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GALERKIN METHODS FOR EVEN-ORDER PARABOLIC EQUATIONS IN ONE SPACE VARIABLE

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Galerkin methods for even-order parabolic equations
in one space variable*)

by

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ABSTRACT

For parabolic equations in one space variable with a strongly coercive self-adjoint 2m-th order spatial operator, a k-th degree Faedo-Galerkin method is developed which has local convergence of order 2(k+1-m) at the knots for the first m-1 spatial derivatives and, if $k \geq 2m$, convergence of order k+2 at specific interior nodal points. These nodal points are the zeros of the Jacobi polynomial $P_n^{m,m}(\sigma)$ $(n=k+1-2m)$ shifted to the segments of the partition. All these convergence properties are preserved if suitable quadrature rules are used.

KEY WORDS AND PHRASES: *parabolic equations, Faedo-Galerkin method, superconvergence, Jacobi polynomials*

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*) This paper will be submitted for publication

1. INTRODUCTION

We consider the 2m-th order initial boundary problem

$$\frac{\partial u}{\partial t}(t,x) + Lu(t,x) = 0; \qquad x \in [-1,+1] = I;$$
$$t \in [0\ ,\ \infty) = J;$$
$$Lu = \sum_{\ell=0}^{m} (-1)^{\ell} \frac{\partial}{\partial x^{\ell}} [p_{\ell}(x) \frac{\partial^{\ell} u}{\partial x^{\ell}}];$$

(1.1)

$$\frac{\partial^{\ell} u}{\partial x^{\ell}} = 0, \quad x = \pm 1,\ \ell = 0,\dots,m-1;\ t \in J;$$

$$u(0,x) = u_0(x).$$

We suppose that $p_0,\dots,p_m$ and u_0 are such that $u(t)$ is sufficiently smooth for every $t \in J$.

1.1 Notations.

For any interval $E \subset I$ we define the Sobolev spaces $W^{\ell}(E)$ and $H^{\ell}(E)$, $\ell \geq 0$, and their norms by

$$W^{\ell}(E) = \{v | D^j v \in L^{\infty}(E), \quad j=0,\dots,\ell\};$$
$$H^{\ell}(E) = \{v | D^j v \in L^{2}(E), \quad j=0,\dots,\ell\};$$

(1.2)

$$\|v\|_{W^{\ell}(E)} = \max_{j=0,\dots,\ell} \|D^j v\|_{L^{\infty}(E)};$$
$$\|v\|_{H^{\ell}(E)} = [\sum_{j=0}^{\ell} (D^j v, D^j v)_E]^{\frac{1}{2}},$$

where D^j denotes d^j/dx^j or $\partial^j/\partial x^j$ and the complexvalued inner product $(,)_E$ is defined by

(1.3) $$(\alpha,\beta)_E = \int_E \alpha(x)\ \overline{\beta(x)}\ dx;\ \alpha,\beta \in L^2(E).$$

For convenience, since we use them frequently, we make the following replacements

$$(1.4) \qquad \|\alpha\|_{\ell} = \|\alpha\|_{H^{\ell}(I)}; \qquad (\alpha,\beta) = (\alpha,\beta)_I.$$

Furthermore, we define $H_0^m(I)$ and the bilinear functional $B:H_0^m(I) \times H_0^m(I) \to \mathbb{C}$ by

$$H_0^m(I) = \{v \mid v \in H^m(I);\ D^{\ell}v(\pm 1) = 0,\ \ell = 0,\ldots,m-1\};$$

(1.5)

$$B(u,v) = (Lu,v) = (u,Lv) = \sum_{\ell=0}^{m} (p_{\ell}D^{\ell}u, D^{\ell}v);\ u,v \in H_0^m(I).$$

We assume that $p_0,\ldots,p_m$ are such that B is strongly coercive, i.e. that there exist positive constants C_1 and C_2 depending on $p_0,\ldots,p_m$ only, such that

$$|B(u,v)| \le C_1 \|u\|_m \|v\|_m; \qquad u,v \in H_0^m(I);$$

(1.6)

$$B(v,v) \ge C_2 \|v\|_m^2; \qquad v \in H_0^m(I).$$

Note that this implies that $p_m(x) > 0$, $x \in I$.

In the sequel, C, C_1, C_2, etc. will be positive generic constants, not necessarily the same.

1.2 The Faedo-Galerkin method.

Let $N \ge 2$ be a constant integer and define the partition $\Delta = \{x_j\}_{j=0}^{N}$ of I by

$$h = 2/N;$$

$$(1.7) \qquad x_j = -1 + hj, \qquad j = 0,\ldots,N;$$

$$I_j = [x_{j-1}, x_j], \qquad j = 1,\ldots,N.$$

Let $k \geq 2m-1$ be a constant integer . Then we define the finite element space $S(\Delta) \subset H_0^m(I)$ by

$$(1.8) \qquad S(\Delta) = \{V \mid V \in H_0^m(I); \quad V \in P_k(I_j), \quad j = 1,\dots,N\},$$

where for any $\ell \geq 0$ $P_\ell(E)$ denotes the class of polynomials of degree at most ℓ defined on the interval E.

In the sequel, we will use the following constant integers associated to k,m and N

$$(1.9) \qquad \begin{aligned} r &= k+1-m; \\ n &= k+1-2m; \\ M &= rN-m. \end{aligned}$$

In (1.9) n is the number of interior nodal points of $S(\Delta)$ on I_j and M is the dimension of $S(\Delta)$.

In connection with Δ, we define the partition spaces $W^\ell(\Delta)$ and $H^\ell(\Delta)$ together with their norms by

$$(1.10) \qquad \begin{aligned} &W^\ell(\Delta) = \{v \mid v \in W^\ell(I_j); \quad j = 1,\dots,N\}; \\ &\|v\|_{W^\ell(\Delta)} = \max_{j=1,\dots,N} \|v\|_{W^\ell(I_j)} ; \\ &H^\ell(\Delta) = \{v \mid v \in H^\ell(I_j); \quad j = 1,\dots,N\}; \\ &\|v\|_{\ell,\Delta} = [\sum_{j=1}^N \|v\|^2_{H^\ell(I_j)}]^{\frac{1}{2}}. \end{aligned}$$

After these preliminary definitions, we can define a finite element solution of (1.1). Let $U: J \to S(\Delta)$ be the solution of the initial boundary problem

$$(1.11)\qquad \begin{aligned} &(\frac{\partial U}{\partial t}, V) + B(U,V) = 0, \qquad V \in S(\Delta),\ t \geq 0; \\ &U(0,x) = U_0(x), \end{aligned}$$

where $U_0 \in S(\Delta)$ is an approximation of u_0 satisfying

$$(1.12)\qquad \|u_0 - U_0\|_\ell \leq Ch^{k+1-\ell}\|u_0\|_{k+1}, \qquad \ell = 0,\ldots,m.$$

LEMMA 1. *Let* $u\colon J \to H_0^m(I) \cap H^{k+1}(I)$ *be the solution of* (1.1) *and let* $U\colon J \to S(\Delta)$ *be the solution of* (1.11) *with condition* (1.12). *Then* $e(t) = u(t)-U(t)$, *has the* L^2 *error bound*

$$(1.13)\qquad \begin{aligned} \|e(t)\|_0 \leq Ch^{k+1} * [\ \|u(t)\|_{k+1} &+ \\ + e^{-\lambda_1 t}\{\|u_0\|_{k+1} &+ \int_0^t e^{\lambda_1 \tau}\|Lu(\tau)\|_{k+1} d\tau\}], \end{aligned}$$

where λ_1 *is the smallest eigenvalue of* L.

PROOF. See [11]. □

1.3 Summary of results in this paper.

In §2 the occurrence of superconvergence at the knots is investigated. It appears that this depends crucially on a proper choice of U_0. A surprisingly simple choice of U_0 is made with the only additional requirement that $u(t) \in H_0^m(I) \cap H^{k+1}(I) \cap W^{2r}(\Delta)$, $t \in J$. In that case $D^\ell e(t,x_j)$ ($\ell = 0,\ldots,m-1$; $j = 1,\ldots,N-1$) is of $O(h^{2r})$ on J. Furthermore, if $n \geq 1$, there are on each I_j n specific interior points, where $e(t)$ is of $O(h^{k+2})$, one order better than the optimal order of convergence.

In §3, it is shown that all the results from §2 remain valid if $B(,)$ is approximated by a proper quadrature rule.

2. SUPERCONVERGENCE PHENOMENA

For m=1 and $k \geq 2$, J. Douglas, jr. et alii [7,8,9,10] have proved that the order of convergence at the knots is 2k, while the optimal order is k+1. We generalize their results for $m > 1$. Also, we establish a minor superconvergence at interior points. For these purposes, the Laplace transforms of u(t) and U(t) are used, because they transform initial boundary problems into boundary problems which are simpler to handle.

2.1. The Laplace transform.

Let V be a class of functions defined on I. Then for any continuous mapping $v\colon J \to V$, we define the Laplace transform $\mathcal{L}\colon C^0(J) \times V \to V$ by

$$(2.1) \qquad \mathcal{L}v(s,x) = \hat{v}(s,x) = \int_0^\infty e^{-st} v(t,x)\,dt,$$

where s lies in the convergence half-plane of v(t).

For the general properties of $\mathcal{L}$ and for the convergence criteria for (2.1), we refer to [6]. If we apply $\mathcal{L}$ to the problems (1.1) and (1.11), we get for $\hat{u}$ the two-point boundary problem (in classical and weak Galerkin form)

$$(2.2a) \qquad L\hat{u} + s\hat{u} = u_0, \quad x \in I;$$

$$(2.2b) \qquad B(\hat{u},v) + s(\hat{u},v) = (u_0,v), \qquad v \in H_0^m(I)$$

and for $\hat{U}$ the weak Galerkin form

$$(2.3) \qquad B(\hat{U},V) + s(\hat{U},V) = (U_0,V), \qquad V \in S(\Delta).$$

Note that (2.3) is *not* the standard finite element solution of (2.2). Since the dependence on s appears from the roof-sign, we will usually omit the argument s.

We first formulate a technical lemma which we will use a couple of times.

LEMMA 2. *Let* x_1 *and* x_2 *be nonnegative numbers; let* μ, γ *and* D *be positive parameters; let* s *be a complex number and let the following inequalities hold*

$$
(2.4)\qquad
\begin{aligned}
|x_1 + s\, x_2| &\le D\sqrt{x_2}\ ;\\
x_1 &\ge \gamma\, x_2;\\
s &= -\alpha + i\beta;\\
\mu \le \alpha &\le |\beta| + \mu;\\
0 < \mu &< \gamma.
\end{aligned}
$$

Then x_1 *and* x_2 *have the bounds*

$$
(2.5)\qquad
\begin{aligned}
x_1 &\le \begin{cases} \dfrac{\gamma D^2}{(\gamma-\alpha)^2+\beta^2}\ , & \text{if } \alpha^2+\beta^2 \le \gamma^2;\\[2ex] \dfrac{D^2}{2\beta^2}\left[\alpha + \sqrt{\alpha^2+\beta^2}\ \right], & \text{if } \alpha^2+\beta^2 \ge \gamma^2;\end{cases}\\[3ex]
x_2 &\le \begin{cases} \dfrac{D^2}{(\gamma-\alpha)^2+\beta^2}\ , & \text{if } \alpha \le \gamma;\\[2ex] \dfrac{D^2}{\beta^2}\ , & \text{if } \alpha \le \gamma.\end{cases}
\end{aligned}
$$

PROOF. We substitute

$$(2.6)\qquad x_1 = y_1 + \alpha y_2;\ x_2 = y_2$$

Then, for y_1 and y_2, we have the inequalities

$$y_1^2 + \beta^2 y_2^2 \le D^2 \; y_2 \;;$$
$$(2.7) \qquad y_1 \ge (\gamma-\alpha) y_2 ; \; y_2 \ge 0 \;;$$
$$\mu \le \alpha \le |\beta| + \mu ; \quad \mu < \gamma ,$$

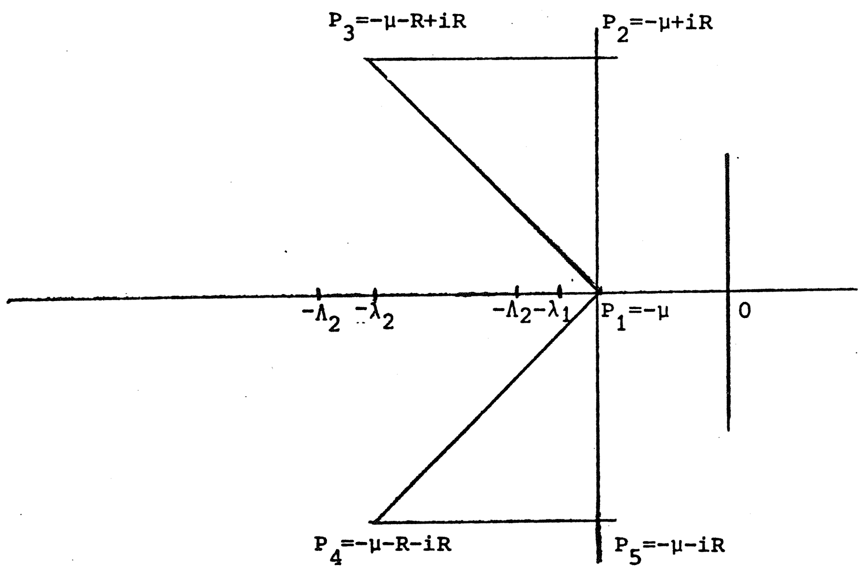
so x_1 and x_2 are linear functions of y_1 and y_2 with constraints (2.7). Elaboration for all possible values of β delivers (2.5). □

We turn to the problems (2.2) and (2.3). Let μ be a positive number with $\mu < \lambda_1$ and define $P_1, P_2, \ldots, P_5$ in the complex plane (see figure 1) by

$$P_1 = -\mu \;;$$
$$(2.8) \qquad P_{2,5} = -\mu \pm iR ;$$
$$P_{3,4} = -(\mu+R) \pm iR, \; R > 0.$$

By $\overline{P_1 \ldots P_n}$, we denote the broken straight line starting in P_1 going to P_2 etc. and ending in P_n.

<u>LEMMA 3</u>. *Let* $e(t) = u(t)-U(t)$ *and* $\hat{e} = \hat{u}-\hat{U}$, *where* $u(t)$, $U(t)$, $\hat{u}$ *and* $\hat{U}$ *are the solutions of* (1.1), (1.11), (2.2) *and* (2.3), *respectively. Then for* $t > 0$ *and* h *sufficiently small, we have*



<u>**Figure 1**</u>

$$D^{\ell}e(t,x) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\overline{P_4P_1P_3}} \hat{e}(s,x)\ \exp(st)ds =$$

(2.9)

$$= \frac{e^{-\mu t}}{\pi} \int_0^\infty e^{-\alpha t}\,\mathrm{Im}[(1-i)e^{-i\alpha t}D^{\ell}\,\hat{e}(-\alpha-\mu-i\alpha,x)]d\alpha, \qquad \ell=0,\dots,m-1.$$

PROOF. It is known [11] that

$$\hat{u}(s,x) = \sum_{i=1}^{\infty} (u_0,\phi_i)\phi_i(x)/(s+\lambda_i);$$

(2.10)

$$\hat{U}(s,x) = \sum_{i=1}^{M} (U_0,\Phi_i)(U_0,\Phi_i)\Phi_i(x) ,$$

where $\lambda_1,\lambda_2,\dots,$ are the positive eigenvalues of L in nondecreasing order, with orthonormal eigenfunctions $\phi_1,\phi_2,\dots,$ and where $\Lambda_1,\Lambda_2,\dots,\Lambda_M$ (in nondecreasing order) and $\Phi_1,\Phi_2,\dots,\Phi_M$ are the positive eigenvalues and eigenfunctions of the problem

$$B(\Phi_i,V) = \Lambda_i(\Phi_i,V),\ V \in S(\Delta), \qquad i=1,\dots,M.$$

Note that

$$(2.11) \qquad \Lambda_1 = \inf_{V\in S(\Delta)} \frac{B(V,V)}{(V,V)} > \inf_{v\in H_0^m(I)} \frac{B(v,v)}{(v,v)} = \lambda_1.$$

From (2.10), we see that $D^{\ell}\hat{e}$ is meromorphic in s with the set $\{-\lambda_i\}_{i=1}^{\infty} \cup \{-\Lambda_i\}_{i=1}^{M}$ as only possible poles. Since these singularities lie outside the contours $\overline{P_1P_2P_3}$ and $\overline{P_1P_4P_5}$ we have by Cauchy's theorem

$$(2.12) \qquad \int_{\overline{P_1P_2P_3}} D^{\ell}\hat{e}(s,x)\ \exp(st)ds = \int_{\overline{P_1P_4P_5}} D^{\ell}\hat{e}(s,x)\ \exp(st)ds = 0.$$

Furthermore, since $\overline{P_5P_1P_2}$ lies in the convergence half-plane of $\hat{e}$, we can apply the complex inversion formula [6] to obtain

$$(2.13)\qquad D^{\ell}e(t,x) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\overline{P_5P_1P_2}} D^{\ell}\hat{e}(s,x)\ \exp(st)ds,$$

Hence we see immediately from (2.12) and (2.13) that we only have to prove that

$$(2.14)\qquad \lim_{R\to\infty} \int_{\overline{P_2P_3}} D^{\ell}\hat{e}(s,x)\ \exp(st)ds = \lim_{R\to\infty} \int_{\overline{P_4P_5}} D^{\ell}\hat{e}(s,x)\ \exp(st)ds = 0.$$

From (2.2), we can derive that

$$(2.15)\qquad |B(\hat{u},\hat{u}) + s(\hat{u},\hat{u})| = |(u_0,\hat{u})| \le \|u_0\|_0\ \|\hat{u}\|_0.$$

Application of lemma 2 for $s = -\mu-\alpha \pm iR$ yields

$$(2.16a)\qquad \begin{aligned} &|B(\hat{u},\hat{u})| \le \tfrac{1}{2}\|u_0\|_0^2\ [\alpha+\sqrt{\alpha^2+R^2}\ ]/R^2; \\ &|D^{\ell}\hat{u}(x)| \le C\|u\|_m \le CR^{-\frac{1}{2}}\|u_0\|_0, \end{aligned}$$

if $R \to \infty$. The last inequality was proved by Sobolev's embedding theorems [11] and by the strong coercivity of B. In a similar way, we can prove from (2.3) that

$$(2.16b)\qquad |D^{\ell}\hat{U}(s,x)| \le CR^{-\frac{1}{2}}\|U_0\|_0, \qquad \ell=0,\ldots,m-1,$$

if $R \to \infty$ and $s = \pm iR-\alpha-\mu$. From (2.16) one easily proves (2.14) and therewith the lemma. □

As in [2], we can exploit (2.9) to transfer local convergence properties of $\hat{e}$ immediately to $e(t)$. Since these properties are not standard if $|s| \to \infty$, we have to prove them here explicitly, of course only for $s = -\alpha-\mu \pm i\alpha$. In the sequel $C(\alpha), C_1(\alpha)$, etc. are positive functions of α which are polynomially bounded on $[0,\infty)$, not necessarily the same ones.

<u>LEMMA 4</u>. *Let* $U_0 \in S(\Delta)$ *be any approximation of* u_0 *satisfying* (1.12). *Then* $\hat{e} = \hat{u}-\hat{U}$ *has the bound*

$$(2.17) \qquad \|\hat{e}\|_\ell \le C(\alpha)h^{k+1-\ell}\|u_0\|_{k+1}, \qquad \ell=0,\dots,m.$$

<u>PROOF</u>. From (2.2b) and (2.3), we find that

$$(2.18) \qquad B(\hat{u}-\hat{U},V) + s(\hat{u}-\hat{U},V) = (u_0-U_0,V), \; V \in S(\Delta).$$

Next, we introduce the elliptic projection $\hat{U}_2 \in S(\Delta)$ of $\hat{u}$ by

$$(2.19) \qquad B(\hat{u}-\hat{U}_2,V) = 0, \qquad V \in S(\Delta).$$

It is standard [11] that $\|\hat{u}-\hat{U}_2\|_\ell \le Ch^{k+1-\ell}\|\hat{u}\|_{k+1}$, $\ell=0,\dots,m$. If we put $V = \hat{\varepsilon} = \hat{U}_2-\hat{U}$ and subtract (2.19) from (2.18), we find

$$(2.20) \qquad \begin{aligned} &|B(\hat{\varepsilon},\hat{\varepsilon}) + s(\hat{\varepsilon},\hat{\varepsilon})| = |(u_0-U_0-s(\hat{u}-\hat{U}_2),\hat{\varepsilon})| \le \\ &\le C\|\hat{\varepsilon}\|_0 h^{k+1}(\|u_0\|_{k+1} + |s|\|\hat{u}\|_{k+1}). \end{aligned}$$

Application of lemma 2 to (2.20) yields

$$(2.21) \qquad \begin{aligned} &B(\hat{\varepsilon},\hat{\varepsilon}) \le C(\alpha)h^{2(k+1)}(\|u_0\|_{k+1} + |s|\|\hat{u}\|_{k+1})^2 ; \\ &\|\hat{\varepsilon}\|_\ell \le \|\hat{\varepsilon}\|_m \le C(\alpha)h^{k+1}(\|u_0\|_{k+1} + |s|\|\hat{u}\|_{k+1}). \end{aligned}$$

We now have

$$(2.22) \qquad \begin{aligned} &\|\hat{e}\|_\ell \le \|\hat{u}-\hat{U}_2\|_\ell + \|\hat{\varepsilon}\|_\ell \le \\ &\le Ch^{k+1-\ell}[\|u\|_{k+1} + C(\alpha)h^\ell(\|u_0\|_{k+1} + |s|\|\hat{u}\|_{k+1}), \qquad \ell=0,\dots,m. \end{aligned}$$

We need an estimation of $\|\hat{u}\|_{k+1}$ yet. From (2.2), we can derive that, since $L\hat{u} \in H_0^m(I)$

$$|B(L\hat{u},L\hat{u}) + s(L\hat{u},L\hat{u})| = |(Lu_0,L\hat{u})| \le \|Lu_0\|_0 \, \|L\hat{u}\|_0.$$

Application of lemma 2 yields

$$\text{(2.23)} \quad \begin{aligned} &\|\hat{u}\|_{3m} \le C\|L\hat{u}\|_m \le C_1(\alpha)\|Lu_0\|_0 \le C_1(\alpha)\|u_0\|_{2m}; \\ &\|\hat{u}\|_{2m} \le C\|L\hat{u}\|_0 \le C_2(\alpha)\|Lu_0\|_0 \le C_2(\alpha)\|u_0\|_{2m}. \end{aligned}$$

Since

$$\|D^{\ell}L\hat{u}\|_0 \le |s|\,\|D^{\ell}\hat{u}\|_0 + \|D^{\ell}u_0\|, \qquad \ell=0,\dots,n,$$

we can prove by induction that

$$\text{(2.24)} \qquad \|\hat{u}\|_{k+1} \le C(\alpha)\|u_0\|_{k+1}.$$

From (2.22) and (2.24), we get (2.17), which proves the lemma. □

REMARK. Although $C(\alpha)$ in (2.17) is polynomially bounded, it tends to be of $O((\lambda_1-\mu)^{-1})$ near $\alpha=0$, as $\mu\uparrow\lambda_1$.

Now that we have established convergence of $\hat{e}$ on the contour $\overline{P_4P_1P_3}$, we can investigate the superconvergence at the knots.

For any $x \in (-1,+1)$ and $\ell \in \{0,1,\dots,m-1\}$, we define the generalized Green's function $\hat{G}_{\ell}(x,\xi) \in H_0^m(I) \cap H^{k+1}(0,x) \cap H^{k+1}(x,1)$ associated to L by

$$\text{(2.25)} \quad \begin{aligned} &L_{\xi}\hat{G}_{\ell}(x,\xi) + \bar{s}\hat{G}_{\ell}(x,\xi) = 0, \qquad \xi \in I\backslash\{x\}; \\ &B(v,\hat{G}_{\ell}(x)) + s(v,\hat{G}_{\ell}(x)) = D^{\ell}v(x), \; v \in H_0^m(I), \end{aligned}$$

where the subscript ξ of L_ξ denotes partial differentiation with respect to ξ. If we denote

$$(2.26) \qquad \hat{G}_{\ell j}(\xi) = \hat{G}_\ell(x_j,\xi), \qquad j=1,\dots,N-1;\ \ell=0,\dots,m-1,$$

we find for $D^\ell\hat{e}(x_j)$ the bound

$$(2.27) \qquad \begin{aligned} |D^\ell\hat{e}(x_j)| &= |B(\hat{e},\hat{G}_{\ell j}) + s(\hat{e},\hat{G}_{\ell j})| \le \\ &\le |B(\hat{e},\hat{G}_{\ell j}-V) + S(\hat{e},\hat{G}_{\ell j}-V)| + |B(\hat{e},V) + s(\hat{e},V)| \le \\ &\le C(\alpha)\|\hat{e}\|_m \|\hat{G}_{\ell j}-V\|_m + |(u_0-U_0,V)|, \quad V \in S(\Delta), \\ &\qquad j=1,\dots,N-1;\ \ell=0,\dots,m-1. \end{aligned}$$

Since $\hat{G}_{\ell j} \in H_0^m(I) \cap H^{k+1}(\Delta)$, we can take V such that

$$(2.28) \qquad \begin{aligned} &\|\hat{G}_{\ell j}-V\|_m \le Ch^r\|\hat{G}_{\ell j}\|_{k+1,\Delta}\ ; \\ &\|V\|_{W^k(\Delta)} \le C\|G_{\ell j}\|_{W^{k+1}(\Delta)}\ . \end{aligned}$$

Then it is easily proved from (2.17) and (2.27) that

$$(2.29) \qquad \begin{aligned} |D^\ell\hat{e}(x_j)| &\le C(\alpha)h^{2r}\|u_0\|_{k+1}\ \|\hat{G}_{\ell j}\|_{k+1,\Delta} + \\ &+ |(u_0-U_0,V)|, \qquad \ell=0,\dots,m-1;\ j=1,\dots,N-1. \end{aligned}$$

We have yet to estimate $|(u_0-U_0,V)|$ and $\|G_{\ell j}\|_{k+1,\Delta}$.

Concerning the first quantity, a seductive choice of U_0 would be the L^2 projection of u_0 which would annihilate $|(u_0-U_0,V)|$. A drawback of this choice, however, is that the superconvergence of $D^\ell e(t,x_j)$ would not be uniform in time: (2.9) is not valid for $t=0$ and $D^\ell e(0,x_j)$ is of $O(h^{k+1-\ell})$, $\ell=0,\dots,m-1$, in stead of $O(h^{2r})$.

In the next sections, we will construct a U_0 which guarantees superconvergence of $D^{\ell}e(t,x_j)$ *uniform in time* and which imposes rather mild extra conditions to u_0 and $u(t)$: they also have to be in $W^{2r}(\Delta)$. Although we chose Δ uniform, for reasons of convenience, it can, of course, also be chosen quasiuniform, if this helps to meet the extra conditions.

2.2 Choice of nodal points; Jacobi polynomials.

In order to construct a proper approximation U_0 of u_0, we first define the ν-th degree Jacobi polynomial $P_\nu^{\alpha,\beta}(x)$ by [1,13]

$$(2.30)\qquad \begin{aligned} P_\nu^{\alpha,\beta}(x) &= [w(x)]^{-1}\, D^\nu[(1-x^2)^\nu\, w(x)];\ \nu \ge 0;\\ w(x) &= (1-x)^\alpha(1+x)^\beta;\ x \in (-1,+1);\ \alpha,\beta > -1. \end{aligned}$$

These polynomials have the properties [1,13]

$$(2.31)\qquad \begin{aligned} &(wP_\mu^{\alpha,\beta},\ P_\nu^{\alpha,\beta}) = \delta_{\mu\nu}(wP_\nu^{\alpha,\beta},P_\nu^{\alpha,\beta})\ ;\ \mu,\nu \ge 0;\\ &P_\nu^{\alpha,\beta}(x_{\mu\nu}) = 0; -1 < x_{1\nu} < x_{2\nu} < \ldots < x_{\nu\nu} < 1, \end{aligned}$$

where $\delta_{\mu\nu}$ is the Kronecker symbol.

Within the context of this paper, we are only interested in the case $\alpha = \beta = m$.

We recall that $r = k+1-m$ and $n = k+1-2m$. Let $\sigma_1,\ldots,\sigma_n$ be the zeros of $P_n^{m,m}(\sigma)$, i.e.

$$(2.32)\qquad P_n^{m,m}(\sigma_\ell) = 0,\quad \ell=1,\ldots,n.$$

Of course, (2.32) only makes sense, if $n \ge 1$. In the sequel, it is tacitly assumed that the formulae which make no sense if $n=0$ are to be omitted.

Given a partition Δ of I, we define the points $\xi_{\ell j}$ by

(2.33) $$\xi_{\ell j} = x_{j-1} + \frac{h}{2}(1+\sigma_\ell); \qquad j=1,\ldots,N;\ \ell=1,\ldots,n.$$

Next, we introduce the linear interpolation Π: $H_0^m(I) \cap W^{2m}(\Delta) \to S(\Delta)$ by

(2.34) $$\begin{aligned} &D^\ell \Pi f(x_j) = D^\ell f(x_j), \quad \ell=0,\ldots,m-1;\ j=1,\ldots,N-1; \\ &\Pi f(\xi_{\ell j}) = f(\xi_{\ell j}),\ \ell=1,\ldots,n;\ j=1,\ldots,N. \end{aligned}$$

<u>LEMMA 5</u>. *For any* $V \in S(\Delta)$ *and* $f \in H_0^m(I) \cap W^{2r}(\Delta)$

(2.35) $$|(f-\Pi f,V)| \le Ch^{2r}\|f\|_{W^{2r}(\Delta)}\|V\|_{W^k(\Delta)}.$$

<u>PROOF</u>. For n=0, (2.35) is trivial [11]. For $n \ge 1$, we consider an arbitrary segment I_j. If we substitute $x = \frac{1}{2}(x_{j-1}+x_j+h\sigma)$, $\sigma \in I$, we find that

$$(f-\Pi f,V)_{I_j} = \tfrac{1}{2}h \int_{-1}^{+1} [(f-\Pi f)V](\tfrac{1}{2}(x_{j-1}+x_j+h\sigma))\,d\sigma =$$
$$= \tfrac{1}{2}h \int_{-1}^{+1} (1-\sigma^2)^m P_n^{m,m}(\sigma)\,(gV)\,(\tfrac{1}{2}(x_{j-1}+x_j+h\sigma))\,d\sigma$$

where g is bounded on I. From (2.31), we conclude that $(f-\Pi f,V)_{I_j} = 0$ if $gV \in P_{n-1}(I_j)$ or $fV \in P_{2r-1}(I_j)$. Application of Bramble and Hilbert's lemma [3] yields

(2.36) $$|(f-\Pi f,V)_{I_j}| \le Ch^{2r+1}\|D^{2r}(fV)\|_{L^\infty(I_j)}; \qquad j=1,\ldots,N.$$

Elaboration of (2.36) and summation over all I_j results in (2.35) and proves the lemma. □

Note that by (2.34) we have defined all the nodal points of $S(\Delta)$.

2.3 Order of convergence at the knots.

We return to (2.29) recalling that

$$|D^{\ell}\hat{e}(x_j)| \le C(\alpha)h^{2r}\|u_0\|_{r+1}\ \|\hat{G}_{\ell j}\|_{k+1,\Delta} + $$

$$+ |(u_0-U_0,V)|, \qquad j=1,\dots,N-1;\ \ell=0,\dots,m-1,$$

where V is an approximation of $\hat{G}_{\ell j}$ satisfying (2.28). If we take $U_0 = \Pi u_0$, Π defined by (2.34), then application of (2.28) and lemma 5 gives

$$(2.37)\qquad \begin{aligned} |D^{\ell}\hat{e}(x_j)| &\le C(\alpha)h^{2r}[\|u_0\|_{k+1}\ \|\hat{G}_{\ell j}\|_{k+1,\Delta} + \|u_0\|_{W^{2r}(\Delta)}\|V\|_{k,\Delta}] \le \\ &\le C(\alpha)h^{2r}\ \|\hat{G}_{\ell j}\|_{W^{k+1}(\Delta)}\|u_0\|_{W^{2r}(\Delta)}, \qquad j=1,\dots,N-1;\ \ell=0,\dots,m-1. \end{aligned}$$

It is easily proved that $\|\hat{G}_{\ell j}\|_{k+1,\Delta}$ is polynomially bounded, hence we can prove by combination of (2.37) and lemma 3 that

$$(2.38)\qquad |D^{\ell}e(t,x_j)| \le h^{2r}e^{-\mu t}\|u_0\|_{W^{2r}(\Delta)}\int_0^{\infty} e^{-\alpha t}C(\alpha)\,d\alpha, \qquad t > 0.$$

There is one last problem: the superconvergence bound (2.38) does not hold down to t=0. This obstacle is immediately removed because the definition of U_0 implies that

$$D^{\ell}e(0,x_j) = 0,\ \ell=0,\dots,m-1;\ j=1,\dots,N-1.$$

That $U_0 = \Pi u_0$ satisfies (1.12) is trivial since Π leaves all members of $S(\Delta)$ invariant. This concludes the proof of

THEOREM 1. *Let* $u: J \to H_0^m(I) \cap H^{k+1}(I) \cap W^{2r}(\Delta)$ *be the solution of* (1.1) *and let* $U: J \to S(\Delta)$ *be the solution of* (1.11) *with* U_0 *defined by* (2.34). *Then the error function* $e(t) = u(t)-U(t)$ *has the global bound* (1.13) *and the local bound*

$$(2.39)\qquad |D^{\ell} e(t,x_j)| \le F(t)e^{-\mu t} h^{2r} \|u_0\|_{W^{2r}(\Delta)},$$

where μ *is a number between* 0 *and* λ_1 *and where* $F(t)$ *is bounded on* J, $F(0) = 0$ *and* $F(t) = O(t^{-1})$ *as* $t \to \infty$. □

2.4. Order of convergence at Jacobi points.

In this section, we will prove that the order of convergence at the points $\xi_{\ell j}$ defined by (2.33) is of $O(h^{k+2}e^{-\mu t})$. Since these points only exist if $n \ge 1$, we confine our attention to the case $k \ge 2m$.

For any $I_j \in \Delta$, we define

$$(2.40)\qquad S(I_j) = \{V | V \in S(\Delta);\ \mathrm{supp}(V) = I_j\}.$$

It is evident that $S(I_j)$ has dimension n and that

$$(2.41)\qquad D^{\ell} V(x) = 0;\ x \in \partial I_j;\ V \in S(I_j);\ \ell=0,\dots,m-1.$$

We define a basis $\{\phi_i\}_{i=1}^{n}$ of $S(I_j)$ by

$$(2.42)\qquad \phi_i(\xi_{\ell j}) = \delta_{i\ell}, \qquad 1 \le i,\ \ell \le n.$$

If we apply (2.18) for $\phi_1,\dots,\phi_n$, we find after partial integration that

$$(2.43)\qquad \begin{aligned} (\hat{e}, L\phi_i + \bar{s}\phi_i) &= (u_0 - U_0, \phi_i) + \\ &+ \sum_{\ell=1}^{m} \sum_{\nu=0}^{\ell-1} [(-1)^{\nu} D^{\nu}(p_{\ell} D^{\ell}\phi_i) D^{\ell-1-\nu}\hat{e}]\Big|_{x_{j-1}}^{x_j}, \qquad i=1,\dots,n. \end{aligned}$$

In order to approximate the inner product (,) by a quadrature rule involving the function values at $\xi_{\ell j}$ which is accurate enough, we define for $f \in W^{2r}(I)$ the approximation

$$(2.44)\qquad \int_{-1}^{+1} f(\sigma)d\sigma = \int_{-1}^{+1} \Pi f(\sigma)d\sigma\,,$$

where Π: $W^{2r}(I) \to P_k(I)$ is defined by (2.34) shifted from I_j to I. Note that in the case m=1, we obtain Lobatto's quadrature rule [1].

LEMMA 7. *Quadrature rule* (2.44) *is exact if* $f \in P_{2r-1}(I)$.

PROOF. Since

$$f(\sigma) - \Pi f(\sigma) = (1-\sigma^2)^m P_n^{m,m}(\sigma) g(\sigma),$$

where $g(\sigma)$ is bounded, it is evident that (2.44) is exact if $g \in P_{n-1}(I)$, i.e. if $f \in P_{2r-1}(I)$. □

Elaboration of (2.44) yields

$$\int_{-1}^{+1} \Pi f(\sigma)d\sigma = \sum_{\ell=0}^{m-1} [\theta_{\ell_1} D^{\ell} f(-1) + \theta_{\ell_2} D^{\ell} f(+1)] \tag{2.45}$$
$$+ \sum_{\ell=1}^{n} \omega_\ell f(\sigma_\ell),$$

where $\sigma_1,\ldots,\sigma_n$ are the zeros of $P_n^{m,m}(\sigma)$ and θ_{ℓ_1}, θ_{ℓ_2} and ω_ℓ are constant weights. By applying (2.44) to $f_\ell(\sigma) = (1-\sigma^2)^m P_n^{m,m}(\sigma)/(\sigma-\sigma_\ell)$, $\ell=1,\ldots,n$, one can prove that [13, ch. XV]

$$\omega_\ell = \mu_\ell(1-\sigma_\ell^2)^{-m}, \qquad \ell=1,\ldots,n,$$

where $\mu_1,\ldots,\mu_n$ are the positive Gauss-Christoffel numbers for the n-point Gauss-Jacobi quadrature formula with weight function $(1-\sigma^2)^m$. This proves that $\omega_\ell > 0$, $\ell=1,\ldots,n$.

Next, we define for $\alpha,\beta \in W^{2r}(I_j)$

$$(\alpha,\beta)^*_{I_j} = \frac{h}{2} \sum_{\ell=1}^{n} \omega_\ell\, \alpha(\xi_{\ell_j})\, \beta(\xi_{\ell_j}) + \tag{2.46}$$
$$+ \frac{h}{2} \sum_{\ell=0}^{m-1} (\frac{h}{2})^{\ell} [\theta_{\ell_1} D^{\ell}(\alpha\beta)(x_{j-1}) + \theta_{\ell_2} D^{\ell}(\alpha\beta)(x_j)].$$

This quadrature rule has the error bound [3]

$$|(\alpha,\beta)_{I_j} - (\alpha,\beta)^*_{I_j}| \le Ch^{2r+1}\|D^{2r}(\alpha\beta)\|_{L^\infty(I_j)}. \tag{2.47}$$

If we apply (2.46) to (2.43) and multiply by $2h^{2m-1}$, we obtain

$$\begin{aligned}
&|\sum_{\ell=1}^{n} h^{2m}\omega_\ell[L\phi_i(\xi_{\ell j}) + \bar{s}\,\delta_{i\ell}]\ \hat{e}(\xi_{\ell j})| \le \\
&\le h^{2m}\sum_{\ell=0}^{m-1}\left(\frac{h}{2}\right)^\ell |\theta_{\ell_1}D^\ell(\hat{e}(\bar{s}\phi_i+L\phi_i))(x_{j-1}) \\
&\qquad + \theta_{\ell_2}D^\ell(\hat{e}(\bar{s}\phi_i + L\phi_i))(x_j)| + \\
&+ C_1h^{2k+2}\|D^{2r}(\hat{e}(\bar{s}\phi_i + L\phi_i))\|_{L^\infty(I_j)} + \\
&+ C_2h^{2k+2}\|D^{2r}(\phi_i(u_0-U_0))\|_{L^\infty(I_j)} + \\
&+ h^{2m}\sum_{\ell=1}^{m}\sum_{\nu=0}^{\ell-1}|[D^\nu(p_\ell D^\ell\phi_i)D^{\ell-1-\nu}\hat{e}]_{j-1}^{x_j}| \le \\
&\le C_1(\alpha)h^{k+2}\|u_0\|_{W^{2r}(\Delta)} + C_2(\alpha)h^{k+2}\|\hat{e}\|_{W^{2r}(I_j)} + \\
&+ C_3h^{k+2}\|u_0-U_0\|_{W^{2r}(I_j)} + C_4(\alpha)h^{k+2}\|u_0\|_{W^{2r}(\Delta)} \le \\
&\le C(\alpha)h^{k+2}(\|u_0\|_{W^{2r}(\Delta)} + \|u_0-U_0\|_{W^{2r}(I_j)} + \|\hat{e}\|_{W^{2r}(I_j)}), \quad i=1,\dots,n.
\end{aligned} \tag{2.48}$$

We have to estimate $\|u_0-U_0\|_{W^{2r}(I_j)}$ and $\|\hat{e}\|_{W^{2r}(I_j)}$.

From [4,11] we know that in virtue of the definition of U_0, we have

$$\|D^\ell(u_0-U_0)\|_{L^\infty(I_j)} \le Ch^{k+1-\ell}\|D^{k+1}u_0\|_{L^\infty(I_j)}, \qquad \ell=0,\dots,k,$$

hence we easily get

$$\|u_0-U_0\|_{W^{2r}(I_j)} \le C\,\|u_0\|_{W^{2r}(I_j)}. \tag{2.49}$$

Let $\Pi\hat{u}$ be the interpolation of $\hat{u}$ defined by (2.34). Then we can prove from [4,11] and [2] that

$$\begin{aligned}
\|\hat{e}\|_{W^{2r}(I_j)} &\le \|\hat{U}-\Pi\hat{u}\|_{W^k(I_j)} + \|\hat{u}-\Pi\hat{u}\|_{W^{2r}(I_j)} \le \\
&\le C_1 h^{-k}\|\hat{U}-\Pi\hat{u}\|_{L^\infty(I_j)} + C_2\|\hat{u}\|_{W^{2r}(I_j)} \le \\
&\le C_1 h^{-k}[\|\hat{e}\|_{L^\infty(I_j)} + \|\hat{u}-\Pi\hat{u}\|_{L^\infty(I_j)}] + C_2\|\hat{u}\|_{W^{2r}(I_j)} \le \qquad (2.50)\\
&\le C_1(\alpha)[\|\hat{u}\|_{k+1} + \|u_0\|_{k+1}] + C_2 h\|D^{k+1}\hat{u}\|_{L^\infty(I_j)} + \\
&+ C_3\|\hat{u}\|_{W^{2r}(I_j)} \le \\
&\le C_1(\alpha)[\|\hat{u}\|_{k+1} + \|u_0\|_{k+1}] + C_2\|\hat{u}\|_{W^{2r}(I_j)}.
\end{aligned}$$

$\|\hat{u}\|_{k+1}$ was already estimated (formula (2.24)), for the estimation of $\|\hat{u}\|_{W^{2r}(I_j)}$, we simply use the differential equation (2.2a) to obtain

$$\|\hat{u}\|_{W^{2r}(I_j)} \le C(\alpha)\|u_0\|_{W^{2r}(I_j)} \quad . \qquad (2.51)$$

Summarily, we have obtained from (2.48)-(2.51) that

$$\begin{aligned}
&\Big|\sum_{\ell=1}^{n} h^{2m}\omega_\ell[L\phi_i(\xi_{\ell j}) + \bar{s}\delta_{i\ell}]\hat{e}(\xi_{\ell j})\Big| \le \qquad (2.52)\\
&\le C(\alpha)h^{k+2}\|u_0\|_{W^{2r}(\Delta)}, \qquad i=1,\dots,n.
\end{aligned}$$

We have to prove the solvability of the linear system (2.52). It is easily proved that

$$\Big|(\omega_\ell L\phi_i(\xi_{\ell j}) - \frac{2}{h}B(\phi_i,\phi_\ell))h^{2m}\Big| \le Ch^2, \qquad (2.53)$$

if h is small enough. Consequently, the matrix $(h^{2m}\omega_\ell L\phi_i(\xi_{\ell j}))$ approximates a symmetric positive definite matrix whose eigenvalues are of $O(h^0)$. This means that its eigenvalues are nearly positive, i.e. the real parts are positive of $O(h^0)$ and the imaginary parts are of $O(h^2)$. Since $\bar{s} \in P_4P_1P_3$, we can show from (2.52) by elementary matrix calculus that

$$(2.54)\qquad |\hat{e}(\xi_{\ell j})| \le C(\alpha) h^{k+2} \|u_0\|_{W^{2r}(\Delta)},$$
$$\ell=1,\dots,n; j=1,\dots,N.$$

Application of lemma 3 to (2.54) plus the fact that $e(0,\xi_{\ell j}) = 0$ lead to

<u>THEOREM 2</u>. *Let the conditions of* Theorem 1 *hold with the restriction that* $k \ge 2m$. *Then* $e(t)$ *has the bounds* (1.12) *and* (2.39) *plus the additional bound*

$$(2.55)\qquad |e(t,\xi_{\ell j})| \le F(t) e^{-\mu t} h^{k+2} \|u_0\|_{W^{2r}(\Delta)},$$
$$j=1,\dots,N;\ \ell=1,\dots,n.$$

where the points $\xi_{\ell j}$ *are defined by* (2.33) *and* $F(t)$ *is bounded on* J, *vanishes if* $t=0$ *and is of* $O(t^{-1})$ *as* $t \to \infty$. □

3. QUADRATURE RULES

When solving (1.11), one is usually forced to approximate $B(U,V)$ by some quadrature [12]. The choice of this rule is, as usual, dictated not only by the accuracy of it but by its impact on the convergence properties. It may sometimes be useful to approximate (U_t,V) by a quadrature rule as well, e.g. in the case m=1 where the choice of (k+1)-point Lobatto quadrature delivers a purely explicit system of ordinary differential equations [2]. However, in this paper, we confine to the numerical quadrature of $B(U,V)$ solely.

<u>3.1 Q-th order Gaussian rules</u>.

Let $q \ge 2r$ be a constant integer and let $-1 \le z_1 < z_2 < \dots < z_p \le 1$ be p distinct points on I and let, for $f \in W^q(I)$

$$(3.1)\qquad \int_{-1}^{1} f(z)\,dz \doteq \sum_{i=1}^{p} w_i f(z_i)$$

be an approximation which is exact if $f \in P_{q-1}(I)$. Given a partition Δ of

I, we define for $\alpha,\beta \in W^q(\Delta)$

$$(\alpha,\beta)_j^* = \frac{h}{2} \sum_{i=1}^{p} w_i (\alpha\beta)(x_{j-1} + \frac{h}{2}(1+z_i));$$

$$(\alpha,\beta)_h = \sum_{j=1}^{N} (\alpha,\beta)_j^*; \tag{3.2}$$

$$B_h(\alpha,\beta) = \sum_{\ell=0}^{m} (p_\ell D^\ell \alpha, D^\ell \beta)_h .$$

As examples, we can take r-point Gauss-Legendre or (r+1)-point Lobatto quadrature.

LEMMA 8. *For any* $U,V \in S(\Delta)$, *we have for sufficiently small* h

$$|B(U,V) - B_h(U,V)| \le Ch^{q-2k+i+j} \|U\|_{i,\Delta} \|V\|_{j,\Delta} ; \tag{3.3}$$
$$0 \le i,\ j \le k.$$

PROOF. Application of Bramble and Hilbert's lemma [3] gives

$$|B_h(U,V) - B(U,V)| \le Ch^{q+1} \sum_{j=1}^{n} \sum_{\ell=0}^{m} \|D^q(p_\ell D^\ell U D^\ell V)\|_{L^\infty(I_j)} \le$$

$$\le Ch^q \|U\|_{k,\Delta} \|V\|_{k,\Delta} \sum_{\ell=0}^{m} \|p_\ell\|_{W^q(\Delta)} \le \tag{3.4}$$

$$\le Ch^{q+i+j-2k} \|U\|_{i,\Delta} \|V\|_{j,\Delta} . \quad \square$$

By applying lemma 8 for $i = j = m$, it is easily proved that

COROLLARY 1. *If* h *is sufficiently small then the bilinear mapping* $B_h : S(\Delta) \times S(\Delta) \to \mathbb{C}$ *is strongly coercive.* $\square$

As a last preliminary of this §, we prove

LEMMA 9. *For* $v \in H^{k+1}(I) \cap H_0^m(I) \cap W^q(\Delta)$, *let* $V \in S(\Delta)$ *be an approximation of* v *with the error bound*

(3.5) $$\|v-V\|_{\ell} \le Ch^{k+1-\ell}\|v\|_{k+1}, \ \ell=0,\dots,m.$$

Then we have

(3.6) $$\|V\|_{k,\Delta} \le C\|v\|_{k+1}.$$

PROOF. Let Π: $H^{k+1}(\Delta) \cap H_0^m(I) \cap W^q(\Delta) \to S(\Delta)$ be defined by (2.34). Then [4]

$$\|V\|_{k,\Delta} \le \|V-\Pi v\|_{k,\Delta} + \|v-\Pi v\|_{k,\Delta} + \|v\|_k \le$$

(3.7) $$\le C_1h^{-k}\|V-\Pi v\|_0 + C_2h\|D^{k+1}v\|_0 + \|v\|_k \le$$

$$\le C\|v\|_{k+1} + C_1h^{-k}[\|v-V\|_0 + \|v-\Pi v\|_0] \le C\|v\|_{k+1}. \quad \Box$$

3.2 Preservation of the orders of convergence.

In this section, we shall prove that the replacement of $B(,)$ by $B(,)_h$ does not affect the validity of theorems 2 and 3 except that the constant μ will be slightly smaller. This is due to the fact that

(3.8) $$\mu < \Lambda_1^* = \inf_{V\in S(\Delta)} \frac{B_h(V,V)}{(V,V)}$$

and Λ_1^* need no longer be greater than λ_1.

Let $Y:J \to S(\Delta)$ be the solution of the initial boundary problem

(3.9)
$$\left(\frac{\partial Y}{\partial t}, V\right) + B_h(Y,V) = 0, \ V \in S(\Delta), \ t \in J;$$
$$Y(0) = U_0 = \Pi u_0,$$

where Π is defined by (2.34) and B_h by (3.2). We define

$$(3.10) \qquad \eta(t) = U(t) - Y(t),$$

where U is the solution of (1.11). We again define the points $P_1, P_2, \ldots, P_5$ by (2.8) where we take care that (3.8) holds, in other words that (see fig. 1) $\hat{\eta} = L\eta(s)$ has no poles inside $\overline{P_1P_2P_3P_4P_5}$. Then we can prove, analogue to lemma 2, that

$$(3.11) \qquad D^{\ell}\eta(t,x) = \frac{e^{-\mu t}}{\pi} \int_0^{\infty} e^{-\alpha t} \mathrm{Im}\,[(1+i)e^{-i\alpha t} D^{\ell}\hat{\eta}(-\alpha-\mu-i\alpha,x)]d\alpha; \quad \ell=0,\ldots,m-1.$$

As before, we are only interested in the case $s \in \overline{P_4P_1P_3}$. By applying L to (3.9) and subtracting the result from (2.3) we get

$$(3.12) \qquad B_h(\hat{\eta},V) + s(\hat{\eta},V) = B_h(\hat{U},V) - B(\hat{U},V), \quad V \in S(\Delta).$$

If we substitute $V = \hat{\eta}$ and apply the lemmas 8 and 9 plus formula (2.24), we get

$$(3.13) \qquad \begin{aligned} &|B_h(\hat{\eta},\hat{\eta}) + s(\hat{\eta},\hat{\eta})| \le Ch^{q-k+m}\|\hat{\eta}\|_m \|\hat{U}\|_{k,\Delta} \le \\ &\le Ch^{q-k+m}\|\hat{\eta}\|_m \|\hat{u}\|_{k+1} \le C(\alpha)h^{q-k+m}\|u_0\|_{k+1} \|\hat{\eta}\|_m. \end{aligned}$$

Since $B_h(\hat{\eta},\hat{\eta}) \ge \Lambda_1^*(\hat{\eta},\hat{\eta})$ and B_h is strongly coercive, we can prove from (3.13) that

$$(3.14) \qquad \|\hat{\eta}\|_m \le C(\alpha)h^{q-k+m} \|u_0\|_{k+1}.$$

For $\hat{\eta}$ we now can prove the local bounds

$$(3.15) \qquad \begin{aligned} &|D^{\ell}\hat{\eta}(x_j)| = |B(\hat{\eta},\hat{G}_{\ell j}) + s(\hat{\eta},\hat{G}_{\ell j})| \le \\ &\le |B(\hat{\eta},G_{\ell j}-V) + s(\hat{\eta},\hat{G}_{\ell j}-V)| + |B_h(\hat{Y},V)-B(\hat{Y},V)| \le \\ &\le C(\alpha)\|\hat{\eta}\|_m \|\hat{G}_{\ell j}-V\|_m + Ch^q\|\hat{Y}\|_{k,\Delta} \|V\|_{k,\Delta}. \end{aligned}$$

We take V such that (2.28) holds. For $\hat{Y}$, we see that

$$\|\hat{u}-\hat{Y}\|_{\ell} \le \|\hat{e}\|_{\ell} + \|\hat{\eta}\|_{\ell} \le C(\alpha)h^{k+1-\ell}\|\hat{u}\|_{k+1},$$

hence after application of lemma 9

$$\|\hat{Y}\|_{k,\Delta} \le C(\alpha)\|u\|_{k+1} \le C(\alpha)\|u_0\|_{k+1}. \tag{3.16}$$

From (3.14) - (3.16), it now easily follows that

$$|D^{\ell}\eta(x_j)| \le C(\alpha)h^q\|u_0\|_{k+1}, \qquad \ell=0,\dots,m-1;\ j=1,\dots,N-1; \tag{3.17}$$

and as an immediate result of (3.11) and (3.17)

$$|D^{\ell}\eta(t,x_j)| \le F(t)e^{-\mu t}h^q\|u_0\|_{k+1}, \tag{3.18}$$

where $F(0) = 0$, $F(t)$ is bounded on J and where $F(t) = O(t^{-1})$ as $t \to \infty$.

For the local bounds of $\eta(t)$ at the Jacobi points, we confine our attention to the case $k \ge 2m$. Let $S(I_j)$ and ξ_{ij} be defined by (2.40) and (2.33). Then for arbitrary j, we can prove from (3.12) that

$$\begin{aligned}(\hat{\eta},LV+\bar{s}V) &= B_h(\hat{Y},V) - B(\hat{Y},V) + \\ &+ \sum_{\ell=1}^{m}\sum_{\nu=0}^{\ell-1}[(-1)^{\nu}D^{\nu}(p_{\ell}D^{\ell}V)D^{\ell-1-\nu}\hat{\eta}]\Big|_{x_{j-1}}^{x_j}, \quad V \in S(I_j).\end{aligned} \tag{3.19}$$

If we apply the quadrature rule (2.44) to (3.19) put $V = \phi_i$, where ϕ_i is defined by (2.42) and multiply by $2h^{2m-1}$, we obtain

$$\begin{aligned}\Big|\sum_{\ell=1}^{n}\omega_{\ell}h^{2m}(L\phi_i(\xi_{\ell j}) + \bar{s}\delta_{i\ell})\hat{\eta}(\xi_{\ell j})\Big| &\le \\ \le 2h^{2m-1}|B_h(\hat{Y},\phi_i) - B(\hat{Y},\phi_i)| \\ + Ch^{2m+2r}\|D^{2r}(\phi_i(L\hat{\eta}+\bar{s}\hat{\eta}))\|_{L^\infty(I_j)} + \\ + C(\alpha)h^{2m+q}\|u_0\|_{k+1}\ \|\phi_i\|_{W^k(I_j)} &\le\end{aligned} \tag{3.20}$$

$$\le C(\alpha)\|\phi_i\|_{W^k(I_j)} h^{2m+2r} *$$
$$* [\|\hat{Y}\|_{W^k(I_j)} + \|\hat{\eta}\|_{W^k(I_j)} + \|u_0\|_{k+1}] \le$$
$$\le C(\alpha)h^{k+2}\|u_0\|_{k+1}, \qquad i=1,\ldots,n.$$

For the last inequality we used lemma 9 and the inequality

$$\|\hat{\eta}\|_{W^k(I_j)} \le Ch^{m-k-1}\|\hat{\eta}\|_{W^{m-1}(I_j)} \le Ch^{m-k-1}\|\hat{\eta}\|_m \le$$
$$\le C(\alpha)h^{q-2k+2m-1}\|u_0\|_{k+1},$$

which can be proved by Sobolev's embedding theorems [11] and (3.7). From (3.20) and the results of §2.4, we easily prove that

$$(3.21) \qquad |\hat{\eta}(\xi_{ij})| \le C(\alpha)h^{k+2}\|u_0\|_{k+1}$$

and application of (3.11) gives

$$(3.22) \qquad |\eta(t,\xi_{ij})| \le F(t)e^{-\mu t}h^{k+2}\|u_0\|_{k+1},$$

where $F(t)$ is bounded on J, $F(0) = 0$ and $F(t) = O(t^{-1})$, as $t \to \infty$.

We have to estimate $\|\eta(t)\|_0$ yet. Since $\eta \in S(\Delta)$, this job is very easy, because all the nodal values of $\eta(t)$: $D^{\ell}\hat{\eta}(t,x_j)$ and $\hat{\eta}(t,\xi_{ij})$ have been shown to be of $O(h^{k+2}F(t)e^{-\mu t})$. This implies automatically that

$$(3.23) \qquad \begin{aligned} \|\eta(t)\|_{L^\infty(I)} &\le F(t)e^{-\mu t}h^{k+2}\|u_0\|_{k+1}; \\ \|\eta(t)\|_0 &\le F(t)e^{-\mu t}h^{k+2}\|u_0\|_{k+1}. \end{aligned}$$

For n=0, we have to replace k+2 by k+1 in (3.23). By this, we proved

THEOREM 3. *Let* $Y: J \to S(\Delta)$ *be the solution of* (3.9) *and let* $u: J \to H_0^m(I) \cap H^{k+1}(I) \cap W^q(\Delta)$ *be the solution of* (1.1) *with* $q \ge 2r$. *Then,* *if* h *is small enough, the error function* $\zeta(t) = u(t)-Y(t)$ *has the bounds*

$$\|\zeta(t)\|_0 \leq \|e(t)\|_0 + F_1(t)e^{-\mu t}h^{\nu}\|u_0\|_{k+1};$$
$$\nu = \min(k+2,2r);$$
$$|\zeta(t,x_j)| \leq F_2(t)e^{-\mu t}h^{2r}\|u_0\|_{W^{2r}(\Delta)};$$
$$j = 1,\ldots,N-1;$$
$$|\zeta(t,\xi_{ij})| \leq F_3(t)e^{-\mu t}h^{k+2}\|u_0\|_{W^{2r}(\Delta)};$$
$$i = 1,\ldots,n;\; j = 1,\ldots,N.$$

where $\|e(t)\|_0$ *has the bound* (1.12), μ *has the bound* (3.8) *and where* F_1, F_2 *and* F_3 *vanish if* $t=0$, *are bounded on* J *and of* $O(t^{-1})$, *as* $t \to \infty$. □

4. CONCLUSIONS

In the preceding sections we saw that earlier superconvergence results [2,7,8,9,10] can be generalized to 2m-th order problems if the spatial operator is independent of time and linear. In that case the Laplace transformation enabled us to transfer the local convergence results of $\hat{e}(x)$ to its object function $e(t,x)$. It also was made clear how the superconvergence of $e(t)$ at the knots and interior nodal points crucially depends on the convergence properties of $e(0)$. Furthermore, it was shown that Gaussian points play an important role in this matter; they are to be chosen as interior nodal points for the Hermite interpolation of $u(0)$ and the local order of convergence is better at these points than at other interior points. En passant, we also gave a proof for superconvergence phenomena in the case of a 2m-th order elliptic problem. That the use of q-th order quadrature rules, necessary to evaluate the stiffness matrix, left all the convergence results of §2 unaltered was to be expected, although the supremum error of $\eta(t)$ is lower than usual.

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