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CONVERGENCE AND STABILITY ANALYSIS OF RUNGE-KUTTA TYPE METHODS FOR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

1980 Mathematics subject classification: 65 Ro5

Convergence and Stability Analysis of Runge-Kutta type Methods for Volterra Integral Equations of the Second Kind *)

by

P.J. van der Houwen

ABSTRACT

Runge-Kutta type methods for Volterra integral equations of the second kind are studied which contain additional terms in order to extend the stability region. The order of convergence is derived and for kernel functions of the form $K(x,y,f) = [A+Bx+Cy]f$ the stability behaviour of the methods is considered by deriving the characteristic equation of the difference equation satisfied by the numerical solution. For a number of Runge-Kutta methods stability regions are given and the stabilizing effect of the additional terms is illustrated.

KEY WORDS & PHRASES: *Numerical Analysis, Volterra Integral Equations, Runge-Kutta Methods, Convergence, Stability*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this note we analyse the order of convergence and the stability of a class of numerical solution methods for the second kind Volterra integral equation

$$(1.1) \qquad f(x) = g(x) + \int_{x_0}^{x} K(x,y,f(y))dy, \qquad x_0 \le y \le x \le X.$$

The numerical schemes to be considered are *Runge-Kutta type methods*. In order to define these methods we write (1.1) in the form

$$(1.1') \qquad f(x) = [g(x) + \int_{x_0}^{x_n} K(x,y,f(y))dy] + \int_{x_n}^{x} K(x,y,f(y))dy = F_n(x) + \Phi_n(x)$$

and we define the numerical approximation f_{n+1} to $f(x_{n+1})$ by

$$(1.2) \qquad f_{n+1} = \tilde{F}_n(x_{n+1}) + \tilde{\Phi}_n(x_{n+1}),$$

where $\tilde{F}_n(x)$ and $\tilde{\Phi}_n(x)$ are numerical approximations to $F_n(x)$ and $\Phi_n(x)$, respectively. Here we assume the "history term" $\tilde{F}_n(x)$ (cf. [3]) in the form

$$(1.3) \qquad \tilde{F}_n(x) = g(x) + h \sum_{j=-1}^{n-1} \sum_{\ell=0}^{m} w_{nj}^{(\ell)} K(x,x_j+\nu_{m\ell}h,f_{j+1}^{(\ell)}), \qquad n \ge 1$$

and the "Runge-Kutta part" $\tilde{\Phi}_n(x_{n+1})$ in the form

$$(1.4) \qquad \tilde{\Phi}_n(x_{n+1}) = h \sum_{\ell=0}^{m} \lambda_{m\ell} K(x_n+\theta_{m\ell}h,\ x_n+\nu_{m\ell}h, f_{n+1}^{(\ell)}).$$

The $f_{n+1}^{(j)}$ are intermediate values defined by

$$(1.5)\quad \begin{aligned} & f_0^{(j)} = 0, \quad j = 0,1,\ldots,m-1, \\ & f_0^{(m)} = f_0,\; f_{n+1}^{(j)} = \tilde{F}_n(x_n+\mu_j h) + h \sum_{\ell=0}^{m} \lambda_{j\ell} K(x_n+\theta_{j\ell}h, x_n+\nu_{j\ell}h, f_{n+1}^{(\ell)}), \\ & \qquad j = 0,1,\ldots,m; \quad \mu_m = \nu_{mm} = 1, \quad \nu_{m0} = 0. \end{aligned}$$

Note that $f_{n+1}^{(m)} = f_{n+1}$. The Runge-Kutta parameters μ_j, $\lambda_{j\ell}$, $\theta_{j\ell}$ and $\nu_{j\ell}$ are determined by accuracy conditions (cf. [2,3,4]). The Runge-Kutta scheme (1.2)-(1.5) is called an ***extended formula*** if $w_{nj}^{(\ell)} = \lambda_{m\ell}$ for all n and j, and a *mixed formula* if $w_{nj}^{(\ell)} = 0$ for $\ell = 0,1,\ldots,m-1$ and for all n and j (cf. [1]).

We now modify the Runge-Kutta scheme by replacing $\tilde{F}_n(x)$ by the new history term

$$(1.6)\qquad \tilde{F}_n^*(x) = \tilde{F}_n(x) + \gamma(x)[f_n - \tilde{F}_n(x_n)],$$

where $\gamma(x)$ is a given function of x. For *mixed Runge-Kutta schemes* where the weights $w_{nj}^{(\ell)}$ have repetition factor 1, the case $\gamma(x) \equiv 1$ was analyzed in the institute reports [4] and was shown to lead to considerably larger stability regions. Here, we extend this analysis to general Runge-Kutta schemes for which the weights $w_{nj}^{(\ell)}$ are related to linear multistep methods for ODE's (cf. section 3). In the derivation of stability regions, however, we concentrate on mixed formulas which are in our opinion more attractive from a computational point of view.

2. CONVERGENCE

Before deriving stability criteria for the Runge-Kutta method (1.2)-(1.5) and its modification according to (1.6), we consider the order of convergence of the modified scheme. For the convergence proof we need the local truncation error of the numerical scheme. Let $\hat{f}_{n+1}^{(\ell)}$, $\ell = 0,1,\ldots,m$, be the solution of (1.5) if we substitute $f(x_n)$ for f_n and $F_n(x)$ for $\tilde{F}_n^*(x)$. Then we define the truncation error at $x_n+\nu_{m\ell}h$ by

(2.1) $$T_n^{(\ell)}(h) = f(x_n+\nu_{m\ell}h) - \hat{f}_{n+1}^{(\ell)}.$$

Furthermore, we define the quadrature error at x_n by

(2.2) $$E_n(x,h) = \int_{x_0}^{x_n} K(x,y,f(y))dy - h\sum_{j=-1}^{n-1}\sum_{\ell=0}^{m} w_{nj}^{(\ell)} K(x,x_j+\nu_{m\ell}h,f(x_j+\nu_{m\ell}h)).$$

In the convergence theorem, and also in the stability analysis, we shall need the vectors $\vec{f}_{n+1}, \vec{f}(x_{n+1})$ and $\vec{T}_n(h)$ the components of which are respectively given by $f_{n+1}^{(\ell)}$, $f(x_n+\nu_{m\ell}h)$ and $T_n^{(\ell)}(h)$ where ℓ runs through the set of integers L defined by

$$L = \{\ell \mid 0 \le \ell \le m;\ w_{ij}^{(\ell)} = 0 \text{ for } \forall i,j \Rightarrow \ell \notin L\}.$$

For these vectors we define the maximum norm $\|\ \|_\infty$.

In the following λ, μ, γ, δ and w will denote the maximum values of the parameters $|\lambda_{j\ell}|$, $|\mu_j|$, $|\gamma(x)|$, $|1-\gamma(x)|$ and $|w_{ij}^{(\ell)}|$, respectively.

LEMMA 2.1. *Let* ε_n, $n = 0,1,\ldots$ *satisfy the inequality*

$$|\varepsilon_{n+1}| \le C_1|\varepsilon_n| + C_2\sum_{i=0}^{n}|\varepsilon_i| + M_1,$$

where C_1, C_2 *and* M_1 *are non-negative constants. Then*

$$|\varepsilon_{n+1}| \le [1+C(n)]^n\,[C(n)|\varepsilon_0| + M(n)]$$

with

$$C(n) = C_2\max_{1\le j\le n}\frac{C_1^{j+1}-1}{C_1-1},\quad M(n) = \max_{1\le j\le n}[C_1^{j+1}|\varepsilon_0| + \frac{C_1^{j+1}-1}{C_1-1}M_1].$$

PROOF. By mathematical induction.

THEOREM 2.1. *Let the function* $K(x,y,f)$ *satisfy a Lipschitz type condition of the form*

$$|[K(x,y,f)-\alpha K(x_n,y,f)] - [K(x,y,f^*)-\alpha K(x_n,y,f^*)]| \leq$$

$$\leq L_1[|1-\alpha|+|\alpha||x-x_n|]|f-f^*|,$$

and let $E_n(x,h)$ *satisfy the condition*

$$|E_n(x,h)-E_n(x_n,h)| \leq L_2|x-x_n||E_n(x_n,h)|,$$

where L_1, L_2 *are constants and* α *is a parameter. Then, for* $\gamma \leq 1$ *and* $f_0 = g(x_0) = f(x_0)$ *we have as* $h \to 0$ *and* nh *fixed*

$$\|\vec{f}_{n+1}-\vec{f}(x_{n+1})\|_\infty \leq A \max_{i\leq n}|E_i(x_i,h)| + \frac{B \max_{i\leq n}\|\vec{T}_i(h)\|_\infty}{1-\gamma+O(h)}, \tag{2.3}$$

where A *and* B *are constants.*

<u>PROOF</u> From (2.1) it follows that

$$|f_{n+1}^{(j)}-f(x_n+\nu_{mj}h)| \leq |f_{n+1}^{(j)}-\hat{f}_{n+1}^{(j)}| + |T_n^{(j)}(h)|, \tag{2.4}$$

and from the definition of $f_{n+1}^{(j)}$, $\hat{f}_{n+1}^{(j)}$ and the Lipschitz condition on K with $\alpha = 0$ it follows that

$$|f_{n+1}^{(j)}-\hat{f}_{n+1}^{(j)}| \leq |\tilde{F}^*(x_n+\mu_jh)-F_n(x_n+\mu_jh)|+L_1\lambda h \sum_{\ell=0}^{m}|f_{n+1}^{(\ell)}-\hat{f}_{n+1}^{(\ell)}|. \tag{2.5}$$

From (1.6) and (2.2) we derive

$$|\tilde{F}_n^*(x)-F_n(x)| \leq \gamma|f_n-f(x_n)|+h\sum_{i=-1}^{n-1}\sum_{\ell=0}^{m}|w_{ni}^{(\ell)}||K(x,x_i+\nu_{m\ell}h,f_{i+1}^{(\ell)})$$

$$- \gamma(x)K(x_n,x_i+\nu_{m\ell}h,f_{i+1}^{(\ell)}) - K(x,x_i+\nu_{m\ell}h,f(x_i+\nu_{m\ell}h))$$

$$+ \gamma(x)K(x_n,x_i+\nu_{m\ell}h,f(x_i+\nu_{m\ell}h))| + |E_n(x,h) - \gamma(x)E_n(x_n,h)|.$$

By using the Lipschitz condition on K with $\alpha = \gamma(x)$ and the Liptschitz condition on E_n it follows

$$(2.6)\qquad |\tilde{F}_n^*(x)-F_n(x)| \le \gamma|e_n^{(m)}| + L_1 h[\delta+\gamma|x-x_n|]\sum_{i=-1}^{n-1}\sum_{\ell=0}^{m}|w_{ni}^{(\ell)}||e_{i+1}^{(\ell)}|$$

$$+[\delta+L_2|x-x_n|]|E_n(x_n,h)|,$$

where we have written $e_{i+1}^{(\ell)} = f_{i+1}^{(\ell)} - f(x_i+\nu_{m\ell}h)$. Substitution of (2.6) into (2.5) and then (2.5) into (2.4) yields

$$(2.4')\qquad |e_{n+1}^{(j)}| \le (\gamma+L_1\lambda h)|e_n^{(m)}| + L_1 h[\delta+\gamma\mu h)\sum_{i=-1}^{n-1}\sum_{\ell=0}^{m}|w_{ni}^{(\ell)}||e_{i+1}^{(\ell)}|$$

$$+[\delta+L_2\mu h]|E_n(x_n,h)| + |T_n^{(j)}(h)| + L_1\lambda h\sum_{\ell=0}^{m}|f_{n+1}^{(\ell)}-\hat{f}_{n+1}^{(\ell)}|.$$

To get rid of the last term in the right-hand side we again use (2.5) and (2.6) to obtain

$$\sum_{\ell=1}^{m}|f_{n+1}^{(\ell)}-\hat{f}_{n+1}^{(\ell)}| \le \frac{m+1}{1-(m+1)L_1\lambda h}\{\gamma|e_n^{(m)}|$$

$$+L_1 h[\delta+\gamma\mu h]\sum_{i=-1}^{n-1}\sum_{\ell=0}^{m}|w_{ni}^{(\ell)}||e_{i+1}^{(\ell)}| + [\delta+L_2\mu h]|E_n(x_n,h)|\},$$

where h is assumed to be sufficiently small. Substitution into (2.4') yields

$$|e_{n+1}^{(j)}| \le A_1|e_n^{(m)}| + A_2\sum_{i=0}^{n-1}\sum_{\ell=0}^{m}|w_{ni}^{(\ell)}||e_{i+1}^{(\ell)}| + A_3|E_n(x_n,h)| + |T_n^{(j)}(h)|,$$

(2.4")

$$A_1 = L_1\lambda h + \frac{\gamma}{1-(m+1)L_1\lambda h},\quad A_2 = \frac{L_1(\delta+\gamma\mu h)h}{1-(m+1)L_1\lambda h},\quad A_3 = \frac{\delta+L_2\mu h}{1-(m+1)L_1\lambda h}.$$

Introducing the vectors $\vec{f}_{n+1}, \vec{f}(x_{n+1})$ and $\vec{T}_n(h)$ we derive from (2.4")

$$(2.7)\qquad \|\vec{f}_{n+1}-\vec{f}(x_{n+1})\|_\infty \le A_1\|\vec{f}_n - f(x_n)\|_\infty + (m+1)wA_2\sum_{i=0}^{n}\|\vec{f}_i-\vec{f}(x_i)\|_\infty$$

$$+A_3|E_n(x_n,h)| + \|\vec{T}_n(h)\|_\infty.$$

The estimate (2.3) is now readily derived by applying lemma 2.1 to (2.7) with $\gamma \leq 1$. □

From this theorem it follows that only for $f_{n+1}^{(j)}$ values used in the history term the local truncation error $T_n^{(j)}$ is needed. Furthermore, if the quadrature error is $O(h^q)$ as $h \to 0$ and the local truncation errors are $O(h^p)$, then the order of convergence is $\min\{q,p\}$ if $\gamma < 1$ and $\min\{q,p-1\}$ if $\gamma = 1$. Thus, an order is lost if $p \leq q$ and $\gamma \to 1$. We observe, however, that in *extended* Runge-Kutta formulas $p = q+1$ so that the order of convergences is preserved if $\gamma = 1$.

3. STABILITY

Consider the simple test equation

of n)

$$f(x) = 1 + \int_{x_0}^{x} (A+Bx+Cy) f(y)\,dy; \quad A,B,C \text{ constant.} \tag{3.1}$$

The numerical scheme reduces to (we assume $\gamma(x_n+\mu_j h) = \gamma_j$ independent of n)

$$\begin{aligned} f_{n+1}^{(j)} &= \gamma_j f_n + (1-\gamma_j)\tilde{F}_n(x_n) + \frac{\mu_j z_2}{h}\tilde{G}_n + \sum_{\ell=0}^{m} \lambda_{j\ell}[z_1+\theta_{j\ell}z_2+\nu_{j\ell}z_3] f_{n+1}^{(\ell)}, \\ \tilde{G}_n &= h\sum_{j=-1}^{n-1}\sum_{\ell=0}^{m} w_{nj}^{(\ell)} f_{j+1}^{(\ell)}, \quad z_1 = (A+Bx_n+Cx_n)h, \quad z_2 = Bh^2, \quad z_3 = Ch^2. \end{aligned} \tag{3.2}$$

In this section we derive a recurrence relation for $\vec{f}_n$, $\tilde{F}_n(x_n)$ and $\tilde{G}_n$ with a fixed number of terms. The corresponding characteristic equation determines the numerical behaviour of these quantities. Recall that $\vec{f}_n$ is the vector of intermediate values $f_n^{(j)}$ which are used in the history term, $\tilde{F}_n(x_n)$ approximates $F_n(x_n) = f(x_n)$ and $\tilde{G}_n$ approximates $\int_{x_0}^{x_n} f(y)\,dy$.

In the first step of our analysis we express $f_{n+1}^{(j)}$ in terms of f_n, $\tilde{F}_n(x_n)$ and $\tilde{G}_n$. It is easily verified that we may write

$$f_{n+1}^{(j)} = (Q_j+\gamma_n R_j) f_n + (1-\gamma_n) R_j \tilde{F}_n(x_n) + \frac{z_2}{h} S_j \tilde{G}_n, \quad j = 0,1,\ldots m, \tag{3.3}$$

where Q_j, R_j and S_j are functions of $\vec{z} = (z_1,z_2,z_3)$ which satisfy the recurrence relations

(3.4)
$$(Q_j+\gamma_j R_j) = \gamma_j + \sum_{\ell=0}^{m} \lambda_{j\ell} z_{j\ell}(Q_\ell+\gamma_\ell R_\ell), \quad z_{j\ell} = z_1 + \theta_{j\ell} z_2 + \nu_{j\ell} z_3,$$
$$(1-\gamma_j)R_j = 1-\gamma_j + \sum_{\ell=0}^{m} (1-\gamma_\ell)\lambda_{j\ell} z_{j\ell} R_\ell, \quad j = 0,1,\dots,m,$$
$$S_j = \mu_j + \sum_{\ell=0}^{m} \lambda_{j\ell} z_{j\ell} S_\ell .$$

Note that Q_j, R_j and S_j are independent of γ_j if γ_j is a constant γ.

In order to derive a recurrence relation for $\tilde{F}_n(x_n)$ and $\tilde{G}_n$ we exploit a property of the weights $w_{nj}^{(\ell)}$ which is satisfied by most quadrature rules used in practice: we will assume that coefficients a_i and $b_i^{(\ell)}$ $(i=0,1,\dots,k;$ $\ell=0,1,\dots,m;$ n fixed) exist such that (cf. [5])

(3.6)
$$\sum_{i=0}^{k} a_i = 0, \quad \sum_{i=0}^{k} a_i w_{n-i+1,j}^{(\ell)} = \begin{cases} 0 & \text{for } j = 0,1,\dots,n-k-1 \\ b_{n-j}^{(\ell)} & \text{for } j = n-k,\dots,n \end{cases} .$$

Without loss of generality we may assume that $w_{nj}^{(0)} = 0$ for $j \geq 1$, hence $b_i^{(0)} = 0$ for $n \geq k+1$.

Two cases will be distinguished $|\gamma_j| < 1$ and $\gamma_j = 1$. For $|\gamma_j| < 1$ we derive by virtue of (3.6)

(3.7)
$$\sum_{i=0}^{k} a_i \tilde{F}_{n-i+1}(x_{n-i+1}) = \sum_{i=0}^{k} \{ \sum_{\ell=0}^{m} b_i^{(\ell)} [z_1 - (i-\nu_{m\ell}) z_3] f_{n+1-i}^{(\ell)} - \frac{1}{h} z_2 a_i (i-1) \tilde{G}_{n+1-i} \}, \quad n \geq k+1 .$$

and

(3.8)
$$\sum_{i=0}^{k} a_i \tilde{G}_{n+1-i} = h \sum_{i=0}^{k} \sum_{\ell=0}^{m} b_i^{(\ell)} f_{n+1-i}^{(\ell)}, \quad n \geq k+1$$

and introducing the vector $\vec{V}_n = (\vec{f}_n, \tilde{F}_n(x_n), \tilde{G}_n)^T$ we may write the relations (3.3), (3.7) and (3.8) in the form

$$(3.9)\qquad \begin{pmatrix} I & 0 & 0 \\ -z_1B_0-z_3\tilde{B}_0 & a_0 & -a_0\frac{z_2}{h} \\ -hB_0 & 0 & a_0 \end{pmatrix} \vec{V}_{n+1} +$$

$$+ \begin{pmatrix} -Q-\Gamma R & (\Gamma-I)\tilde{R} & -\frac{z_2}{h}\tilde{S} \\ (z_3-z_1)B_1-z_3\tilde{B}_1 & a_1 & 0 \\ -hB_1 & 0 & a_1 \end{pmatrix} \vec{V}_n +$$

$$+ \sum_{i=2}^{k} \begin{pmatrix} 0 & 0 & 0 \\ (iz_3-z_1)B_i-z_3\tilde{B}_i & a_i & (i-1)a_i\frac{z_2}{h} \\ -hB_i & 0 & a_i \end{pmatrix} \vec{V}_{n+1-i} = \vec{0}, \quad n \ge k+1,$$

where Q, R, $\tilde{R}$, $\tilde{S}$, B_i and $\tilde{B}_i$ are matrices defined by

$$Q = (Q_j\delta_{j,j+\ell-m}),\ R = (R_j\delta_{j,j+\ell-m}),\ \tilde{R} = (R_j),\ \tilde{S} = (S_j),$$

$$B_i = (b_i^{(\ell)}),\ \tilde{B}_i = (\nu_{m\ell}b_i^{(\ell)}),\quad \Gamma = (\gamma_j\delta_{j\ell}),\ I = (\delta_{i\ell})\ j \text{ and } \ell \in L.$$

Here, δ denotes the Kronecker symbol and j,ℓ are the row and column index, respectively.

Defining the characteristic polynomials

$$\rho(\zeta) = \sum_{i=0}^{k} a_i\zeta^{k-1},\ \sigma(\zeta) = \sum_{i=0}^{k} B_i\zeta^{k-i},\ \tilde{\sigma}(\zeta) = \sum_{i=0}^{k} \tilde{B}_i\zeta^{k-i},$$

we may express the characteristic equation of (3.9) in the form (note that $\sigma(\zeta)$ and $\sigma(\zeta)$ have matrix coefficients)

$$(3.10)\qquad \det \begin{bmatrix} \zeta^{k-1}[\zeta-Q-\Gamma R] & (\Gamma-I)\tilde{R}\zeta^{k-1} & -\frac{z_2}{h}\tilde{S}\zeta^{k-1} \\ z_3\tilde{\sigma}(\zeta)+(z_1-kz_3)\sigma(\zeta)+z_3\zeta\sigma'(\zeta) & -\rho(\zeta) & \frac{z_2}{h}[(1-k)\rho(\zeta)+\zeta\rho'(\zeta)] \\ -h\,\sigma(\zeta) & 0 & \rho(\zeta) \end{bmatrix} = 0$$

where $\sigma'(\zeta)$ and $\rho'(\zeta)$ are polynomials obtained by differentiation of $\sigma(\zeta)$ and $\rho(\zeta)$.

In (3.10) it is assumed that $|\gamma_j| < 1$. For $\gamma_j = 1$ the $\tilde{F}_n(x_n)$ term vanishes in (3.3), hence we need no recurrence relation for $\tilde{F}_n(x_n)$. Proceeding in the same manner as above we find for $\gamma_j = 1$ a characteristic equation which can be obtained from (3.10) by omitting the second row and column in the determinant.

To the characteristic equation (3.10) we associate the *stability region* consisting of the set of points $\vec{z} = (z_1,z_2,z_3)$ in the $\vec{z}$-space where the roots ζ of (3.10) are on the unit disk. The vector $\vec{V}_n$ is certainly bounded as $n \to \infty$ and h fixed if the following conditions are satisfied:

(i) *Root condition*: The roots $\zeta(\vec{z})$ of (3.10) are on the unit disk those on the unit circle being simple roots.

(ii) *Constant-coefficient condition*: The coefficients in (3.9) do not depend on n.

Evidently, a necessary condition for satisfying the root condition is that $\vec{z}$ lies in the stability region.

If the constant-coefficient condition is not satisfied the stability region may still be of some value if the coefficients change slowly with n (compare the situation in ODE stability theory).

3.1 Mixed Runge-Kutta schemes

Mixed Runge-Kutta schemes arise when the set L only contains the integer m. The quantities Q, R, $\tilde{R}$, S and σ become scalar functions and the characteristic equation (3.10) reduces to (note that $\tilde{\sigma} = \sigma$)

$$(3.11)\qquad \zeta^{k-1}\{\rho(\zeta)[\rho(\zeta)(\zeta-Q_m(\vec{z})-\gamma_m R_m(\vec{z}))-z_2\sigma(\zeta)S_m(\vec{z})] + (1-\gamma_m)R_m(\vec{z})[-z_1\rho(\zeta)\sigma(\zeta) + z_2((k-1)\rho(\zeta)-\zeta\rho'(\zeta))\sigma(\zeta) + z_3((k-1)\sigma(\zeta) - \zeta\sigma'(\zeta))\rho(\zeta)]\} = 0$$

where $|\gamma_m| < 1$. For $\gamma_m = 1$ we obtain the equation

$$(3.11')\qquad \zeta^{k-1}[\rho(\zeta)(\zeta-Q_m(\vec{z})-R_m(\vec{z}))-z_2\sigma(\zeta)S_m(\vec{z})] = 0.$$

We derive stability regions for a few history terms, characterized by the characteristic polynomials $\{\rho,\sigma\}$ listed in table 3.1, and a few Runge-Kutta parts defined by the polynomials Q_m, R_m and S_m listed in table 3.2. For a detailed discussion of the evaluation of the history terms by backward differentiation quadrature rules (BD rules) we refer to [6]. The polynomials Q_m, R_m and S_m correspond to Runge-Kutta formulas with $(\gamma_\ell) = (1,\gamma,\ldots,\gamma)$. Note that by choosing $\gamma_0 = 1$ an F_n evaluation is saved in all formulas listed in table 3.2 (cf. [4], [8]).

Table 3.1 History terms defined by the polynomials $\{\rho,\sigma\}$

Formula	k	$\rho(\zeta)$	$\sigma(\zeta)$
Repeated trapezium rule	1	$\zeta-1$	$\frac{1}{2}(\zeta+1)$
Third order Gregory rule	2	$\zeta(\zeta-1)$	$\frac{1}{12}[5\zeta^2+8\zeta-1]$
Second order BD rule [6]	2	$\frac{1}{3}[3\zeta^2-4\zeta+1]$	$\frac{2}{3}\zeta^2$
Third order BD rule [6]	3	$\frac{1}{11}[11\zeta^3-18\zeta^2+9\zeta-2]$	$\frac{6}{11}\zeta^3$

Table 3.2 Runge Kutta parts defined by $\{Q_m,R_m,S_m\}$

Formula	Q_m	R_m	S_m
Forward Euler	z_1+z_2	1	R_1
Backward Euler	0	$[1-z_1-z_2-z_3]^{-1}$	R_0
Trapezium rule	$[z_1+z_2][2-z_1-z_2-z_3]^{-1}$	$2[2-z_1-z_2-z_3]^{-1}$	R_1
Fourth order Beltjukov [2,3,4]	$\frac{1}{36}[2z_1+z_2][6z+3z^2-z_3(1+2z)]$	$\frac{1}{18}[18+18z+3z^2-z_3(9+2z)]$	$R_3-\frac{3z-2z_3}{6}$
	$z = z_1+z_2+z_3$		
Fourth order Newton-Cotes [4,formula (2.50)]	$\frac{z_1+z_2+(z_1+\frac{1}{2}z_2)Z}{6-z_1-z_2-z_3}$	$\frac{6+4Z}{6-z_1-z_2-z_3}$	$\frac{6+2Z}{6-z_1-z_2-z_3}$
	$Z = 4\,\frac{2z_1+2z_2+z_3}{8-2z_1-z_2-z_3}$		

3.1.1. The test kernel K = Af

This kernel implies that $z_2 = z_3 = 0$ in the characteristic equations (3.11) and (3.11'), hence the stability region reduces to a stability interval along the z_1-axis. In table 3.3 these intervals are listed for a few combinations of the history terms and Runge-Kutta parts specified above and for a few values of γ (we observe that both the root condition and the constant coefficient condition are satisfied in all cases of table 3.3). For $\gamma = 1$ it follows from (3.11') that the stability interval only depends on the Runge-Kutta part of the method. The stabilizing effect of the additional terms is clearly demonstrated by the figures in table 3.3.

Table 3.3 Stability intervals for various values of γ

	$\gamma = 0$	$\gamma = \frac{1}{2}$	$\gamma = 1$
Rep. Trap. Rule - Euler Forward	(-1,0]	(-1½,0]	(-2,0]
Gregory - Trap. Rule	(-∞,0]	(-∞,0]	(-∞,0]
Gregory - Beltjukov	(-1.23,0]	(-1.23,0]	(-2.51,0]
Second order BD - Euler Forward	(-0.77,0]	(-1.31,0]	(-2,0]
Third order BD - Beltjukov	(-1.23,0]	(-1.23,0]	(-2.51,0]

3.1.2. The convolution kernel K = [A+B(x-y)]f

In this case $z_3 = -z_2 = -Bh^2$ and $z_1 = Ah$. Hence, the constant-coefficient is satisfied. The root condition is satisfied in all points of the stability regions given in figure 3.1.

3.1.3. The test kernel K = (A+Bx)f

For this kernel $z_1 = (A+Bx_n)h$ changes with n so that the constant-coefficient condition is not satisfied. Therefore, the stability regions have only a local meaning. However, when $|Bx_n| \ll |A|$ the stability region determined by (3.10) may be considered as a first approximation to the true region of stability. In figure 3.2 a few of these approximate regions are given (further examples, e.g. Simpson-Kutta formulas, may be found in [4]).

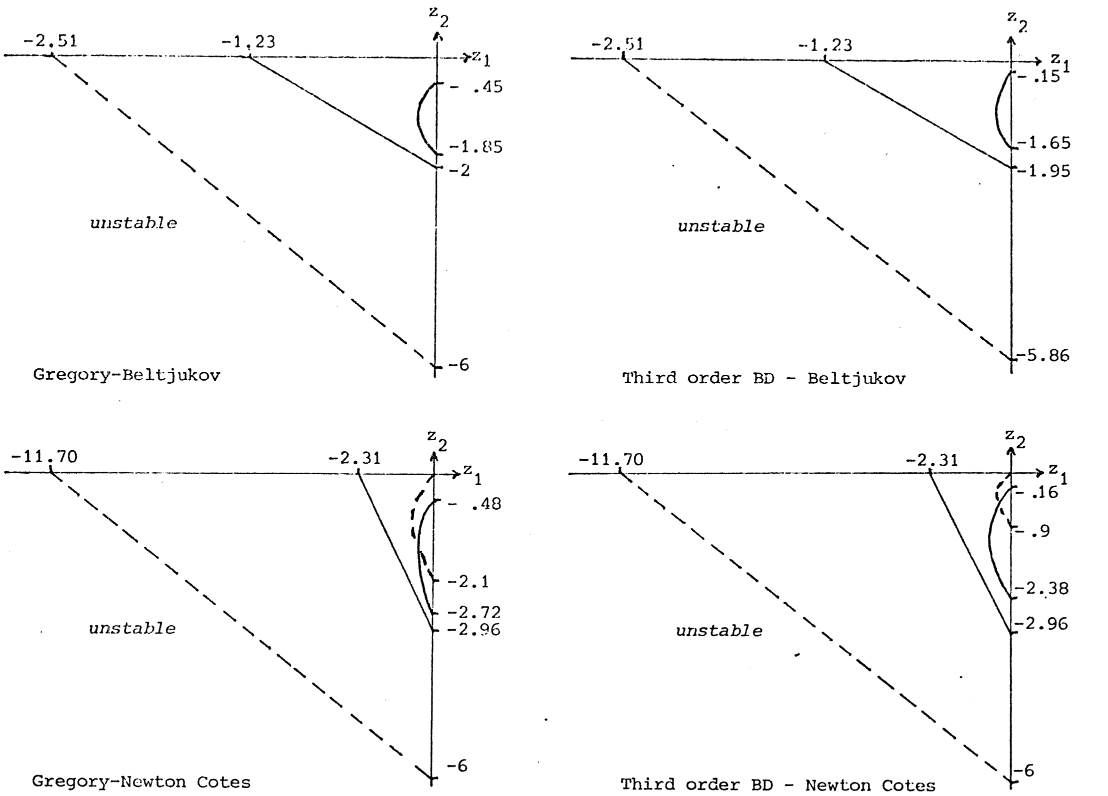

Fig. 3.1 Stability regions for K = [Ax+B(x-y)]f; ——γ =0; ---γ =1

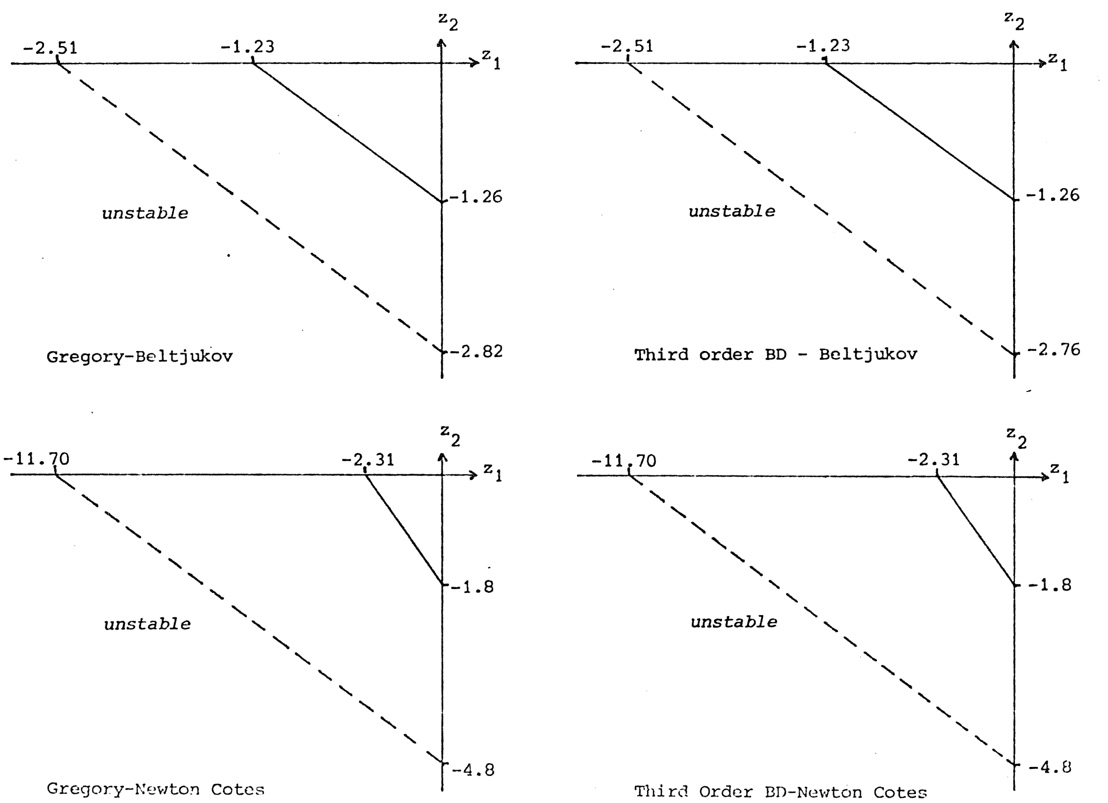

Fig. 3.2 Stability regions for K = [A+Bx]f; ——γ =0; ---γ =1

3.2 Alternative analysis of stability

In the case of the convolution test equation (C = -B) it is not difficult to derive a recurrence relation only involving the vectors $\vec{f}_n$. For $\gamma_j = 1$ this derivation is particularly simple, because by forming the linear combinations of the relations (3.3) we obtain

$$(3.12) \qquad \sum_{i=0}^{k} a_i f_{n+1-i}^{(j)} = \sum_{i=0}^{k} a_i \{[Q_j + R_j] f_{n-i} + \frac{z_2}{h} S_j \tilde{G}_{n-i}\},$$

and substituting (3.8) immediately leads to ($n \geq k+2$)

$$(3.12') \qquad \sum_{i=0}^{k} [a_i \vec{f}_{n+i-1} - (a_i (Q+R) + z_2 \tilde{S} B_i) \vec{f}_{n-i}] = \vec{0}$$

with the characteristic equation (cf. (3.11'))

$$(3.13') \qquad \det\{\rho(\zeta)[\zeta - Q - R] - z_2 \tilde{S} \sigma(\zeta)\} = 0.$$

For $\gamma_j \neq 1$ we form twice a linear combination of the relations (3.3) and derive in a similar way as was done in [7] for direct quadrature methods the characteristic equation

$$(3.13) \qquad \det\{\rho(\zeta)[\rho(\zeta)(\zeta - Q - \Gamma R) - z_2 \tilde{S} \sigma(\zeta)] + (1-\Gamma)\tilde{R}[-z_1 \rho(\zeta)\sigma(\zeta) + z_2((k-1)\rho(\zeta) - \zeta\rho'(\zeta))\sigma(\zeta) + z_3(k\sigma(\zeta) - \tilde{\sigma}(\zeta) - \zeta\sigma'(\zeta))\rho(\zeta)]\} = 0.$$

Note that this equation has a strong resemblance with equation (3.11).

It should be observed that (3.12'), and therefore the characteristic equations (3.13') and (3.13), only applies rigourously to the convolution case $C = -B(z_3 = -z_2)$. In the non-convolution case, the functions Q_j, R_j and S_j depend on n which should be taken into account when forming the linear combination of the relations (3.3). It can straightforwardly be shown that (3.13) then assumes the form

$$\det\Big\{ \sum_{i=0}^{k} \sum_{j=0}^{k} a_i a_j M_n^{-1} M_{n-i-j} [\zeta - Q^{(n-i-j)} - \Gamma R^{(n-i-j)}]\zeta^{2k-i-j} - z_2 \tilde{S}^{(n)} \rho\sigma$$

$$+ (1-\Gamma)\tilde{R}^{(n)}[-z_1\rho\sigma + z_2(2k\rho - 2\zeta\rho')\sigma \tag{3.14}$$

$$+ z_3(2k\rho\sigma - \zeta(\rho\sigma)' + \rho(\sigma - \tilde{\sigma}))]\Big\} = 0,$$

where $\underset{\sim}{M}_{\underset{\sim}{n}} = (\delta_{j\ell} - \lambda_{j\ell} z_{j\ell})$ and $Q^{(n)}$, $R^{(n)}$, $\tilde{R}^{(n)}$, $\tilde{S}^{(n)}$ denote the matrices $Q, R, \tilde{R}, \tilde{S}$ at the point x_n. A comparison of (3.13) and (3.14) reveals that (3.13) can be considered as an approximation to the exact equation (3.14) if $|z_2| << |z_1|$ and $|z_3| << |z_1|$.

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ERRATA IN MATHEMATICAL CENTRE REPORT NW 83/80

page 4	*line* 5^+:	$\gamma \leq 1$	*should read*	$\gamma_j < 1$ or $\gamma_j = 1$
page 5	*line* 1^+:	Liptschitz	*should read*	Lipschitz
page 6	*line* 8^+:	convergences	*should read*	convergence
	formula (3.3):	γ_m	*should read*	γ_j
page 9	*line* 2^-:	γ_m	*should read*	γ_j