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AFDELING NUMERIEKE WISKUNDE NW 87/80 AUGUSTUS
(DEPARTMENT OF NUMERICAL MATHEMATICS)

H.J.J. TE RIELE

HYPERPERFECT NUMBERS WITH MORE THAN TWO DIFFERENT PRIME FACTORS

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Hyperperfect numbers with more than two different prime factors*)

by

H.J.J. te Riele

ABSTRACT

Very recently, MINOLI [2] has defined n-hyperperfect numbers as positive integers m satisfying the equation

$$m = 1 + n[\sigma(m) - m - 1]$$

for some $n \in \mathbb{N}$. He wondered whether all hyperperfect numbers might have the form $p^{\alpha}q$, where p and q are prime numbers, $p < q$ and $\alpha \in \mathbb{N}$.

In this report we answer this question in the negative by constructing eleven hyperperfect numbers with three and one with four different prime factors.

KEY WORDS & PHRASES: *hyperperfect numbers*

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*) A shorter version of this report will appear in the January 1981 issue of Mathematics of Computation [3].

Very recently, MINOLI [2] has defined n-hyperperfect numbers as positive integers m such that for some $n \in \mathbb{N}$

(1) $$m = 1 + n[\sigma(m) - m - 1].$$

For $n = 1$, this yields the well-known perfect numbers. Minoli gives a table of all (38) n-hyperperfect numbers up to 1,500,000 with $n \geq 2$. These numbers all have the form $p^\alpha q$, where p and q are prime numbers, $p < q$ and $\alpha \in \mathbb{N}$. Minoli wonders whether *all* hyperperfect numbers might have this form.

In this report we shall construct eleven hyperperfect numbers with three and one with four different prime factors, thus answering Minoli's question in the negative. Our technique is well-known, and was used, for instance, by EULER [1] to compute amicable number pairs.

Let $m = apq$ ($a \in \mathbb{N}$, p and q different prime numbers, $(a,pq) = 1$) be an n-hyperperfect number. By (1) we have

$$apq = 1 + n[\sigma(a)(p+1)(q+1) - apq - 1].$$

We assume a and n to be given, define $\bar{a} := \sigma(a)$, and rewrite it as

$$[a - n(\bar{a} - a)]pq - n\bar{a}(p+q) = 1 + n(\bar{a} - 1).$$

Multiplying by $a - n(\bar{a}-a)$ and adding $n^2\bar{a}^2$ to both sides, we obtain

(2) $$\{[a - n(\bar{a}-a)]p - n\bar{a}\}\{[a - n(\bar{a}-a)]q - n\bar{a}\}$$

$$= [a - n(\bar{a}-a)][1 + n(\bar{a}-1)] + n^2\bar{a}^2.$$

If AB, $1 \leq A < B$, is a factorization of the (known) right hand side, then we can write

(3) $$p = \frac{n\bar{a}+A}{a-n(\bar{a}-a)}, \quad q = \frac{n\bar{a}+B}{a-n(\bar{a}-a)}.$$

There may not be integer solutions of (3), but if solutions do exist such that both p and q are primes with $(a,pq) = 1$, then $m = apq$ is an n-hyperperfect number.

In order to be sure that p and q in (3) be integers, we shall require a and n to satisfy

(4) $$a - n(\bar{a}-a) = 1.$$

(Note the resemblance of this equation with (1)!). Then (2) and (3) reduce to

(2') $$(p-n\bar{a})(q-n\bar{a}) = 1 + n(\bar{a}-1) + n^2\bar{a}^2 =: AB, \qquad 1 \le A < B,$$

and

(3') $$p = n\bar{a} + A, \quad q = n\bar{a} + B.$$

Equations (2'), (3') and (4) form our starting point for a number of special cases to be considered.

CASE 1. $a=r^\alpha$, r prime, $\alpha \in \mathbb{N}$

For this choice of a it follows from (4) that $n = (a-1)/(\bar{a}-a) = r-1$ is always an integer. From (2') and (3') we find that

(5) $$\begin{cases} p = r^{\alpha+1} - 1 + A, \quad q = r^{\alpha+1} - 1 + B, \quad \text{p and q primes,} \\ AB = r^{2\alpha+2} - r^{\alpha+1} - r + 2, \qquad 1 < A < B. \end{cases}$$

Note that we have excluded A = 1 since this gives $p = r^{\alpha+1}$, which is not a prime.

CASE 1.1. $\alpha = 1$

If $r \equiv 2 \pmod 3$, then $r^2 - 1 \equiv 0 \pmod 3$ and $r^4 - r^2 - r + 2 \equiv 0 \pmod 3$ so that, by (5), 3 divides at least one of A and B, hence also at least one of p and q. Now it is easy to see that both p and q are >3. It follows that at least one of them is composite.

For all primes r < 300 which are $\equiv 1 \pmod 3$ we have checked (5) for all possible factorizations AB. In this way we have found five hyperperfect numbers of the form rpq (viz., for r = 13, 223, 229, 277 and 283; see Table 1).

CASE 1.2. $\alpha = 2$

If $r \equiv 2 \pmod 3$, then $r^3-1 \equiv 1 \pmod 3$ and $r^6-r^3-r+2 \equiv 2 \pmod 3$. It follows from (5) that one of A and B is $\equiv 2 \pmod 3$, hence one of p and q is $\equiv 0 \pmod 3$. Excluding these r we have checked (5) for all primes $r < 160$, and we have found five hyperperfect numbers of the form r^2pq (viz., for r = 7, 13, 19, 43 and 97; see Table 1).

CASE 1.3. $\alpha = 3$

Excluding $r \equiv 2 \pmod 3$ and $r \equiv 2 \pmod 5$ (similarly as in the Case 1.1) we have checked (5) for all primes $r < 100$ and we have found one hyperperfect number of the form r^3pq (viz., for r = 73; see Table 1).

CASE 2. $a = rs$, r and s primes, $r \neq s$

From (4) it follows that for this choice of a

$$(6) \qquad n = (rs-1)/(r+s+1) = r - (r^2+r+1)/(r+s+1).$$

This is an integer provided that $(r+s+1)$ divides (r^2+r+1). Equations (2') and (3') become

$$(7) \qquad \begin{cases} p = n(r+1)(s+1)+A, & q = n(r+1)(s+1)+B, \quad p \text{ and } q \text{ primes}, \\ AB = 1-n+n(r+1)(s+1)[1+n(r+1)(s+1)], & 1 \le A < B. \end{cases}$$

Suppose now that $r \equiv 2 \pmod 3$, then $r^2+r+1 \equiv 1 \pmod 3$. It is not difficult to prove that for this r all prime divisors, hence all divisors, of r^2+r+1 are $\equiv 1 \pmod 3$. It follows that $(r^2+r+1)/(r+s+1) \equiv 1 \pmod 3$ and from (6), that $n \equiv 1 \pmod 3$. From (7) we conclude that $AB \equiv 0 \pmod 3$, hence one of p and q is $\equiv 0 \pmod 3$.

For all primes $r \equiv 1 \pmod 3$, $r < 2000$, we have determined the primes s, if existing, such that $r+s+1$ divides r^2+r+1, yielding an integer n, by (6). For these r, s and n we have checked (7), and we have found one hyperperfect number of the form rspq (viz., for r = 1327, s = 6793; see Table 1).

REMARK 1. One may construct many more hyperperfect numbers with more than two different prime factors by extending the ranges of r in the cases considered above or by considering other cases. However, the numbers in (5) and (7) to be factorized grow fast with r, hence also the amount of computer time. Another possibility to find more hyperperfect numbers may emerge from dropping the condition (4) (taking into account that *small* values of $a - n(\bar{a}-a)$ in (3) may be favourable to find *integer* values of p and q).

REMARK 2. Minoli's table shows six n-hyperperfect numbers with *odd* n (n = 3, 11, 19, 31, 35 and 59). These are all instances of the following rule: if both $p = 6k-1$ and $q = 12k+1$ are prime numbers for some $k \in \mathbb{N}$, then p^2q is a $(4k-1)$-hyperperfect number. Based on this rule, we conjecture that there are infinitely many hyperperfect numbers.

m	n
1570153 = 13.269.449	12
60110701 = $7^2$383.3203	6
13544168521 = $13^2$2347.34147	12
3675965445337 = 229.67187.238919	228
8898807853477 = 283.112087.280537	282
8992165119733 = $19^2$6871.3625243	18
72315968283289 = 277.78541.3323977	276
217158581600773 = $43^2$84319.1392883	42
348231627849277 = 223.49807.31352557	222
7972299196816043329 = $97^2$913571.927465611	96
36320978727037068273 = $73^3$31293799.306914431	72
160510813295957612416000257198 1 = 1327.6793.10020547039.17769709449589	1110

Table 1. Eleven hyperperfect numbers with three and one with four different prime factors.

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