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P.J. VAN DER HOUWEN

MODIFIED NYSTRÖM METHODS FOR SEMI-DISCRETE HYPERBOLIC DIFFERENTIAL EQUATIONS

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Modified Nyström methods for semi-discrete hyperbolic differential equations *)

by

P.J. van der Houwen

ABSTRACT

First and second order Nyström type methods are derived for second order differential equations without first derivatives possessing the following properties: (i) the stability interval equals $4m^2$, m denoting the number of stages per integration step; (ii) the method is internally stable irrespective the value of m; (iii) the storage requirements are limited; (iv) the costs per integration step are one right hand side evaluation, one evaluation of the Jacobian matrix and m-1 matrix-vector multiplications. These four properties are of interest in the integration of the usually very large systems of ordinary differential equations resulting from the semi-discretization of partial differential equations which are of hyperbolic type and of second order in time.

KEY WORDS & PHRASES: *Numerical analysis, Nyström formulas, stability, hyperbolic initial-boundary value problems*

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*) This report is submitted for publication elsewhere.

1. INTRODUCTION

Consider the system of ordinary differential equations

$$(1.1) \qquad \frac{d^2\vec{y}}{dt^2} = \vec{f}(t,\vec{y}),$$

where the Jacobian matrix J of the right hand side function has eigenvalues which are located in a large, narrow strip along the negative axis. Such equations often arise when a second order hyperbolic differential equation is semi-discretized with respect to its space variables.

In [4] explicit Nyström-Runge-Kutta methods (or briefly Nyström methods) were analysed for the integration of (1.1). It was shown that stabilized formulas can be constructed with a real stability interval $[-\beta,0]$ where the stability boundary $\beta \cong 4m^2$, m being the number of $\vec{f}$-evaluations per integration step. Since the stability condition for the integration step τ in Nyström methods is of the form

$$(1.2) \qquad \tau \leq \sqrt{\frac{\beta}{\sigma(J)}},$$

where $\sigma(J)$ denotes the spectral radius of J, we see that the *effective* integration step, i.e. the step per $\vec{f}$-evaluation which is maximally allowed, does not increase with m. However, by realizing that in a stabilized formula almost all $\vec{f}$-evaluations are introduced for the sake of stability and hardly affect the accuracy of the formula, one may economize the formula by replacing these $\vec{f}$-evaluations by $\vec{f}^*$-evaluations which can be obtained with less computational effort. Evidently, as m is larger the gain factor is also larger. In particular, we choose for $\vec{f}^*$ the linearization of $\vec{f}$. Instead of m evaluations of $\vec{f}$ the linearized formula requires one evaluation of $\vec{f}$, one Jacobian and m-1 matrix-vector multiplications. In [4] a few experiments were reported carried out with such *modified Nyström methods*. Although the m-values used were relatively low (only four-stage formulas) the results obtained were rather encouraging.

In this paper modified methods are considered for *large* m-*values*. Because of the danger of internal instability which is frequently exhibited by Runge-Kutta type schemes with many stages [2, 5], we first derive in

Section 2 Nyström formulas which are stable in all stages of the integration step. These formulas will be called *Nyström-Chebyshev formulas* as they are based on properties of Chebyshev polynomials. In Section 3, modified formulas are considered and their order in τ with respect to the unmodified formulas is derived. In Section 4, numerical experiments are reported which show that the modified formulas produce roughly the same accuracy and are more stable than the original Nyström-Chebyshev formulas.

2. INTERNALLY STABLE NYSTRÖM FORMULAS

2.1. Preliminaries

In [4] we analyzed a class of Nyström formulas of the form

$$
\begin{aligned}
&\vec{y}_{n+1}^{(0)} = \vec{y}_n \\
(2.1)\qquad &\vec{y}_{n+1}^{(j)} = \vec{y}_n + \mu_j \tau \dot{\vec{y}}_n + \tau^2 \sum_{\ell=1}^{j-1} \lambda_{j\ell} \vec{f}(t_n + \mu_\ell \tau, \vec{y}_{n+1}^{(\ell)}), \qquad j = 1,2,\ldots,m \\
&\vec{y}_{n+1} = \vec{y}_{n+1}^{(m)}, \quad \dot{\vec{y}}_{n+1} = \dot{\vec{y}}_n + \tau \sum_{\ell=1}^{m-1} \beta_\ell \vec{f}(t_n + \mu_\ell \tau, y_{n+1}^{(\ell)}), \quad \mu_m = 1.
\end{aligned}
$$

In particular we considered the relations obtained by applying (2.1) to the model equation

$$
(2.2)\qquad \frac{d^2\vec{y}}{dt^2} = J\vec{y},
$$

that is

$$
\begin{pmatrix} \vec{y}_{n+1}^{(j)} \\ \tau \dot{\vec{y}}_{n+1}^{(j)} \end{pmatrix} = R_j(\tau^2 J) \begin{pmatrix} \vec{y}_n \\ \tau \dot{\vec{y}}_n \end{pmatrix}, \qquad j = 1,2,\ldots,m.
$$

where the matrices $R_j(\cdot)$ are defined by

$$
R_j(z) = \begin{pmatrix} A_j(z) & B_j(z) \\ 0 & 0 \end{pmatrix}, \qquad j = 1,2,\ldots,m-1;
$$

$$R_m(z) = \begin{pmatrix} A_m(z) & B_m(z) \\ z\sum_{\ell=1}^{m-1} \beta_\ell A_\ell(z) & 1+z\sum_{\ell=1}^{m-1} \beta_\ell B_\ell(z) \end{pmatrix}$$

$A_j(z)$ and $B_j(z)$ being polynomials satisfying the recurrence relations

$$A_0(z) = 1, \quad A_j(z) = 1 + \sum_{\ell=1}^{j-1} \lambda_{j\ell} z A_\ell(z)$$

$$B_0(z) = 0, \quad B_j(z) = \mu_j + \sum_{\ell=1}^{j-1} \lambda_{j\ell} z B_\ell(z) \qquad j = 1,2,\ldots,m. \tag{2.3}$$

Usually (cf. [1, 4]) the scheme (2.1) is called *stable* when the eigenvalues of $R_m(\tau^2 J)$ with $J \cong \partial\vec{f}/\partial\vec{y}$ are within the unit circle (*strong stability*) or on the unit disk *(weak stability)*. The polynomials $A_j(z)$ and $B_j(z)$ will be called *stability polynomials*.

In view of the large values of m considered in this paper, we shall now require that the eigenvalues of *all* matrices $R_j(\tau^2 J)$ are within or on the unit circle. In such cases we will call the scheme *internally stable* (note that internal stability automatically means stability in the usual sense).

Let $\alpha^{(j)}(z)$ denote the eigenvalues of $R_j(z)$, then we may define the *stability region*

$$\{z \mid |\alpha^{(j)}(z)| < 1, \; j = 1,2,\ldots,m\}. \tag{2.4}$$

In this paper we will only be concerned with the negative stability interval. The quantities $\alpha^{(j)}(\tau^2\delta)$, δ being an eigenvalue of J, will be called the *amplification factors*.

The consistency conditions for the scheme (2.1) are (see e.g. [3])

$$\begin{aligned} &\sum_{\ell=1}^{m-1} \beta_\ell = 1 \text{ for first order consistency} \\ &\frac{1}{2}\sum_{\ell=1}^{m-1} \beta_\ell = \sum_{\ell=1}^{m-1} \lambda_{m\ell} = \sum_{\ell=1}^{m-1} \beta_\ell \mu_\ell = \frac{1}{2} \text{ for second order consistency.} \end{aligned} \tag{2.5}$$

2.2. Consistency and internal stability conditions

In this section we express the consistency and internal stability conditions in terms of the stability polynomials $A_j(z)$.

THEOREM 2.1. *Let the parameters* μ_j *and* β_ℓ *in* (2.1) *be such that*

$$\mu_j = \mu, \quad j = 1,2,\ldots,m-1; \qquad \mu z \sum_{\ell=1}^{m-1} \beta_\ell A_\ell(z) = A_m(z) - 1. \tag{2.6}$$

(i) *The scheme is first order consistent if* $\mu = A_m'(0)$ *and second order consistent if* $\mu = A'(0) = \frac{1}{2}$.

(ii) *The amplification factors* $\alpha^{(j)}(z)$, $j = 1,2,\ldots,m-1$ *are given by*

$$\alpha_+^{(j)}(z) = A_j(z), \quad \alpha_-^{(j)}(z) = 0, \qquad j = 1,2,\ldots,m-1. \tag{2.7}$$

(iii) *Let* $r(z)$ *be a function with* $0 < r(z) \le 1$, *then the amplification factors* $\alpha^{(m)}(z)$ *are bounded by* $r(z)$ *if*

$$-\frac{1-\mu(1-r^2(z))}{2\mu(r(z)+1)-1} \le A_m(z) \le \frac{\mu(1+r^2(z))-1}{2\mu-1} \tag{2.8a}$$

$$\frac{1}{2} < \mu \le \frac{1}{2(1-r(z))}.$$

(iv) *For* $\mu = \frac{1}{2}$, $\alpha^{(m)}(z)$ *is bounded by* 1 *if*

$$|A_m(z)| \le 1. \tag{2.8b}$$

□

PROOF.

(i) From (2.6) it follows that $A_m'(0) = \mu \sum_{\ell=1}^{m-1} \beta_\ell$, hence substitution in (2.5) yields part (i) of the theorem.

(ii) By virtue of (2.6) the $\alpha^{(j)}(z)$ satisfy the characteristic equations

$$\alpha(\alpha - A_j(z)) = 0, \qquad j = 1,2,\ldots,m-1$$

$$\alpha^2 - S(z)\alpha + P(z) = 0, \qquad j = m \tag{2.9}$$

where

$$S(z) = A_m(z) + 1 + z \sum_{\ell=1}^{m-1} \beta_\ell B_\ell(z) = 2A_m(z),$$

$$P(z) = A_m(z) + z \sum_{\ell=1}^{m-1} \beta_\ell [A_m(z)B_\ell(z) - A_\ell(z)B_m(z)] =$$

$$= \frac{2\mu-1}{\mu} A_m(z) + \frac{1-\mu}{\mu} .$$

The amplification factors corresponding to the first m-1 stages are immediate from (2.9). For $\alpha^{(m)}(z)$ we use the Hurwitz type criterion which states that $\alpha^{(m)}(z)$ is bounded by a function r(z) if [4]

$$|S(z)| \le \frac{P(z)}{r(z)} + r(z), \quad P(z) \le r^2(z), \quad 0 < r(z) \le 1.$$

Working out these conditions for the case (2.9) we arrive at the inequalities (2.8). □

By substitution of r(z) = 1 in (2.8a) it follows from this theorem that the class of Nyström formulas satisfying (2.6) are internally stable with real stability interval [-β,0] if $A_j(z)$ and μ satisfy the inequalities

$$|A_j(z)| \le 1, \qquad j = 1,2,\ldots,m-1;$$

$$\frac{1}{1-4\mu} \le A_m(z) \le 1; \tag{2.10}$$

$$\mu \ge \frac{1}{2}$$

for $-\beta \le z \le 0$ (note that second order accuracy implies $\mu = \frac{1}{2}$ and therefore *weak stability*). The corresponding condition for the integration step τ is given by (1.2). Thus, in order to have stability for large integration steps we need polynomials $A_j(z)$ which satisfy (2.10) in rather large intervals [-β,0]. Such polynomials will be given in Section 2.4.

2.3. Derivation of Nyström formulas with prescribed stability polynomials

When the stability polynomials $A_j(z)$ are prescribed, the relations (2.3) and (2.6) completely define scheme (2.1) and therefore enable us to construct internally stable Nyström methods by solving (2.3) and (2.6) for $\lambda_{j\ell}$ and β_ℓ. In general, however, this will be rather difficult. Things simplify considerably if the polynomials $A_j(z)$ satisfy a recurrence relation.

THEOREM 2.2. *Let the polynomials* $A_j(z)$ *satisfy the recurrence relation*

$$A_1(z) = 1; \quad A_2(z) = 1 + b_1 z;$$

$$A_{j+1}(z) = (a_j + b_j z)A_j(z) + (1 - a_j)A_{j-1}(z), \qquad j = 2,3,\ldots,m \tag{2.11}$$

and let $\beta_\ell^{(j)}$ *be parameters generated by the recurrence relations*

$$\beta_\ell^{(\ell+1)} = \frac{b_\ell}{\mu}; \quad \beta_\ell^{(\ell+2)} = \frac{b_\ell a_{\ell+1}}{\mu}; \quad \beta_\ell^{(j+1)} = a_j\beta_\ell^{(j)} + (1 - a_j)\beta_\ell^{(j-1)}, \tag{2.12}$$

$$j = \ell+2,\ldots,m-1$$

where $\ell = 1,2,\ldots,m-1$. *Then the corresponding Nyström method can be written as*

$$\vec{y}_{n+1}^{(1)} = \vec{y}_n + \tau\mu\dot{\vec{y}}_n; \quad \vec{y}_{n+1}^{(2)} = \vec{y}_{n+1}^{(1)} + b_1\tau^2\vec{f}(t_n + \mu\tau, \vec{y}_{n+1}^{(1)});$$

$$\vec{y}_{n+1}^{(j+1)} = a_j\vec{y}_{n+1}^{(j)} + (1-a_j)\vec{y}_{n+1}^{(j-1)} + b_j\tau^2\vec{f}(t_n + \mu\tau, \vec{y}_{n+1}^{(j)}), \quad j = 2,3,\ldots,m-2$$

$$\vec{y}_{n+1} = a_{m-1}\vec{y}_{n+1}^{(m-1)} + (1-a_{m-1})\vec{y}_{n+1}^{(m-2)} + b_{m-1}\tau^2\vec{f}(t_n + \mu\tau, \vec{y}_{n+1}^{(m-1)}) + \tag{2.13}$$

$$+ (1-\mu)\tau\dot{\vec{y}}_n$$

$$\dot{\vec{y}}_{n+1} = \dot{\vec{y}}_n + \tau\sum_{\ell=1}^{m-1}\beta_\ell^{(m)}\vec{f}(t_n + \mu\tau, \vec{y}_{n+1}^{(\ell)})$$

and satisfies (2.6). □

PROOF. It is easily verified that (2.13) is a Nyström method belonging to the class (2.1) and that the corresponding stability polynomials $A_j(z)$ as defined by (2.3) satisfy the recurrence relation (2.11).

In order to prove (2.6) we show that

$$(2.14) \qquad A_{j+1}(z) = 1 + \mu z \sum_{\ell=1}^{j} \beta_\ell^{(j+1)} A_\ell(z), \qquad j = 0,1,\ldots,m-1.$$

For $j = 0$ and $j = 1$ this equation is satisfied. For $j > 1$ we substitute (2.14) into (2.11) to obtain

$$1 + \mu z \sum_{\ell=1}^{j} \beta_\ell^{(j+1)} A_\ell(z) = b_j z A_j(z) + a_j [1 + \mu z \sum_{\ell=1}^{j-1} \beta_\ell^{(j)} A_\ell(z)] +$$
$$+ (1-a_j)[1 + \mu z \sum_{\ell=1}^{j-2} \beta_\ell^{(j-1)} A_\ell(z)]$$

By virtue of the recurrence relations (2.12) for the $\beta_\ell^{(j)}$ this equation is easily seen to be satisfied. □

2.4. Nyström-Chebyshev methods

In this section we consider Nyström methods satisfying (2.6) and which are generated by the polynomials

$$(2.15) \qquad A_j(z) = \frac{T_{j-1}(w_0 + \frac{w_0+1}{\beta} z)}{T_{j-1}(w_0)}, \qquad w_0 \geq 1, \; j = 1,2,\ldots,m,$$

where w_0 and are parameters to be determined and T_j denotes the first kind Chebyshev polynomial of degree j. These methods will be called *Nyström-Chebyshev methods*. The polynomials (2.15) are chosen firstly because of their property to lead to optimal real stability intervals (cf. [4]) and secondly, because they satisfy a three-terms recurrence relation as required by Theorem 2.2. From the recurrence relation for $T_j(x)$ it easily follows that (2.15) satisfies (2.11) with

$$(2.15') \qquad a_j = 2w_0 \frac{T_{j-1}(w_0)}{T_j(w_0)}, \quad b_1 = \frac{w_0+1}{\beta w_0}, \quad b_j = 2\frac{w_0+1}{\beta}\frac{T_{j-1}(w_0)}{T_j(w_0)},$$
$$j = 2,3,\ldots,m-1.$$

In order to have at least first order accuracy we require that (cf. Theorem 2.1(i))

$$A_m'(0) = \frac{w_0+1}{\beta}\,\frac{T_{m-1}'(w_0)}{T_{m-1}(w_0)} = \mu.$$

Using the identity

$$T_{m-1}'(w_0) = (m-1)\sqrt{\frac{T_{m-1}^2(w_0)-1}{w_0^2-1}}$$

we find for β (it is convenient to leave μ free)

$$\beta = \frac{m-1}{\mu}\sqrt{\frac{w_0+1}{w_0-1}}\sqrt{1-\frac{1}{T_{m-1}^2(w_0)}} = \frac{m-1}{\mu}\sqrt{\frac{w_0+1}{w_0-1}}\tanh[(m-1)\ln(w_0+\sqrt{w_0^2-1})]$$

$$\cong \frac{2}{\mu}(m-1)^2\left[1-\frac{4(m-1)^2-1}{6}(w_0-1)\right] \qquad \text{as } w_0 \to 1.$$

We recall that we obtain second order accuracy (but also weak stability) for $\mu = \frac{1}{2}$.

Before stating the stability theorem for Nyström-Chebyshev methods we introduce the following abbreviations:

$$\tilde{w}_0 = \cosh\left[\frac{\ell n(T+\sqrt{T^2-1})}{m-1}\right] \qquad \theta = \frac{w_0-1}{w_0+1}\beta,$$

$$T = \frac{2\tilde{\mu}-1}{\tilde{\mu}(1+r^2)-1}, \qquad \tilde{\mu} = \begin{cases} \dfrac{1}{2(1-r)} & \text{for } r \le 2\sqrt{3}-3 \\[2ex] \dfrac{r+3+\sqrt{(r+1)^2-r^3}}{2(r^3+r+2)} & \text{for } r \ge 2\sqrt{3}-3 \end{cases} . \tag{2.17}$$

THEOREM 2.3. *Let* $w_0 \ge \tilde{w}_0$, r *be a constant* $\in [\sqrt{2}-1, 1)$ *and*

$$\mu = \frac{T_{m-1}(w_0)-1}{(1+r^2)T_{m-1}(w_0)-2}. \tag{2.18}$$

Then the amplification factors $\alpha^{(j)}(z)$ *satisfy the inequalities*

$$|\alpha^{(j)}(z)| \le T_{j-1}^{-1}(w_0) \qquad \text{for } -\beta \le z \le -\theta,\ j = 1,2,\ldots,m-1$$

$$|\alpha^{(m)}(z)| \le r \qquad \text{for } -\beta \le z \le -\theta, \tag{2.19}$$

$$|\alpha^{(j)}(z)| \le 1 \qquad \text{for } -\theta \le z \le 0,\ j = 1,2,\ldots,m. \qquad \square$$

PROOF. Since $T_j(w_0 + (w_0 + 1)z/\beta)$ is bounded by 1 for $-\beta \le z \le -\theta$ the first inquality of (2.19) follows immediately from (2.15) and Theorem 2.1.

In order to prove the second inequality of (2.19) we first determine the set of (μ,w_0) points where $|\alpha^{(m)}(z)| \le r$ for $-\beta \le z \le -\theta$. It is convenient to express this region in the variables μ and $T_{m-1}(w_0)$. From Theorem 2.1 we find that in the interval $-\beta \le z \le -\theta$, where $A_m(z)$ is bounded by $T_{m-1}^{-1}(w_0)$, the amplification factor $\alpha^{(m)}(z)$ is bounded by r if (see figure 2.1)

$$\mu \le \frac{1}{2(1-r)}\ , \quad T_{m-1}^{-1}(w_0) \le \frac{\mu(1+r^2)-1}{2\mu-1}\ , \quad T_{m-1}^{-1}(w_0) \le \frac{1-\mu(1-r^2)}{2\mu(1+r)-1}\ . \tag{2.20}$$

Evidently, r should be greater than or equal to $\sqrt{2} - 1$ in order to have a nonempty region where $|\alpha^{(m)}(z)| \le r$. It is also clear from figure 2.1 and (2.16) that β is maximal in the points $(\mu,T_{m-1}^{-1}(w_0))$ on the line PQ. This leads to the expression (2.18) for μ with $T_{m-1}(w_0) \ge T_{m-1}(\tilde{w}_0)$ or equivalently $w_0 \ge \tilde{w}_0$, where $\tilde{w}_0$ is defined by

$$T_{m-1}(\tilde{w}_0) = \frac{2\tilde{\mu}-1}{\tilde{\mu}(1+r^2)-1}\ ,$$

$\tilde{\mu}$ being the μ-coordinate of the point Q. A straightforward calculation yields for w_0 the expression given in (2.17).

The third inequality in (2.19) is proved by verifying condition (2.10) for the polynomials (2.15). For $j = 1,2,\ldots,m-1$ this condition is satisfied, for $j = m$ we have to show that $(1-4\mu)^{-1} \le -T_m^{-1}(w_0)$ when μ is given by (2.18). A straightforward calculation reveals this to be true. $\square$

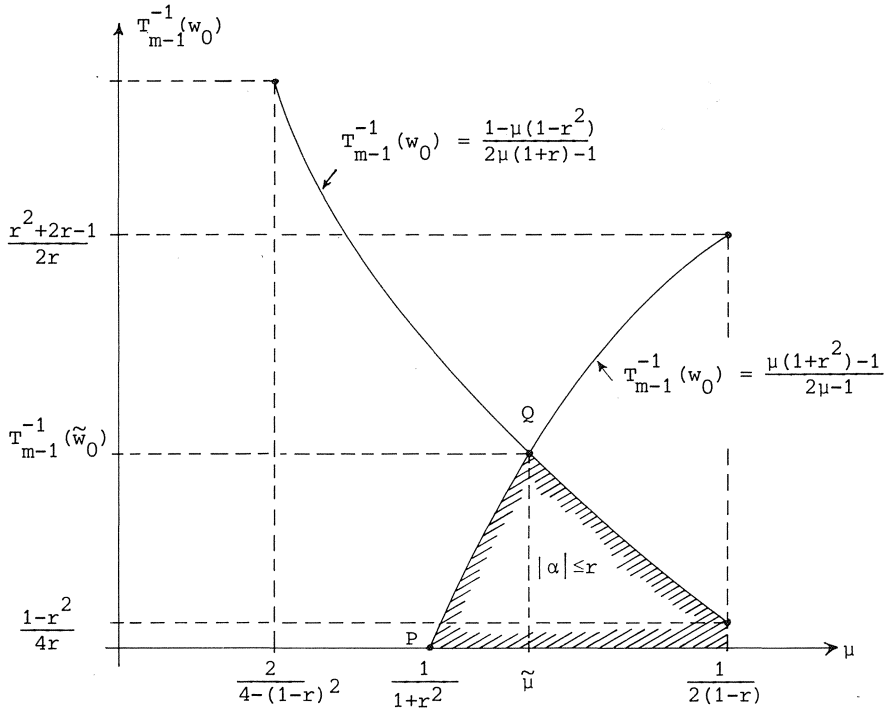

Fig. 2.1a The region (2.20) for
$r \ge 2\sqrt{3} - 3 = .46...$

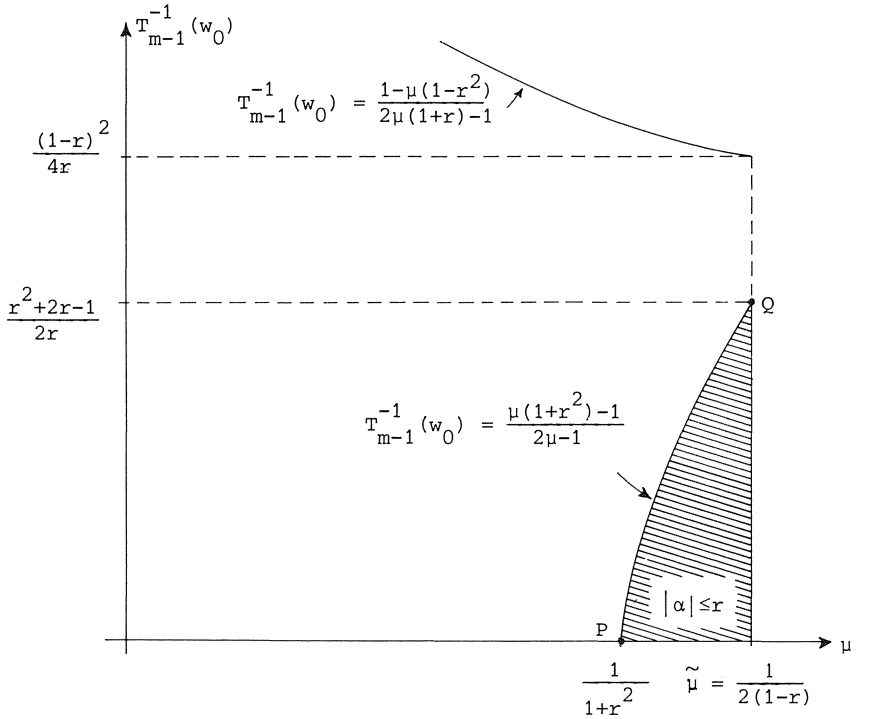

Fig. 2.1b The region (2.20) for
$.41... = \sqrt{2} - 1 \le r \le 2\sqrt{3} - 3$

From the proof of this theorem it may be concluded that the stability boundary β defined by (2.16) and (2.18), i.e.

$$\beta = \beta(r,m,w_0) = (m-1)\sqrt{\frac{w_0+1}{w_0-1}}\ \frac{(1+r^2)T_{m-1}(w_0)-2}{T_{m-1}(w_0)(T_{m-1}(w_0)-1)}\sqrt{T_m^2(w_0)-1}, \tag{2.21}$$

is optimal for a priori given values of $r \in [\sqrt{2}-1,1)$, m and $w_0 \geq \tilde{w}_0$. For fixed values of m and r one may try to find the value of $w_0 \geq \tilde{w}_0$ which maximizes β. A few numerical calculations showed that the optimal value of w_0 is very close to $\tilde{w}_0$, so that in our numerical experiments we took $w_0 = \tilde{w}_0$. For m we chose the smallest integer which satisfies the stability condition (1.2), i.e.

$$\tau \leq \sqrt{\frac{\beta(r,m,\tilde{w}_0)}{\sigma(J)}}$$

For $r \cong 1$ it can be derived that m is approximately given by

$$m \cong 1 + \tau\sqrt{\frac{\tilde{\mu}\sigma(J)T\ln(T+\sqrt{T^2-1})}{2\sqrt{T^2-1}}}, \tag{2.22}$$

where T and $\tilde{\mu}$ are given in (2.17). (Note that $w_0 = \tilde{w}_0$ implies $\mu = \tilde{\mu}$.)

Finally, the damping parameter r was chosen such that the damping in a unit interval equals some prescribed value η, thus $r = \eta^{\tau}$.

Summarizing, the Nyström-Chebyshev method defined by (2.12), (2.13), (2.15'), (2.21), (2.22) with $w_0 = \tilde{w}_0$ and $\mu = \tilde{\mu}$ as given in (2.17) presents an internally stable method which is "almost" of second order as $r \cong 1$. The limited storage requirements and the relatively large stability interval make this method a potential candidate for the time-integration of semi-discrete second order hyperbolic equations when implicit time-integrators offer difficulties. However, to exploit the property of the Nyström-Chebyshev method that m can be chosen arbitrarily large, we will consider modified methods in the next section.

3. MODIFIED NYSTRÖM-CHEBYSHEV METHODS

According (2.22) the number of stages needed in a Nyström-Chebyshev method to remain stable for a given integration step τ, may be very large if the spectral radius $\sigma(J_n)$ is large. Since a semi-discrete hyperbolic equation usually does have a large $\sigma(J_n)$-value, the Nyström-Chebyshev method is only an efficient method if the computational work involved to evaluate the m-1 functions $\vec{f}(t_n + \mu\tau, \vec{y}_{n+1}^{(j)})$, $j = 1,2,\ldots,m-1$, slowly increases with m. For instance, if the time-dependent part in the function $\vec{f}(t,\vec{y})$ forms the major part of the computational effort to calculate $\vec{f}(t,\vec{y})$, then (because of the fixed t-argument $t_n + \mu\tau$) the Nyström-Chebyshev method derived in the preceding section, will require considerably less effort per integration step than m-1 arbitrary $\vec{f}$-evaluations would require. In other cases, one may economize the method by replacing $\vec{f}(t,\vec{y})$ by a local linearization

$$
\begin{aligned}
&\vec{f}^{*}(t,\vec{y}) = \vec{f}(t_n + \mu\tau, \vec{y}_n + \mu\tau\dot{\vec{y}}_n) + J_n^{*}(\vec{y} - \vec{y}_n - \mu\tau\dot{\vec{y}}_n) \\
&J_n^{*} = \frac{\partial \vec{f}}{\partial \vec{y}}(t_n + \mu\tau, \vec{y}_n + \mu\tau\dot{\vec{y}}_n), \qquad t_n \le t \le t_{n+1}
\end{aligned}
\tag{3.1}
$$

One integration step of the Nyström-Chebyshev method modified in this way now requires one $\vec{f}$-evaluation, one Jacobian evaluation and (m-2) matrix-vector multiplications. (The modification (3.1) is more accurate than the one proposed in [4] without increase of the computational effort.)

Evidently, the stability region of the modified Nyström-Chebyshev method is identical to that of the original method. Hence, we may expect the same stability behaviour or even better because the modified method is linear in all stages of an integration step so that the (linear) stability theory can be rigorously applied. In order to see the effect of the linearization (3.1) on the accuracy we consider the scheme for the quantities $\vec{y}_{n+1}^{(j)} - \vec{y}_{n+1}^{*(j)}$ where $\vec{y}_{n+1}^{*(j)}$ denotes the solution of the modified scheme and where we assume that $\vec{y}_n^{*} = \vec{y}_n$ and $\dot{\vec{y}}_n^{*} = \dot{\vec{y}}_n$. A straightforward expansion reveals that

$$
\begin{aligned}
&\vec{y}^{(j)}_{n+1} - \vec{y}^{*(j)}_{n+1} = \begin{cases} 0 & \text{for } j = 0,1,2 \\ O(\tau^6) & \text{for } j = 3,4,\ldots,m \end{cases}, \\
&\dot{\vec{y}}_{n+1} - \dot{\vec{y}}^{*}_{n+1} = O(\tau^5).
\end{aligned} \tag{3.2}
$$

Thus, we may hope that the modification (3.1) will hardly decrease the accuracy of the Nyström - Chebyshev method (cf. Tables 4.1 and 4.2).

4. NUMERICAL EXPERIMENTS

In our numerical experiments we wanted to test two aspects of the Nyström-Chebyshev methods and the modification according to (3.1):

(i) The effect of the linearization on the accuracy;

(ii) The stability for large values of m

The initial-boundary value problems tested are given by

$$
\begin{cases}
u_{tt} = 100\cos^2[(x_1+x_2)u]\Delta u + e^{-t}\{x_1^2+x_2^2 \\
\qquad -400\cos^2[(x_1+x_2)(1+e^{-t}(x_1^2+x_2^2))]\}, \\
\qquad\qquad \vec{x} = (x_1,x_2) \in \Omega,\ t \in [0,1] \\
u = 1 + e^{-t}(x_1^2+x_2^2) \qquad \text{for } (t,\vec{x}) \in (0,\Omega) \text{ and } (t,\vec{x}) \in (t,\partial\Omega)
\end{cases} \tag{4.1}
$$

and

$$
\begin{cases}
u_{tt} = 100\,\Delta u \qquad \text{for } \vec{x} \in \Omega,\ t \in [0,1] \\
u = 1 \qquad \text{for } (t,\vec{x}) \in (t,\partial\Omega), \quad u = 1 + e(\vec{x})_{10}^{-8} \qquad \text{for } (t,\vec{x}) \in (0,\Omega)
\end{cases} \tag{4.2}
$$

where $e(\vec{x})$ assumes randomly values $\in [-1,+1]$ and where Ω represents the unit square $0 < x_1 < 1$, $0 < x_2 < 1$ with boundary $\partial\Omega$. It is easily verified that the function u used to define the initial and boundary conditions in (4.1) is also the exact solution of (4.1).

By discretizing the unit square with meshes of width h = 1/5 and h = 1/20, and by replacing differentiations with the usual standard difference

approximations we obtain initial value problems of the type (1.1) for a system of 16 and 361 ordinary differential equations, respectively. The solution of problem (4.1) is determined by the function u also given in (4.1) and is obtained by restricting u to the grid points in Ω. The eigenvalues of the Jacobian matrices are negative and bounded by the spectral radius

$$\sigma(J) \cong \frac{800}{h^2} . \tag{4.3}$$

In the tables of results below the accuracies A, defined by

$$A = -{}^{10}\log \, |\text{maximum absolute error at } t = 1|, \tag{4.4}$$

are listed produced by the Nyström-Chebyshev method (NC method) and its modified form (MNC method). We recall that the NC method is defined by (2.12), (2.13), (2.15'), (2.21), (2.22) and the MNC method is obtained

Table 4.1. Results obtained for problem (4.1) by the NC and MNC methods for h = 1/5 and various values of τ and η

method	η	$\tau = 1/8$ m	$\tau = 1/8$ A	$\tau = 1/16$ m	$\tau = 1/16$ A	$\tau = 1/32$ m	$\tau = 1/32$ A	$\tau = 1/64$ m	$\tau = 1/64$ A
NC	.99	11	2.07	6	2.75	4	3.75	3	4.23
MNC	.99	11	1.69	6	3.17	4	3.75	3	4.23
NC	.90	11	2.24	6	2.52	4	3.61	3	4.06
MNC	.90	11	2.23	6	2.47	4	3.61	3	4.06
NC	.80	12	2.36	6	3.13	4	3.44	3	3.90
MNC	.80	12	2.04	6	2.82	4	3.44	3	3.90

from the NC method by replacing $\vec{f}$ by $\vec{f}^*$ as defined in (3.1).

The results in Table 4.1 show that in the low accuracy range the MNC

method is only slightly less accurate than the NC method and delivers the same accuracies for small τ. It also shows that an increase of the damping improves the accuracy for larger value of τ but decreases the accuracy for small values of τ. This may be explained by the fact that μ differs more from 1/2 as η becomes larger, so that the scheme becomes "more" first order than second order.

Table 4.2. Results obtained for problem (4.1) by the NC and MNC methods for h = 1/20 and various values of τ and η

method	η	τ = 1/8 m	τ = 1/8 A	τ = 1/16 m	τ = 1/16 A	τ = 1/32 m	τ = 1/32 A	τ = 1/64 m	τ = 1/64 A
NC	.99	38	-	20	-	10	2.23	6	3.23
MNC	.99	38	.02	20	.42	10	2.17	6	3.23
NC	.90	41	-	20	-	11	2.62	6	3.48
MNC	.90	41	1.10	20	1.55	11	2.67	6	3.48
NC	.80	42	.58	21	-	11	3.00	6	3.69
MNC	.80	42	.96	21	1.62	11	3.13	6	3.69
NC	.70	44	1.30	22	2.05	11	3.17	6	3.73
MNC	.70	44	.69	22	1.90	11	3.13	6	3.73

In Table 4.2 the results are listed for the highly stiff case which arises when we put h = 1/20 (the spectral radius may become as large as 320000). For τ = 1/8 and τ = 1/16 the NC method behaves unstable unless the damping is sufficiently high ($\eta \leq .70$). The MNC method, however, remains stable (although inaccurate) for rather low damping (η = .99 means "almost" weak stability). As already observed in Section 3 this can be explained by the fact that the linear stability theory does not rigorously apply to the NC method whereas the MNC method satisfies the condition that the Jacobian matrix is constant in all stages. Furthermore, as in the experiment with

$h = 1/5$, we see that the NC and MNC method produce roughly the same accuracies.

Our last experiment demonstrates the highly stable behaviour of the (M)NC method for linear problems when m is large. Problem (4.2) would have both its exact solution and its numerical solution identical to 1 if $e(\vec{x}) \equiv 0$, but by perturbing the initial condition $u(0,\vec{x}) = 1$ randomly by a term of magnitude 10^{-8} (which is large compared with the accuracy of 14 digits of the CDC Cyber 73-28 on which the calculations are performed), the numerical solution will also be perturbed and one may ask how large this perturbation after one step becomes for large values of m. Denoting the perturbation at $t = \tau = 1$ by $\Delta\vec{y}_1$ we have listed in Table 4.3 the amplification factor $\alpha = \|\Delta\vec{y}\|/10^{-8}$, where $\|\cdot\|$ is the maximum norm. These results show that the stability theory is in complete agreement with the actual performances of

Table 4.3. Results obtained for problem (4.2) by the NC method for $h = 1/20$ and various values of η

η	.99	.90	.80	.70
m	310	381	439	494
α	1.12	.86	.65	.39

the NC algorithm on a computer.

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