k) + Ch^{2k}D^{2k}(eL\phi_i)(\xi \in I_j),$$

$$i = 1,\ldots,k-1.$$

This is a linear system for $e(\xi_1),\ldots,e(\xi_{k-1})$. We have to prove the nonsingularity of $(w_\ell L\phi_i(\xi_\ell))$ and to compute the order of the solution.

Take a $V \in M_0^k(I_j)$ represented by

$$V(x) = \sum_{i=1}^{k-1} v_i \phi_i(x).$$

Suppose

$$\sum_{h=1}^{k-1} w_\ell L\phi_i(\xi_\ell)v_\ell = 0, \qquad i = 1,\ldots,k-1.$$

Then

$$0 = \frac{h}{2}\sum_{\ell=1}^{k-1} w_\ell v_\ell \sum_{i=1}^{k-1} v_i L\phi_i(\xi_\ell) = \frac{h}{2}\sum_{\ell=1}^{k-1} w_\ell V(\xi_\ell)LV(\xi_\ell)$$

$$= (V,LV) + Ch^{2k+1}D^{2k}(VLV)(\xi \in I_j),$$

Hence

$$\|V\|^2_{H^1(I_j)} \le CB(V,V) \le Ch^{2k+1}\|D^{2k}(VLV)\|_{L^\infty(I_j)} \le Ch^{2k+1}\|V\|^2_{W^{k,\infty}(I_j)}$$

$$\le Ch^{2k+1}\|V\|^2_{H^k(I_j)} \le Ch^3\|V\|^2_{H^1(I_j)},$$

which is only true if $V \equiv 0$, which proves the nonsingularity of $(w_\ell L\phi_i(\xi_\ell))$, if h is small enough. Since $h^2 w_\ell L\phi_i(\xi_\ell) \sim -h^2 w_\ell p(\xi_\ell)\phi_i''(\xi_\ell) = c_{i\ell}$, where $c_{i\ell}$ is of $O(1)$, as $h \to 0$, all the entries of $(w_\ell L\phi_i(\xi_\ell)$ are of $O(h^{-2})$, hence the entries of its inverse are of $O(h^2)$.

We turn to the second part of our problem. The first three terms of the right hand side of (14) are of $O(h^{2k-2}\|u\|_{k+1})$. For the last term, we prove that

$$(16) \qquad \|D^{2k}(eL\phi_i)\|_{L^\infty(I_j)} \le C\|e\|_{W^{2k,\infty}(I_j)}\|L\phi_i\|_{W^{2k,\infty}(I_j)}.$$

From [3], it can be proved that

$$(17) \qquad \|D^{\ell}e\|_{L^\infty(I_j)} \le \begin{cases} Ch^{k+1-\ell}\|u\|_{k+1}, & \ell \le k; \\ \|D^{\ell}u\|_{L^\infty(I_j)}, & \ell > k. \end{cases}$$

Furthermore,

$$(18) \qquad \|L\phi_i\|_{W^{2k,\infty}} \le Ch^{-k},$$

hence we summarily have

$$(19) \qquad \left|\sum_{\ell=1}^{k-1} w_\ell L\phi_i(\xi_\ell)e(\xi_\ell)\right| \le Ch^k[\|u\|_{k+1}h^{k-2} + \|u\|_{W^{2k,\infty}(I_j)}],$$

$$i = 1,\ldots,k-1.$$

This was the last step in the proof of

THEOREM 1. *Let* $u \in H_0^1(I) \cap H^{k+1}(I) \cap \bigcap_{j=1}^{N} W^{2k,\infty}(I_j)$ *be the solution of* (1) *and let* $U \in M_0^k(\Delta)$ *be the solution of* (3). *Then the error function has the local error bounds.*

$$(20) \qquad |e(\xi_{j\ell})| \le Ch^{k+2}[\|u\|_{k+1}h^{k-2} + \|u\|_{W^{2k,\infty}(I_j)}],$$

$$j = 1,\ldots,N; \quad \ell = 1,\ldots,k-1. \quad \square$$

3. LOBATTO QUADRATURE

Usually, $B(,)$ and $(,)$ are to be evaluated by numerical quadrature. We will show that Lobatto quadrature leaves the order of convergence at the

Lobatto points invariant.

We define

$$(21) \qquad B_h(\alpha,\beta) = (p\alpha',\beta')_h + (q,\alpha,\beta)_h; \qquad \alpha,\beta \in \bigcap_{j=1}^{N} W^{2k,\infty}(I_j),$$

where $(,)_h$ is defined by (9).

LEMMA 2. *Let* $Y \in M_0^k(\Delta)$ *be the solution of*

$$(22) \qquad B_h(Y,V) = (f,V)_h, \qquad V \in M_0^k(\Delta)$$

and let $u \in H_0^1(I) \cap H^{k+1}(I) \cap \bigcap_{j=1}^{N} W^{2k,\infty}(I_j)$ *be the solution of* (1). *Then the error function* $\eta = u - Y$ *has the bounds*

$$|\eta(x_j)| \le Ch^{2k}\|f\|_{2k,\Delta}; \qquad j = 1,\dots,N-1,$$

if h *is small enough, with*

$$(23) \qquad \|f\|_{\ell,\Delta} = \Big[\sum_{j=1}^{N} \|f\|^2_{H^{\ell}(I_j)}\Big]^{\frac{1}{2}}.$$

PROOF. See [4]. □

We now consider $\varepsilon(x) = U(x) - Y(x)$, where U is the solution of (3). From (3) and (22), we obtain for every I_j

$$|B(\varepsilon,V)| \le |(f,V)-(f,V)_h| + |B_h(Y,V) - B(Y,V)|$$

$$\le Ch^{2k+1}\|V\|_{H^k(I_j)}\big[\|f\|_{H^{2k}(I_j)} + \|Y\|_{H^k(I_j)}\big], \qquad V \in M_0^k(I_j).$$

If we take for V any of the basis functions ϕ_i of $M_0^k(I_j)$, as defined by (12), we have

$$(25) \qquad |B(\varepsilon,\phi_i)| \le Ch^{k+1}\big[\|f\|_{H^{2k}(I_j)} + \|Y\|_{H^k(I_j)}\big], \qquad i = 1,\dots,k-1.$$

Since

$$\sum_{\ell=1}^{k-1} w_\ell \varepsilon(\xi_\ell) L\phi_i(\xi_\ell) = 2h^{-1} B(\varepsilon,\phi_i)$$

(26)
$$- w_0 \varepsilon(\xi_0) L\phi_i(\xi_0) - w_k \varepsilon(\xi_k) L\phi_i(\xi_k)$$

$$- \frac{2}{h}[p(x)\varepsilon(x)\phi_i'(x)]_{\xi_0}^{\xi_k} + Ch^{2k} D^{2k}(\varepsilon L\phi_i)(\xi \in I_j)$$

and

$$\|D^{2k}(\varepsilon L\phi_i)\|_{L^\infty(I_j)} \le C\|\varepsilon\|_{W^{k,\infty}(I_j)} \|\phi_i\|_{W^{k,\infty}(I_j)}$$

(27)
$$\le Ch^{-2k}\|\varepsilon\|_{L^\infty(I_j)} \le Ch^{-k+1}\|f\|_{2k,\Delta},$$

we have

$$\Big|\sum_{\ell=1}^{k-1} w_\ell \varepsilon(\xi_\ell) L\phi_i(\xi_\ell)\Big| \le C_1 h^k [\|f\|_{H^{2k}(I_j)} + \|Y\|_{H^k(I_j)}]$$

(28)
$$+ C_2 h^{2k-2}\|f\|_{2k,\Delta} + C_3 h^{k+1}\|f\|_{2k,\Delta}.$$

The nonsingularity of $(w_\ell L\phi_i(\xi_\ell))$ has already been proved, its inverse is of $O(h^2)$, hence we have

(29)
$$|\varepsilon(\xi_\ell)| \le C_1 h^{k+2}[\|f\|_{H^{2k}(I_j)} + \|Y\|_{H^k(I_j)}] + C_2 h^{k+3}\|f\|_{2k,\Delta}.$$

Since (see [3]).

$$\|Y\|_{H^k(I_j)} \le \|\eta\|_{H^k(I_j)} + \|u\|_{H^k(I_j)} \le Ch\|u\|_{k+1} + \|u\|_{H^k(I_j)}$$

(30)
$$\le C\|u\|_{k+1},$$

we can prove by combination of (20), (29) and (30)

THEOREM 2. *Let* $u \in H_0^1(I) \cap H^{k+1}(I) \cap \bigcap_{j=1}^{N} W^{2k,\infty}(I_j)$ *be the solution of* (1) *and let* $Y \cap M_0^k(\Delta)$ *be the solution of* (22). *Then the error function* η *has the bounds*

$$|\eta(\xi_{\ell j})| \leq C_1 h^{k+2}[\|f\|_{H^{2k}(I_j)} + \|u\|_{k+1}] + C_2 h^{k+3}\|f\|_{2k,\Delta};$$

$$j = 1,\ldots,N; \qquad \ell = 1,\ldots,k-1. \;\square$$

4. CONCLUSIONS

We have found a lighter form of superconvergence at other points than the knots. The findings of this paper stress the important part that Lobatto points play in the C^0 Galerkin method for two-point boundary problems. This is especially true for $k = 2$, since in that case the error is of $O(h^4)$ at all Lobatto points.

The results of this paper can be easily applied to the case of two-point initial boundary problems, (see [2]) and probably to other cases, as nonlinear boundary problems.

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