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# STABILITY ANALYSIS OF REDUCIBLE QUADRATURE METHODS FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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Stability analysis of reducible quadrature methods for Volterra integral equations of the second kind*

by

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ABSTRACT

Direct quadrature methods, reducible to linear multistep methods for solving ODEs, are applied to a test equation of the convolution type. The difference equation for the numerical solution and the associated stability polynomial are derived. A definition of $A_0$-stability is given and it is shown that the direct quadrature methods we consider cannot be $A_0$-stable. The boundary locus method is used to determine the regions of absolute stability. For the backward differentiation methods diagrams of such regions are presented.

KEY WORDS & PHRASES: *Numerical analysis, Volterra integral equations of the second kind, stability*

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* This report will be submitted for publication elsewhere

# 1. INTRODUCTION

Consider the Volterra integral equation of the second kind

$$(1.1) \qquad f(x) = g(x) + \int_0^x K(x,y,f(y))dy, \quad 0 \le x \le X,$$

where $f(x)$ is the unknown function, and where $g(x)$ and the kernel $K(x,y,f)$ are given functions. We assume that the conditions for the existence of a unique continuous solution are satisfied (see e.g. [1, p.80]).

Direct quadrature methods for the numerical solution of (1.1) are obtained by applying quadrature rules of the form

$$\int_0^{x_n} \phi(y)dy \simeq h \sum_{j=0}^{n} w_{nj}\phi(x_j), \qquad x_j = jh,$$

to discretize (1.1), and yield equations of the form

$$(1.2) \qquad f_n = g(x_n) + h \sum_{j=0}^{n} w_{nj}K(x_n,x_j,f_j), \qquad n \ge k \ge 1,$$

for values $f_n$ approximating $f(x_n)$. Here, the value of $k$ depends on the desired accuracy. If the required starting values $f_0$ $(= g(0))$, $f_1,\dots,f_{k-1}$ are known, the values $f_k, f_{k+1},\dots$ can be computed in a step-by-step fashion. BAKER [1, ch.6] discusses a wide variety of numerical methods for (1.1) including the methods (1.2) for various choices of the weights $w_{nj}$.

In this paper we consider the class of quadrature methods which are reducible to linear multistep methods for solving ODEs. The construction and analysis of such quadrature methods is treated in [10]. We shall use subsequently an important property of the weights $w_{nj}$: it holds, for $n \ge 2k$, that

$$(1.3) \qquad \sum_{i=0}^{k} a_i w_{n-i,j} = \begin{cases} 0 & \text{if } 0 \le j \le n-k-1, \\ b_{n-j} & \text{if } n-k \le j \le n, \end{cases}$$

where we have defined $w_{nj} = 0$ for $j > n$. The real constants $a_i$ and $b_i$ $(i = 0(1)k)$ are the coefficients of a convergent linear multistep method. We assume that $a_0 \neq 0$. From the theory of linear multistep methods for ODEs (see e.g. [8, p.30]) we recall the characteristic polynomials $\rho$ and $\sigma$, defined by

$$(1.4) \qquad \rho(\zeta) := \sum_{i=0}^{k} a_i \zeta^{k-i}, \quad \sigma(\zeta) := \sum_{i=0}^{k} b_i \zeta^{k-i},$$

the consistency conditions $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$, and the fact that $\rho$ and $\sigma$ have no common factors. Furthermore, the polynomial $\rho$ is assumed to satisfy the root condition, that is $\rho(\zeta) = 0$ implies $|\zeta| \leq 1$ and $\rho(\zeta) = \rho'(\zeta) = 0$ implies $|\zeta| < 1$. The quadrature method (1.2) associated with the linear multistep method $(\rho,\sigma)$ through (1.3) is said to be $(\rho,\sigma)$-*reducible*.

The stability behaviour (for fixed $h \neq 0$ and $n \to \infty$) of various methods of the form (1.2) has been analyzed by BAKER & KEECH [2] with respect to the test equation

$$(1.5) \qquad f(x) = 1 + \lambda \int_0^x f(y)dy.$$

It can be derived (see [10]) that the stability behaviour with respect to (1.5) of $(\rho,\sigma)$-reducible quadrature methods is determined by the roots of the characteristic equation

$$(1.6) \qquad \rho(\zeta) - h\lambda\, \sigma(\zeta) = 0.$$

The stability behaviour is well-known from the ODE-theory, since (1.6) is identical to the characteristic equation of the linear multistep method $(\rho,\sigma)$ applied to the ODE test equation $f' = \lambda f$ (to which (1.5) is equivalent). Thus the stability analysis based upon (1.5) is straightforward, which is a consequence of the fact that the kernel in (1.5) is independent of $x$.

The main purpose of this paper is to analyze the stability behaviour of $(\rho,\sigma)$-reducible quadrature methods with respect to the convolution test equation

$$(1.7)\qquad f(x) = 1 + \int_0^x \{\lambda+\mu(x-y)\}f(y)dy, \qquad \lambda,\mu \in \mathbb{R},$$

which is an extension of (1.5). The choice of this test equation was motivated by the requirement that the kernel in (1.7) depends upon x on the one hand. On the other hand we chose the convolution, because Volterra integral equations of the convolution type occur frequently in applications such as demography [7] and renewal theory [5]. The stability analysis based on (1.7) yields more information than the analysis based on (1.5), and an important result we obtain is that $(\rho,\sigma)$-reducible quadrature methods which are A-stable with respect to (1.5) are not even $A_0$-stable with respect to (1.7).

In section 2 we discuss the test equation (1.7) and derive the recurrence relation for the numerical solution $f_n$ and its associated stability polynomial. In section 3 we adapt the boundary locus method (see [8]) for the determination of stability regions. In section 4 we prove that $(\rho,\sigma)$-reducible quadrature methods cannot be $A_0$-stable, and indicate a constrast to corresponding methods for Volterra integro-differential equations. In section 5 we actually present plots of the stability regions of the quadrature methods which are reducible to the backward differentiation methods. We conclude in section 6, with some additional remarks.

## 2. STABILITY ANALYSIS WITH RESPECT TO THE CONVOLUTION TEST EQUATION

By differentiating (1.7) twice, it is readily seen that the solution of the convolution test equation is identical to the solution of the second order differential equation

$$f'' = \lambda f' + \mu f, \quad f(0) = 1,\ f'(0) = 0.$$

As a consequence, the solution f(x) of (1.7) tends to zero as $x \to \infty$ if and only if both $\lambda$ and $\mu$ are negative. In order to have a stable method, the same asymptotic property is now required for the numerical solution. That is, if $f_n$ is obtained with the method (1.2) applied, with a fixed positive h, to the equation (1.7), then $f_n$ must tend to zero as $n \to \infty$.

Application of the direct quadrature method (1.2) to the test equation (1.7) yields the equations

$$(2.1) \qquad f_n = 1 + h\lambda \sum_{j=0}^{n} w_{nj} f_j + h^2\mu \sum_{j=0}^{n} w_{nj}(n-j) f_j .$$

Instead of differentiating twice, as we did in the continuous case, we now apply twice a differencing technique to the equations (2.1). We take a weighted sum of successive equations (2.1) to obtain

$$(2.2) \qquad \begin{aligned} \sum_{i=0}^{k} a_i f_{n-i} &= h\lambda \sum_{i=0}^{k} b_i f_{n-i} + h^2\mu \sum_{j=0}^{n} \sum_{i=0}^{k} a_i w_{n-i,j}(n-i-j) f_j \\ &= h\lambda \sum_{i=0}^{k} b_i f_{n-i} + h^2\mu \sum_{i=0}^{k} i b_i f_{n-i} \\ &\quad - h^2\mu \sum_{j=0}^{n} \sum_{i=0}^{k} i a_i w_{n-i,j} f_j , \end{aligned}$$

where we have used (1.3) and the equality $\rho(1) = \Sigma a_i = 0$ (recall that, by definition, $w_{nj} = 0$ for $j > n$). Notice that for $\mu = 0$, we obtain the difference equation associated with the test equation (1.5).

Applying the same differencing technique again to successive equations (2.2) we find

$$(2.3) \qquad \begin{aligned} \sum_{\ell=0}^{k} a_\ell \sum_{i=0}^{k} a_i f_{n-i-\ell} = h\lambda \sum_{\ell=0}^{k} a_\ell \sum_{i=0}^{k} b_i f_{n-i-\ell} &+ \\ h^2\mu \sum_{\ell=0}^{k} a_\ell \sum_{i=0}^{k} i b_i f_{n-i-\ell} &- \\ h^2\mu \sum_{j=0}^{n} \sum_{i=0}^{k} \sum_{\ell=0}^{k} i a_i a_\ell w_{n-i-\ell,j} f_j &. \end{aligned}$$

In order to rewrite the last term of the right-hand side of (2.3) we define $b_j^* = b_j$ for $j = 0(1)k$ and $b_j^* = 0$ for $j < 0$ or $j > k$. Thus, using (1.3), we obtain:

$$\sum_{j=0}^{n} \sum_{i=0}^{k} i a_i \sum_{\ell=0}^{k} a_\ell w_{n-i-\ell,j} f_j = \sum_{j=0}^{n} \sum_{i=0}^{k} i a_i b^*_{n-i-j} f_j =$$

$$= \sum_{i=0}^{k} ia_i \sum_{j=0}^{n} b^*_{n-i-j} f_j = \sum_{i=0}^{k} ia_i \sum_{\ell=-i}^{n-i} b^*_\ell f_{n-i-\ell} =$$

$$= \sum_{i=0}^{k} ia_i \sum_{\ell=0}^{k} b_\ell f_{n-i-\ell} = \sum_{\ell=0}^{k} \ell a_\ell \sum_{i=0}^{k} b_i f_{n-i-\ell}.$$

Substitution of this expression in (2.3) yields the following recurrence relation for the numerical solution $f_n$

(2.4) $$\sum_{\ell=0}^{k} a_\ell \{ \sum_{i=0}^{k} a_i - b_i [h\lambda + h^2\mu(i-\ell)]\} f_{n-i-\ell} = 0.$$

The solution $f_n$ of this difference equation tends to zero as $n \to \infty$ if and only if the roots of the characteristic equation

(2.5) $$\rho(\zeta)[\rho(\zeta) - h\lambda\sigma(\zeta)] - h^2\mu \sum_{\ell=0}^{k} a_\ell \zeta^{k-\ell} \sum_{i=0}^{k} b_i (i-\ell)\zeta^{k-i} = 0$$

lie inside the unit circle. We give the following definitions.

DEFINITION 1. The quadrature method (1.2) applied to (1.7) is said to be *absolutely stable* for a given $h\lambda$ and $h^2\mu$ if, for these values of $h\lambda$ and $h^2\mu$, the roots $\zeta_i$ of (2.5) satisfy $|\zeta_i| < 1$, $i = 1(1)k$.

DEFINITION 2. A region $\mathcal{R}$ in the $(h\lambda, h^2\mu)$-plane is said to be the *region of absolute stability* of the method (1.2) if (1.2) is absolutely stable for all $(h\lambda, h^2\mu) \in \mathcal{R}$.

DEFINITION 3. The quadrature method (1.2) applied to (1.7) is said to be $A_0$*-stable* if its region of absolute stability includes the third quadrant, that is, if $\{(h\lambda, h^2\mu) \mid h\lambda < 0,\ h^2\mu < 0\} \subset \mathcal{R}$.

Since $\lambda$ and $\mu$ assume only real values, Definition 3 is readily seen to be an adaptation of the $A_0$-stability concept in ODE theory (cf. [4]).

We shall refer to the left-hand side of (2.5) as the *stability polynomial* of the method, and denote it by $S(\zeta; h\lambda, h^2\mu)$. This polynomial can be expressed completely in terms of the polynomials $\rho$ and $\sigma$ and their first derivatives; to be specific

(2.6) $$S(\zeta; h\lambda, h^2\mu) := \rho^2(\zeta) - h\lambda\rho(\zeta)\sigma(\zeta) - h^2\mu\zeta[\sigma(\zeta)\rho'(\zeta) - \rho(\zeta)\sigma'(\zeta)].$$

From (2.6) explicit conditions for absolute stability can be derived, for example by means of the Routh-Hurwitz or the Schur criterion [8, §3.7]. Since the degree of S is 2k in general, such a procedure becomes increasingl complicated, and, therefore, it is more convenient to determine the stabilit region by other means.

## 3. THE BOUNDARY LOCUS METHOD

In this section we shall determine the region $\mathcal{R}$ in the $(h\lambda, h^2\mu)$-plane where the roots $\zeta_i$ of the stability polynomial (2.6) are in modulus less than unity. To this end we define the set $\Gamma$:

$$\Gamma := \{(h\lambda, h^2\mu) \mid \exists \zeta \text{ with } |\zeta| = 1 \text{ and } S(\zeta; h\lambda, h^2\mu) = 0\}.$$

In view of this definition, $(h\lambda, h^2\mu) \in \Gamma$ when at least one of the zeros of (2.6) lies on the boundary of the unit disk or, equivalently, when

$$S(e^{i\phi}; h\lambda, h^2\mu) = 0, \tag{3.1}$$

with $\phi$ running through the interval $[-\pi, \pi]$. Therefore, the set $\Gamma$ (which is a curve or a set of curves) is determined by finding $h\lambda$ and $h^2\mu$ from (3.1) when $\phi \in [-\pi, \pi]$. Since the zeros of (2.6) are continuous functions of $h\lambda$ and $h^2\mu$, the boundary $\partial\mathcal{R}$ of the stability region $\mathcal{R}$ is a subset of $\Gamma$.

The $(h\lambda, h^2\mu)$-plane is divided by $\Gamma$ into subregions, and in order to determine which of the subregions are regions of absolute stability, it is necessary to compute the roots of $S(\zeta; h\lambda, h^2\mu) = 0$ at a number of appropriate values of $h\lambda$ and $h^2\mu$.

This technique (also used for ODE methods, see [8, §3.7]) is called the *boundary locus method* for finding the region of absolute stability. In contrast to the ODE case, however, the stability polynomial (2.6), associated with quadrature methods for solving integral equations, comprises two real parameters $h\lambda$ and $h^2\mu$, and, as can be seen below, degenerate solutions to (3.1) may arise. Therefore we will consider (3.1) (for a fixed value of $\phi$) in more detail and write it, according to (2.6), in the more transparent form

(3.2) $\rho^2(e^{i\phi}) - h\lambda\rho(e^{i\phi})\sigma(e^{i\phi}) - h^2\mu[\sigma(e^{i\phi})\rho^*(e^{i\phi}) - \rho(e^{i\phi})\sigma^*(e^{i\phi})] = 0$

where $\rho^*(\zeta) = \zeta\rho'(\zeta)$ and $\sigma^*(\zeta) = \zeta\sigma'(\zeta)$. For subsequent use, we write (3.2) also in the form

(3.3) $(A+iB) - h\lambda(C+iD) - h^2\mu(E+iF) = 0,$

with obvious definitions for A,B,...,F. We distinguish between the following cases.

<u>Case 1</u>: $\rho(e^{i\phi}) = 0$.

Since $\rho$ and $\sigma$ are assumed to have no common factor, $\sigma(e^{i\phi}) \neq 0$ for this value of $\phi$. Furthermore $\rho'(e^{i\phi}) \neq 0$, since the zeros of $\rho$ on the unit circle are simple. From (3.2) we then derive that the line $h^2\mu = 0$ is part of $\Gamma$. Note that this case always occurs for $\phi = 0$, since consistency of the linear multistep method requires that $\rho(1) = 0$. ∎

In the following cases we assume that $\rho(e^{i\phi}) \neq 0$. Equation (3.3) for finding the real values $h\lambda$ and $h^2\mu$ is equivalent to the linear system

(3.4) $\begin{bmatrix} C & E \\ D & F \end{bmatrix} \begin{bmatrix} h\lambda \\ h^2\mu \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix},$

where $(A,B) \neq (0,0)$, since $\rho(e^{i\phi}) \neq 0$. At this point, one can readily see that it is sufficient to take $\phi$ only in the interval $[0,\pi]$, since for $\psi = -\phi$, $\phi\in[0,\pi]$, the values of B,D and F change sign, whereas the values of A,C and E remain unchanged.

<u>Case 2</u>: $CF - DE \neq 0$, $\rho(e^{i\phi}) \neq 0$.

In this case the system (3.4) has a *unique* solution $(h\lambda, h^2\mu)$, which is a point on $\Gamma$. ∎

The more interesting cases occur, when the system (3.4) is singular. We distinguish between three (mutually exclusive) cases.

Case 3.1: $CF - DE = 0$, $\rho(e^{i\phi}) \neq 0$, $\sigma(e^{i\phi}) \neq 0$.

In this case $(C,D) \neq (0,0)$, and from (3.3) and (3.2) we then derive that (3.4) has a degenerate solution (which is a straight line in the $(h\lambda, h^2\mu)$-plane) if and only if $\rho(e^{i\phi})/\sigma(e^{i\phi})$ is real and non-zero. ■

Case 3.2: $CF - DE = 0$, $\rho(e^{i\phi}) \neq 0$, $\sigma(e^{i\phi}) = 0$, $\sigma^*(e^{i\phi}) \neq 0$.

Now $(C,D) = (0,0)$ and $(E,F) \neq (0,0)$, and we find the degenerate solution $h^2\mu = A/E$ (if $E \neq 0$) or, equivalently, $h^2\mu = B/F$ (if $F \neq 0$) if and only if $\rho(e^{i\phi})/\sigma^*(e^{i\phi})$ is real and non-zero. ■

Case 3.3: $CF - DE = 0$, $\rho(e^{i\phi}) \neq 0$, $\sigma(e^{i\phi}) = 0$, $\sigma^*(e^{i\phi}) = 0$.

Since $(A,B) \neq (0,0)$, no solution of (3.4) exists in this case. ■

If for $\phi = \phi_0$ the system (3.4) has no solution, a small perturbation of $\phi$ yields a solvable system and the values of $h\lambda$ or $h^2\mu$, or both, tend to infinity as $\phi \to \phi_0$.

The occurrence of degenerate solutions, as mentioned in the cases 3.1 and 3.2, is not exceptional: for $\phi = \pi$ the imaginary parts B,D and F vanish, and we obtain from (3.2) the degenerate solution

$$\rho^2(-1) - h\lambda\rho(-1)\sigma(-1) + h^2\mu[\sigma(-1)\rho'(-1) - \rho(-1)\sigma'(-1)] = 0. \quad (3.5)$$

As an illustration we derive, in the following examples, the stability region of two simple quadrature methods.

EXAMPLE 1. (the repeated trapezium rule). For the quadrature method (1.2) we choose the repeated trapezium rule ($w_{n0} = w_{nn} = 1/2$, $w_{nj} = 1$ for $j = 1(1)n-1$), which is reducible to the trapezoidal rule for solving ODEs. For this rule, the polynomials $\rho$ and $\sigma$ are $\rho(\zeta) = \zeta-1$ and $\sigma(\zeta) = (\zeta+1)/2$. For $\phi = 0$ and $\phi = \pi$, we find the degenerate solutions $h^2\mu = 0$ and $h^2\mu = -4$, respectively. For $0 < \phi < \pi$, the determinant $CF - DE = -\sin\phi$ does not vanish, and we find the unique solution $(h\lambda, h^2\mu) = (0, 2\cos\phi - 2)$. The set $\Gamma$ is now completely determined, and from it one easily derives that the stability region (with respect to the convolution test equation (1.7)) of the repeated trapezium rule is

$$\mathcal{R}_{trap} = \{(h\lambda,h^2\mu) \mid h\lambda < 0, -4 < h^2\mu < 0\}.$$

We emphasize that the trapezoidal rule, in view of definition 3, is *not* $A_0$-stable. ●

EXAMPLE 2. (the repeated rectangle rule). We now choose the repeated rectangle rule ($w_{n0} = 0$, $w_{nj} = 1$ for $j = 1(1)n$) which is reducible to the first order backward differentiation (BD) method or backward Euler rule for solving ODEs ($\rho(\zeta) = \zeta-1$ and $\sigma(\zeta) = \zeta$). For $\phi = 0$ and $\phi = \pi$, we find the degenerate solutions $h^2\mu = 0$ and $4-2h\lambda+h^2\mu = 0$, respectively, and for $0 < \phi < \pi$ we find the unique solution $(h\lambda,h^2\mu) = (0, 2\cos\phi - 2)$ from which we derive that the stability region is

$$\mathcal{R}_{BD}^{k=1} = \{(h\lambda,h^2\mu) \mid h\lambda < 0, 2h\lambda - 4 < h^2\mu < 0\}$$
$$\cup \{(h\lambda,h^2\mu) \mid h\lambda > 2, 0 < h^2\mu < 2h\lambda - 4\}.$$

Again we observe that this quadrature method is *not* $A_0$-stable. ●

The above examples illustrate a general result which is given in the following section.

## 4. A RESULT CONCERNING THE EXISTENCE OF $A_0$-STABLE QUADRATURE METHODS

The two examples given in the previous section raise the question: Do there exist $A_0$-stable $(\rho,\sigma)$-reducible quadrature methods at all? The following theorem provides the answer.

THEOREM 1. *$(\rho,\sigma)$-reducible quadrature methods cannot be $A_0$-stable.*

PROOF. Choose, in (2.6), $h = 1$ and a fixed negative $\lambda$ and define

$$\rho_1(\zeta) := \rho(\zeta)[\rho(\zeta) - \lambda\sigma(\zeta)],$$

(4.1)

$$\sigma_1(\zeta) := \zeta[\sigma(\zeta)\rho'(\zeta) - \rho(\zeta)\sigma'(\zeta)].$$

With this definition of $\rho_1$ and $\sigma_1$, the stability polynomial (2.6) is identical to the stability polynomial of the linear multistep method $(\rho_1,\sigma_1)$ applied with stepsize $h = 1$ to the ODE $f' = \mu f$.

The coefficient of $\zeta^{2k}$ in $\rho_1(\zeta)$ is $a_0(a_0-\lambda b_0)$ and does not vanish for a suitable negative $\lambda$ (recall that $a_0 \neq 0$). The coefficient of $\zeta^{2k}$ in $\sigma_1(\zeta)$ is $b_0ka_0 - a_0kb_0 = 0$, and therefore the linear multistep method $(\rho_1,\sigma_1)$ is *explicit*. If $\rho_1$ and $\sigma_1$ have common factors, they are divided out to yield polynomials $\rho_2$ and $\sigma_2$ with no common factors. The method $(\rho_2,\sigma_2)$, however, *remains explicit*. CRYER [4, theorem 3.1] has shown that explicit linear multistep methods for ODEs cannot be $A_0$-stable. This result implies the existence of a negative $\mu$ such that $\rho_2(\zeta) - \mu\sigma_2(\zeta)$ has zeros outside the open unit disk. For this $\mu$ and the suitable choice of $\lambda$, the stability polynomial (2.6) has, therefore, zeros outside the open unit disk. Hence we have shown the existence of a point in the third quadrant of the $(h\lambda,h^2\mu)$-plane, which is outside the stability region. ∎

We end this section with two remarks.

REMARKS.

4.1. The result of the above Theorem can also be derived in a more heuristic way. Consider the equation (2.1) for $h\lambda \to 0$ and $h^2\mu$ fixed. In this case, all $(\rho,\sigma)$-reducible quadrature methods for finding $f_n$ become explicit, and, clearly the region of absolute stability cannot have the entire negative $h^2\mu$-axis as part of its boundary.

4.2. We also want to consider the result of Theorem 1 in connection with similar methods for solving Volterra integro-differential equations (VIDE). Consider the class of numerical methods $(\rho,\sigma: Q)$ (see [3,9,10]) where $(\rho,\sigma)$ is a linear multistep method for ODEs and where Q represents the set of quadrature formulae. Choosing for Q $(\rho,\sigma)$-reducible quadrature formulae, we obtain the class of methods $(\rho,\sigma;\rho,\sigma)$.

The stability analysis of such methods is based on the test equation (cf. [3])

$$f'(x) = \lambda f(x) + \mu \int_0^x f(y)\,dy. \tag{4.2}$$

*Although this test equation is equivalent to the convolution test equation* (1.7)*, the stability behaviour with respect to* (4.2) *and* (1.7) *of the same basic method* $(\rho,\sigma)$ *is different.* To be specific, it is known ([3,9]) that $A_0$-stable methods of the form $(\rho,\sigma;\rho,\sigma)$ exist for VIDEs. For example, choosing for $(\rho,\sigma)$ the trapezoidal rule or the first or second order BD methods (which are A-stable for ODEs) yields an $A_0$-stable method $(\rho,\sigma;\rho,\sigma)$ for VIDEs. However, for the same underlying method $(\rho,\sigma)$ used to solve Volterra integral equation of the second kind, *the property of* $A_0$*-stability is lost.*

## 5. THE STABILITY REGIONS OF THE BACKWARD DIFFERENTIATION METHODS

In this section we determine the stability regions of the quadrature methods which are reducible to the well-known backward differentiation (BD) methods for $k = 1(1)6$. The coefficients $a_i$ and $b_0$ ($b_i = 0$ for $i = 1(1)k$) are listed in [8, p.242]. For a discussion of the corresponding quadrature method we refer to [10].

For the BD methods $\rho(e^{i\phi}) = 0$ only for $\phi = 0$, and $\sigma(e^{i\phi}) \neq 0$ for all $\phi$. Therefore only the Cases 1,2 and 3.1 of §3 can occur. For $\phi = 0$ (Case 1 in §3), we obtain the straight line $h^2\mu = 0$. Furthermore, since for the BD methods $\rho(e^{i\phi})/\sigma(e^{i\phi})$ is real and non-zero if and only if $\phi = \pi$, we derive from (3.5), for $\phi = \pi$, the straight line $c_1 + c_2 h\lambda + c_3 h^2\mu = 0$ which is part of the set $\Gamma$. The values of $c_1, c_2$ and $c_3$, computed for different values of k, are listed below.

| k | $c_1$ | $c_2$ | $c_3$ |
|---|---|---|---|
| 1 | 4 | -2 | 1 |
| 2 | 16 | -4 | 3 |
| 3 | 400 | -60 | 63 |
| 4 | 1024 | -96 | 135 |
| 5 | 6975 | -3840 | 65536 |
| 6 | 14175 | -6240 | 173056 |

These lines bisect the third quadrant of the $(h\lambda,h^2\mu)$-plane, and therefore, in view of Definition 3, the quadrature methods reducible to the BD methods are not $A_0$-stable.

For $0 < \phi < \pi$ and $k \leq 5$ the system (3.4) has a unique solution, which we have computed for $\phi = j\pi/100$, $j = 1(1)99$. For $k = 6$ we have found that the system (3.4) has no solution for $\phi = 60^\circ$ and $\phi \simeq 77^\circ 35'$. Consequently, for $\phi$ in the neighbourhood of these values, $h\lambda$ or $h^2\mu$ tend to infinite values. For this reason the set of curves $\Gamma$ for $k = 6$ was totally different from those of the other k values. Furthermore the resulting diagram did not permit any surveyable representation which led us to omit the case $k = 6$.

For $k = 1$, the stability region has been given already in Example 2. For $k = 2(1)5$ we give in Figure 1 diagrams of the stability region in the (x,y)-plane, where $x = h\lambda$ and $y = G(h^2\mu)$ with G defined by $G(z) = (\text{if } z \geq 0 \text{ then } \sqrt{z} \text{ else } -\sqrt{-z})$. In Figure 2 a close-up of Figure 1 near the origin is given.

The reason for choosing this (x,y)-scale is the following. Suppose that $\lambda$ and $\mu$ are fixed, and suppose that one is interested in the value of $h_0$ such that the points $(h\lambda,h^2\mu)$ lie within the stability region for $0 < h < h_0$ ($h_0$ may be interpreted as the maximal stable stepsize for that $\lambda$ and $\mu$). If the stability region is given in the $(h\lambda,h^2\mu)$-plane, then, for fixed $\lambda$ and $\mu$, the point $(h\lambda,h^2\mu)$ moves, as h increases from 0, along the parabola $h^2\mu = (h\lambda)^2\mu/\lambda^2$ away from the origin. The first intersection point of this parabola with the boundary curve then determines the value of $h_0$. If the stability region is given in the (x,y)-plane, then the point (x,y) moves, as h increases, along the straight line $y = x\, G(\mu)/\lambda$ away from the origin which is the line through the points $(\lambda,G(\mu))$ and (0,0) in the (x,y)-plane. The first intersection point with the boundary curve then determines $h_0$. In the case of a straight line, however, *such an intersection can be read directly from the diagram of the stability region.*

Due to this transformation the straight lines $c_1+c_2h\lambda+c_3h^2\mu = 0$ appear as parabolas in the (x,y)-plane. Note also that the scales on the x-axis and y-axis are different.

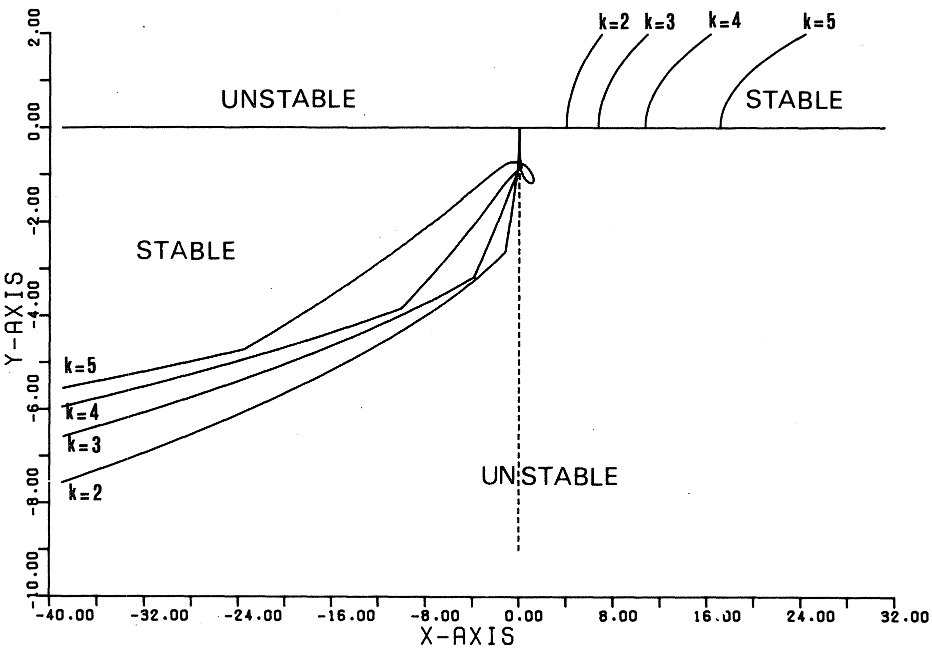


Figure 1.

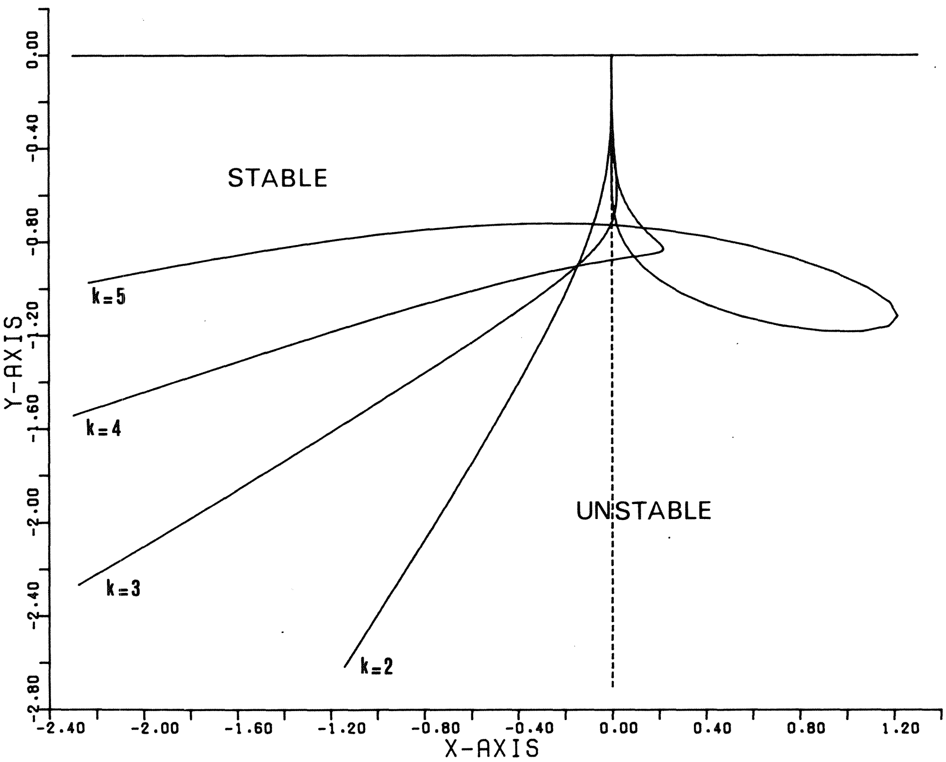


Figure 2.

## 6. CONCLUDING REMARKS

In this paper we have investigated the stability behaviour of a class of direct quadrature methods, viz. the $(\rho,\sigma)$-reducible quadrature methods, for the solution of Volterra integral equations of the second kind. An important result is that within this class no $A_0$-stable methods exist. The question whether there exist more general direct quadrature methods which are $A_0$-stable, remains open. On the basis of the heuristic argument mentioned in remark 4.1, however, *we conjecture that a direct quadrature method cannot be $A_0$-stable.*

Numerical methods for solving (1.1) which are $A_0$-stable do exist. An example is the second order method discussed in [6]. Their method (which is essentially a VIDE method adapted to solve second kind integral equations) is outside the class of direct quadrature methods, and hence does not contradict our conjecture.

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## REFERENCES

[1] C.T.H. BAKER, *The numerical treatment of integral equations*, Oxford: Clarendon Press, 1977.

[2] C.T.H. BAKER & M.S. KEECH, *Stability regions in the numerical treatment of Volterra integral equations*, SIAM J. Numer. Anal. 15 (1978), 394-417.

[3] H. BRUNNER & J.D. LAMBERT, *Stability of numerical methods for Volterra integro-differential equations*, Computing 12 (1974), 75-89.

[4] C.W. CRYER, *A new class of highly-stable methods: $A_0$-stable methods*, BIT 13 (1973), 153-159.

[5] W. FELLER, *On the integral equation of renewal theory*, Ann. Math. Stat. 12 (1941), 243-267.

[6] P.J. van der HOUWEN & H.J.J. te RIELE, *Backward differentiation formulas for Volterra integral equations of the second kind, I Convergence and stability*, Report NW 48/77, Mathematisch Centrum, Amsterdam (1977).

[7] N. KEYFITZ, *Introduction to the mathematics of population*, Reading: Addison-Wesley, 1968.

[8] J.D. LAMBERT, *Computational methods in ordinary differential equations*, London: John Wiley, 1973.

[9] J. MATTHYS, *A-stable linear multistep methods for Volterra integro-differential equations*, Numer. Math. 27 (1976), 85-94.

[10] P.H.M. WOLKENFELT, *Linear multistep methods and the construction of quadrature formulae for Volterra integral and integro-differential equations*, Report NW 76/79, Mathematisch Centrum, Amsterdam (1979).